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INDEPENDENT-TYPE STRUCTURES AND THE NUMBER OF CLOSED SUBSETS OF A SPACE

Franco **Obersnel**

Dipartimento di Matematica e Informatica, Università degli Studi di Trieste, via A. Valerio 12/1, I-34127 Trieste, Italia

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Abstract: Three different notions of an independent family of sets are considered, and it is shown that they are all equivalent under certain conditions. In particular it is proved that in a compact space X in which there is a dyadic system of size τ there exists also an independent matrix of closed subsets of size $\tau \times 2^\tau$. The cardinal function $M(X,\kappa)$ counting the number of disjoint closed subsets of size larger than or equal to κ is introduced and some of its basic properties are studied.

1. Introduction

The notion of an independent family of sets is a well known tool from set theoretic combinatorics, already used in topology by Hausdorff and even before ([2]). To approach different kinds of problems, different notions of set independence have been defined. In [3], [4], [5] for instance, the idea of an independent collection of subsets of a set is used. In [6] the notion of an independent matrix is used (for example) to show that in ω^* there exists an R-point. In [1] the notion of a dyadic system is introduced and a strong use of this (generalized) concept is made in [7] to

E-mail address: obersnel@units.it

give a characterization of the spaces that can be mapped onto a Tykhonov cube.

Meeting these notions in so different contexts, it is natural to ask if these structures might have something in common. In Sec. 2 we establish that they are all equivalent under certain conditions.

As the existence of a dyadic system, an independent matrix or an independent family of closed sets implies the existence of many disjoint closed sets in a topological space X, these notions can be used to count the cardinalities of families of pairwise disjoint closed sets of a given cardinality. In Sec. 3 we introduce the cardinal function $M(X,\kappa)$, counting the number of the disjoint closed subsets of cardinality κ of a space X, and study some of its basic properties.

2. Independent families, independent matrices, dyadic systems

Definition 2.1. An independent family of subsets of a set X is a family $\mathcal{F} = \{F_\alpha : \alpha \in A\}$ such that, for any finite collection of elements $F_{\alpha_1}, F_{\alpha_2}, ..., F_{\alpha_n}$ and $F_{\beta_1}, F_{\beta_2}, ..., F_{\beta_m}$, with distinct indices $\alpha_1, \alpha_2, ..., \alpha_n$, $\beta_1, \beta_2, \ldots, \beta_m$, we have

$$\bigcap_{i=1}^{n} F_{\alpha_i} \cap \bigcap_{i=1}^{m} (X \setminus F_{\beta_i}) \neq \emptyset.$$

Definition 2.2. An indexed family $\{A^i_j : i \in I, j \in J\}$ of closed subsets of a space X is called an $I \times J$ independent matrix if

(1) for all distinct $j_0, j_1 \in J$ and fixed $i \in I$, $A^i_{j_0} \cap A^i_{j_1} = \emptyset$;

(2) for any choice of finitely many rows $T = \{i_0, i_1, \ldots, i_n\}$, and for any function $f : T \to J$,

$$\bigcap \{A^i_{f(i)} : i \in T\} \neq \emptyset.$$

Definition 2.3. A dyadic system in a topological space X is a family of pairs of closed sets $\{\{A^0_\alpha, A^1_\alpha\} : \alpha \in A\}$ such that

(1) $A^0_\alpha \cap A^1_\alpha = \emptyset$ for any α;

(2) for any choice of finitely many α, say $T = \{\alpha_1, \alpha_2, \ldots \alpha_n\}$ and for any function $f : T \to 2$ we have

$$\bigcap \{A^{f(\alpha)}_\alpha : \alpha \in T\} \neq \emptyset.$$

It is easily seen that a dyadic system is an $I \times 2$ independent matrix. Moreover it is clear that if $\mathcal{F} = \{F_\alpha : \alpha \in A\}$ is an independent family of clopen sets in a space X then the system $\{\{F_\alpha, X \setminus F_\alpha\} : \alpha \in A\}$ is dyadic. Also, if there is a dyadic system of size κ in a space X, then there is an independent family of the same size (for, if $\mathcal{D} = \{\{D^0_\alpha, D^1_\alpha\} : : \alpha < \kappa\}$ is the dyadic system, both $\{D^0_\alpha : \alpha < \kappa\}$ and $\{D^1_\alpha : \alpha < \kappa\}$ are independent families of closed sets in X). Finally, if you are given an independent matrix $\{A^i_j : i \in I, j \in J\}$, fixing any two columns j_0 and j_1 gives the dyadic system (of size $|I|$) $\mathcal{D} = \{\{D^0_i, D^1_i\} : i \in I\}$ where $D^0_i = A^i_{j_0}$ and $D^1_i = A^i_{j_1}$. So we can state the following.

Proposition 2.1. *In any topological space X*

(1) *if there is an $I \times J$ independent matrix, then there is also a dyadic system of size $|I|$, and an independent family of closed sets of the same size;*

(2) *if there is a dyadic system of size κ, then there is an independent family of size κ and also an independent matrix of size $\kappa \times 2$;*

(3) *if there is an independent family of clopen sets of size κ, then there is a dyadic system of the same size and also an independent matrix of size $\kappa \times 2$.*

It is evident that the existence of an independent matrix with a large number of rows is stronger than the existence of a simple dyadic system or an independent family. The following result, however, allows us to produce large matrices starting from a dyadic system in a compact space.

Theorem 2.1. *Let X be a compact space and suppose that there exists in X a dyadic system $\mathcal{D} = \{\{A^0_\alpha, A^1_\alpha\} : \alpha < \tau\}$. Then there is in X a $\tau \times 2^\tau$ independent matrix of closed sets.*

Proof. Pick a family D of τ pairwise disjoint subsets of τ, each of cardinality τ. Put $D = \{d_\alpha : \alpha < \tau\}$. For any fixed $d_\alpha \in D$ consider the set ${}^{d_\alpha}2$ of the functions from d_α to 2. We clearly have $|{}^{d_\alpha}2| = 2^\tau$, so there exists a bijection $f_\alpha : 2^\tau \to {}^{d_\alpha}2$. Now, for any $\alpha \in \tau$ and any $\beta \in 2^\tau$ set

$$B^\alpha_\beta = \bigcap_{\gamma \in d_\alpha} A^{f_\alpha(\beta)(\gamma)}_\gamma.$$

Note that B^α_β is a nonempty closed subset of X. Indeed pick any finite collection of elements of d_α, say $\gamma_1, \gamma_2, \ldots, \gamma_n$; since $\mathcal{D}$ is a dyadic system,

$$A^{f_\alpha(\beta)(\gamma_1)}_{\gamma_1} \cap A^{f_\alpha(\beta)(\gamma_2)}_{\gamma_2} \cap \ldots \cap A^{f_\alpha(\beta)(\gamma_n)}_{\gamma_n} \neq \emptyset.$$

Since the space X is compact, B^α_β is not empty.

Let us show that the family $\{B^\alpha_\beta : \alpha < \tau, \beta < 2^\tau\}$ is an independent matrix of closed subsets of X. Fix α and pick $\beta_1 \neq \beta_2$. We must check that $B^\alpha_{\beta_1} \cap B^\alpha_{\beta_2} = \emptyset$. Since $\beta_1 \neq \beta_2$ we have $f_\alpha(\beta_1) \neq f_\alpha(\beta_2)$, so there exists a $\gamma' \in d_\alpha$ such that

$$f_\alpha(\beta_1)(\gamma') = |1 - f_\alpha(\beta_2)(\gamma')|.$$

Therefore $\emptyset = A^0_{\gamma'} \cap A^1_{\gamma'} \supset B^\alpha_{\beta_1} \cap B^\alpha_{\beta_2}$ and the claim is proved.

Now pick any finite subset F of τ, say $F = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$, and pick any function $g : F \to 2^\tau$. We must check that $\bigcap_{1\le i\le n} B^{\alpha_i}_{g(\alpha_i)} \neq \emptyset$. Note that $d_{\alpha_1}, d_{\alpha_2}, \ldots, d_{\alpha_n}$ are pairwise disjoint. Put $d = \bigcup_{1\le i\le n} d_{\alpha_i}$ and $h = \bigcup_{1\le i\le n} f_{\alpha_i}(g(\alpha_i))$ (note that $h \in {}^d2$). We have

$$\bigcap_{1\le i\le n} B^{\alpha_i}_{g(\alpha_i)} = \bigcap_{1\le i\le n} \bigcap_{\gamma\in d_{\alpha_i}} A_\gamma^{f_{\alpha_i}(g(\alpha_i))(\gamma)} = \bigcap_{\gamma\in d} A_\gamma^{h(\gamma)} \neq \emptyset,$$

where the last inequality is true since $\mathcal{D}$ is dyadic. $\Diamond$

3. The number of disjoint closed subsets in a space

Suppose that in a space X there exists a $\kappa \times \lambda$ independent matrix $\{A^\alpha_\beta : \alpha < \kappa, \beta < \lambda\}$. Then we have at least λ closed pairwise disjoint subsets each of cardinality larger than or equal to λ. In fact, consider any row $\{A^{\alpha'}_\beta : \beta < \lambda\}$, where α' is fixed. All of its elements are pairwise disjoint. Moreover, since any fixed $A^{\alpha'}_\beta$ meets all the $A^{\alpha''}_\beta$ for a fixed $\alpha'' \neq \alpha'$ and for all β, and these are pairwise disjoint, we have that $|A^{\alpha'}_\beta| \geq \lambda$. In particular (Th. 2.1) if there is a dyadic system of size τ in a compact space X, then there is a $\tau \times 2^\tau$ independent matrix, so that there are at least 2^τ pairwise disjoint closed subsets of size larger than or equal to 2^τ. We state this observation in the following.

Proposition 3.1. *Let X be a compact space. If there exists a dyadic system of size τ in X, then there are at least 2^τ pairwise disjoint closed subsets of size larger than or equal to 2^τ.*

The following well-known fact can be helpful to find a dyadic system in a given space.

Proposition 3.2. *Let X be any topological space and assume that there exists a continuous map $f : X \to I^\tau$ (or $f : X \to 2^\tau$) onto. Then there is a dyadic system of size τ in X.*

Proof. Let $\mathcal{A} = \{\{A^0_\alpha, A^1_\alpha\} : \alpha < \tau\}$, where $A^i_\alpha = \{h \in 2^\tau : h(\alpha) = i\}$ for $i = 0, 1$. It is straightforward to check that $\mathcal{A}$ is a dyadic system in 2^τ. Put $\mathcal{D} = \{\{f^{-1}(A^0_\alpha), f^{-1}(A^1_\alpha)\} : \alpha < \tau\}$. Since f^{-1} preserves intersections and f is onto $\mathcal{D}$ is dyadic in X. $\Diamond$

It is also possible to prove [7] that, in a compact space X, the existence of a dyadic system of size τ is equivalent to the existence of a continuous surjection $f : X \to I^\tau$.

Example 3.1. In the unit interval I there are $\mathfrak{c}$ pairwise disjoint closed sets of size $\mathfrak{c}$.

The existence of a countable dyadic system in I can be easily proved by using a construction similar to that one of the triadic Cantor set. Subdivide the unit interval into three equal parts and call A_1^l the left subinterval, A_1^r the right subinterval. Divide now both A_1^l and A_1^r into three equal parts. Call A_2^l the union of the left subinterval in A_1^l and the left subinterval in A_1^r. Call A_2^r the union of the right subinterval in A_1^l and the right subinterval in A_1^r. Suppose you have defined all A_k^l and A_k^r for $k < n$. Subdivide each subinterval of size $\frac{1}{3^{n-1}}$ into three equal parts. Call A_n^l the union of all the left pieces and A_n^r the union of all the right pieces. By induction over ω you get a countable family of pairs of closed sets $\mathcal{D} = \{\{A_n^l, A_n^r\} : n < \omega\}$. It is straightforward to check that $\mathcal{D}$ is a dyadic system. By Prop. 3.1 we conclude that in I there are at least $\mathfrak{c}$ pairwise disjoint closed subsets of size larger than or equal to $\mathfrak{c}$.

Example 3.2. In $\beta\omega$ and in ω^* there are $2^{\mathfrak{c}}$ pairwise disjoint closed subsets of size $2^{\mathfrak{c}}$. In particular there are $2^{\mathfrak{c}}$ disjoint copies of $\beta\omega$.

To apply Prop. 3.1 we need to construct a dyadic system of size $\mathfrak{c}$ in $\beta\omega$. Since the space $2^{\mathfrak{c}}$ is separable there exists a countable dense set $D \subset 2^c$. Let $f : \omega \to D$ be a bijection. Let $f^\beta : \beta\omega \to 2^c$ be its Stone–Čech extension. f^β is onto. By Prop. 3.2 there exists in X a dyadic system of size $\mathfrak{c}$. Let $\{K_\alpha : \alpha < 2^c\}$ be a family of pairwise disjoint closed subsets of $\beta\omega$ of size $2^{\mathfrak{c}}$. Clearly $\{K_\alpha \cap \omega^* : \alpha < 2^c\}$ are $2^{\mathfrak{c}}$ pairwise disjoint closed subsets of ω^* of size $2^{\mathfrak{c}}$.

Prop. 3.1 suggests the definition of a cardinal function that counts the number of large disjoint closed subsets of a space.

Definition 3.1. For any space X and any cardinal number κ we define

$$M(X,\kappa) = \sup\big\{|\mathcal{F}| : (F \in \mathcal{F} \Rightarrow F \subset X \text{ closed } |F| \geq \kappa) \\ (F, G \in \mathcal{F} \Rightarrow F \cap G = \emptyset)\big\}.$$

With this new terminology we can write the statements in Examples 3.1 and 3.2 as follows:

$M(I, \mathfrak{c}) = \mathfrak{c}$.

$M(\beta\omega, 2^{\mathfrak{c}}) = 2^{\mathfrak{c}}$.

We now list some properties of $M(X,\kappa)$. We recall that $o(X)$ denotes the cardinality of the topology of the space X.

Proposition 3.3. (1) *For any space X and any cardinal number κ, $M(X,\kappa) \le |X|$ and $M(X,\kappa) \le o(X)$.*

(2) *For any cardinal numbers κ and λ with $\kappa \le \lambda$, $M(X,\kappa) \ge \ge M(X,\lambda)$.*

(3) *Suppose that X is a completely regular locally compact space, and call αX its one-point compactification. Then, for any cardinal number κ, $M(\alpha X,\kappa) \le M(X,\kappa)$, and the inequality can be strict.*

(4) *$M(X,\kappa)$ is not monotone with respect to the first parameter. However, if $A \subseteq X$ is a closed subset, then $M(A,\kappa) \le M(X,\kappa)$ for any κ.*

(5) *Let $f : X \to Y$ be a continuous onto function. Then $M(X,\kappa) \ge \ge M(Y,\kappa)$.*

(6) *Let X and Y be any spaces. Then $M(X \times Y,\kappa) \ge M(X,\kappa) \cdot \cdot M(Y,\kappa)$ and the inequality can be strict.*

(7) *Let $X = \prod_{\alpha<\tau} X_\alpha$, with $|X_\alpha| \ge 2$ for all $\alpha < \tau$. Then $M(X,\kappa) \ge \sup\{M(X_\alpha,\kappa) : \alpha < \tau\}$. Moreover, if $\kappa \le 2^\tau$, then we also have $M(X,\kappa) \ge 2^\tau$. Finally, suppose that there is an X_β such that $\kappa \le |X_\beta|$. Then we have $M(X,\kappa) \ge \sup\{|X_\alpha| : \alpha < \tau, \alpha \ne \beta\}$.*

Proof. To explain the claim in (3) consider $X = \omega$ with its one-point compactification $\alpha X = \omega + 1$. We clearly have $M(X,\omega) = \omega$ and $M(\alpha X,\omega) = 1$, since any infinite closed set of ω has ω as a limit point.

It is not difficult to obtain an example where the inequality in (6) is strict. Note that if $\kappa \le |X|$ then $M(X \times Y,\kappa) \ge |Y|$. Take X, Y and κ such that $\kappa \le |X|$ and $|Y| > M(X,\kappa) \cdot M(Y,\kappa)$; then $M(X \times Y,\kappa) > > M(X,\kappa) \cdot M(Y,\kappa)$.

To explain the claims in (7) notice that it is possible to embed 2^τ into X, and in the compact space 2^τ there exists a dyadic system of cardinality τ (see Prop. 3.2). Hence, by Prop. 3.1 there are at least 2^τ disjoint compact subsets of X of size larger than or equal to 2^τ, or $M(X,2^\tau) \ge 2^\tau$. $\Diamond$

We recall that the index of a space is the cardinal function

$$i(X) = \sup\{\tau : \exists f : X \to I^\tau \text{onto}\}.$$

Let X be any space. Suppose that $i(X) \ge \kappa$, then there is a continuous surjection $f : X \to I^\kappa$. Since, by Prop. 3.2, $M(I^\kappa, 2^\kappa) = 2^\kappa$, by (5) in Prop. 3.3 $M(X,2^\kappa) \ge 2^\kappa$. Therefore we can state the following:

Proposition 3.4. *Let X be any space. If $i(X) \ge \kappa$, then*

$$M(X,2^\kappa) \ge M(X,2^{i(X)}) \ge 2^{i(X)} \ge 2^\kappa.$$

It is often the case that $M(X,\kappa) = |X|$ for any $\kappa \le |X|$. However, easy examples show that $M(X,\kappa)$ can be any number between 1 and $|X|$. For instance, for any cardinal number τ with cof(τ)$\neq \omega$ we have $M(\tau,\tau) = 1$, because two unbounded closed sets of τ cannot be disjoint. Also $M(\mu,\tau) = \tau$ for any $\mu < \tau$.

Example 3.3. A space X for which $M(X,\omega) = \omega_2$, $M(X,\omega_1) = \omega_1$ and $M(X,\omega_2) = 1$.

Let Y be a discrete space of cardinality ω_1. Let Z be the one-point Lindelöfization of a discrete space D of cardinality ω_2, i.e. $Z = D \cup \{\infty\}$ where the neighbourhoods of ∞ are all sets of the form $\{\infty\} \cup C$, with $D \setminus C$ countable. The disjoint union $X = Y \cup Z$ is the desired space.

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