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ON THE MARCINKIEWICZ–FEJÉR MEANS OF DOUBLE WALSH–KACZMARZ–FOURIER SERIES

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Abstract: In this paper we prove that the maximal operator of the Marcinkiewicz–Fejér means of the 2-dimensional Fourier series with respect to the Walsh–Kaczmarz system is not bounded from the Hardy space $H_{2/3}\left(G^2\right)$ to the space $L_{2/3}\left(G^2\right)$.

The second author [5] proved that the maximal function of Marcinkiewicz–Fejér means with respect to the two dimensional Walsh–Kaczmarz system is of weak type $(1,1)$ and of type (p,p) for all $p>1$. Consequently, for any integrable function f the Marcinkiewicz–Fejér means with respect to the two dimensional Walsh–Kaczmarz system converge almost everywhere to the function itself. This theorem was extended in [2] by the authors and G. Gát. Namely, for $p>2/3$, the maximal oper-

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ator $\mathcal{M}^{\kappa *}$ is bounded from the Hardy space $H_p(G^2)$ to the space $L_p(G^2)$. The main aim of this paper is to prove that the assumption $p > 2/3$ is essential. Namely, the maximal operator $\mathcal{M}^{\kappa *}$ is not bounded from the Hardy space $H_{2/3}(G^2)$ to the space $L_{2/3}(G^2)$.

Let $\mathbf{P}$ denote the set of positive integers, $\mathbf{N} := \mathbf{P} \cup \{0\}$. Denote Z_2 the discrete cyclic group of order 2, that is $Z_2 = \{0, 1\}$, where the group operation is the modulo 2 addition and every subset is open. The Haar measure on Z_2 is given such that the measure of a singleton is $1/2$. Let G be the complete direct product of the countable infinite copies of the compact groups Z_2. The elements of G are of the form $x = (x_0, x_1, \ldots, x_k, \ldots)$ with $x_k \in \{0, 1\}$ $(k \in \mathbf{N})$. The group operation on G is the coordinate-wise addition, the measure (denote by μ) and the topology are the product measure and topology. The compact Abelian group G is called the Walsh group. A base for the neighborhoods of G can be given in the following way:

$$\begin{aligned} I_0(x) := G, \quad I_n(x) &:= I_n(x_0, \ldots, x_{n-1}) := \\ &:= \{y \in G : \, y = (x_0, \ldots, x_{n-1}, y_n, y_{n+1}, \ldots)\}, \\ &(x \in G, \; n \in \mathbf{N}). \end{aligned}$$

These sets are called dyadic intervals. Let $0 = (0 : i \in \mathbf{N}) \in G$ denote the null element of G, $I_n := I_n(0)$ $(n \in \mathbf{N})$. Set $e_n := (0, \ldots, 0, 1, 0, \ldots) \in G$, the nth coordinate of which is 1 and the rest are zeros $(n \in \mathbf{N})$.

For $k \in \mathbf{N}$ and $x \in G$ denote

$$r_k(x) := (-1)^{x_k}$$

the kth Rademacher function. If $n \in \mathbf{N}$, then $n = \sum_{i=0}^{\infty} n_i 2^i$, where $n_i \in \{0, 1\}$ $(i \in \mathbf{N})$, i.e. n is expressed in the number system of base 2. Denote $|n| := \max\{j \in \mathbf{N} : n_j \neq 0\}$, that is $2^{|n|} \leq n < 2^{|n|+1}$.

The Walsh–Paley system is defined as the sequence of Walsh–Paley functions:

$$w_n(x) := \prod_{k=0}^{\infty} (r_k(x))^{n_k} = r_{|n|}(x) (-1)^{\sum_{k=0}^{|n|-1} n_k x_k} \quad (x \in G, \; n \in \mathbf{P}).$$

The Walsh–Kaczmarz functions are defined by $\kappa_0 := 1$ and for $n \geq 1$

$$\kappa_n(x) := r_{|n|}(x) \prod_{k=0}^{|n|-1} (r_{|n|-1-k}(x))^{n_k} = r_{|n|}(x) (-1)^{\sum_{k=0}^{|n|-1} n_k x_{|n|-k-1}}.$$

For $A \in \mathbf{N}$ define the transformation $\tau_A : G \to G$ by

$$\tau_A(x) := (x_{A-1}, x_{A-2}, \dots, x_0, x_A, x_{A+1}, \dots).$$

By the definition of τ_A (see [9]), we have

$$\kappa_n(x) = r_{|n|}(x) w_{n-2^{|n|}}(\tau_{|n|}(x)) \quad (n \in \mathbf{N},\ x \in G).$$

The σ-algebra generated by the dyadic 2-dimensional cube I_k^2 of measure $2^{-k} \times 2^{-k}$ will be denoted by F_k $(k \in \mathbf{N})$.

The space $L_p(G^2)$, $0 < p \leq \infty$ with norms or quasi-norms $\|\cdot\|_p$ is defined in the usual way (For details see e.g. Weisz [12].)

Denote by $f = (f_n,\ n \in \mathbf{N})$ the one-parameter martingale with respect to $(F_n,\ n \in \mathbf{N})$. The maximal function of a martingale f is defined by

$$f^* = \sup_{n \in \mathbf{N}} |f_n| .$$

For $0 < p \leq \infty$ the Hardy martingale space $H_p(G^2)$ consists all martingales for which

$$\|f\|_{H_p} = \|f^*\|_p < \infty.$$

The Dirichlet kernels are defined by

$$D_n^\alpha(x) := \sum_{k=0}^{n-1} \alpha_k(x),$$

where $\alpha_k = w_k$ or κ_k. Recall that (see e.g. [1, 7])

$$D_{2^n}(x) := D_{2^n}^w(x) = D_{2^n}^\kappa(x) = \begin{cases} 2^n, & \text{if } x \in I_n(0), \\ 0, & \text{if } x \notin I_n(0). \end{cases} \tag{1}$$

The Fejér kernels are defined as follows

$$K_n^\alpha(x) := \frac{1}{n} \sum_{k=0}^{n-1} D_k^\alpha(x).$$

The Kroneker product $(\alpha_{m,n} : n, m \in \mathbf{N})$ of two Walsh(–Kaczmarz) system is said to be the two-dimensional Walsh(–Kaczmarz) system. Thus,

$$\alpha_{m,n}\left(x^1, x^2\right) = \alpha_n\left(x^1\right) \alpha_m\left(x^2\right).$$

If $f \in L(G^2)$, then the number $\hat{f}^\alpha(n,m) := \int_{G^2} f \alpha_{m,n}$ $(n, m \in \mathbf{N})$ is said to be the (n,m)th Walsh(–Kaczmarz)–Fourier coefficient of f. We can extend this definition to martingales in the usual way (see Weisz [12, 13]). Denote by $S_{n,m}^\alpha$ the (n,m)th rectangular partial sum of the Walsh–Fourier series of a martingale f, namely,

$$S_{n,m}^{\alpha}(f;x^1,x^2):=\sum_{k=0}^{n-1}\sum_{i=0}^{m-1}\hat{f}^{\alpha}(k,i)\alpha_{k,i}(x^1,x^2).$$

The Marcinkiewicz–Fejér means of a martingale f are defined by

$$\mathcal{M}_n^{\alpha}\left(f;x^1,x^2\right):=\frac{1}{n}\sum_{k=0}^{n-1}S_{k,k}^{\alpha}(f,x^1,x^2).$$

The 2-dimensional Dirichlet kernels and Marcinkiewicz–Fejér kernels are defined by

$$D_{k,l}^{\alpha}(x^1,x^2):=D_k^{\alpha}(x^1)D_l^{\alpha}(x^2),\quad K_n^{\alpha}(x^1,x^2):=\frac{1}{n}\sum_{k=0}^{n-1}D_{k,k}^{\alpha}(x^1,x^2).$$

For the martingale f we consider the maximal operators

$$\mathcal{M}^{*\kappa}f(x^1,x^2)=\sup_n\left|\mathcal{M}_n^{\kappa}(f,x^1,x^2)\right|.$$

In 1939 for the two-dimensional trigonometric Fourier partial sums $S_{j,j}(f)$ Marcinkiewicz [6] has proved for $f\in L\log L([0,2\pi]^2)$ that the means

$$\mathcal{M}_nf=\frac{1}{n}\sum_{j=1}^{n}S_{j,j}(f)$$

converge a.e. to f as $n\to\infty$. Zhizhiashvili [14] improved this result for $f\in L([0,2\pi]^2)$.

For the two-dimensional Walsh–Fourier series Weisz [11] proved that the maximal operator

$$\mathcal{M}^{*w}f=\sup_{n\geq 1}\frac{1}{n}\left|\sum_{j=0}^{n-1}S_{j,j}^{w}(f)\right|$$

is bounded from the two-dimensional dyadic martingale Hardy space H_p to the space L_p for $p>2/3$ and is of weak type (1,1). The first author [3] proved that the assumption $p>2/3$ is essential for the boundedness of the maximal operator $\mathcal{M}^{w*}$ from the Hardy space $H_p(G^2)$ to the space $L_p(G^2)$.

In 1974 Schipp [7] and Young [10] proved that the Walsh–Kaczmarz system is a convergence system. Gát [1] proved, for any integrable functions, that the Fejér means with respect to the Walsh–Kaczmarz system converge almost everywhere to the function itself. Gát's Theorem was extended by Simon [8] to H_p spaces, namely that the maximal operator of Fejér means of one-dimensional Fourier series is bounded from Hardy space $H_p(G^2)$ into the space $L_p(G^2)$ for $p > 1/2$.

The second author [5] proved, that for any integrable functions, the Marcinkiewicz–Fejér means with respect to the two dimensional Walsh–Kaczmarz system converge almost everywhere to the function itself. This theorem was extended in [2]. Namely, the following is true:

Theorem A1. *Let $p > 2/3$, then the maximal operator $\mathcal{M}^{\kappa *}$ of the Marcinkiewicz–Fejér means of double Walsh–Kaczmarz–Fourier series is bounded from the Hardy space $H_p(G^2)$ to the space $L_p(G^2)$.*

The aim of this paper is to prove that the assumption $p > 2/3$ is essential for the boundedness of the maximal operator $\mathcal{M}^{\kappa *}$ from the Hardy space $H_p(G^2)$ to the space $L_p(G^2)$. Namely, the following theorem holds:

Theorem 1. *The maximal operator $\mathcal{M}^{\kappa *}$ of the Marcinkiewicz–Fejér means of double Walsh–Kaczmarz–Fourier series is not bounded from the Hardy space $H_{2/3}(G^2)$ to the space $L_{2/3}(G^2)$.*

Proof. Let

$$f_A(x^1, x^2) := (D_{2^{A+1}}(x^1) - D_{2^A}(x^1))(D_{2^{A+1}}(x^2) - D_{2^A}(x^2)).$$

It is simple to calculate

$$\hat{f}_A^{\kappa}(i,k) = \begin{cases} 1, & \text{if } i,k = 2^A, \dots, 2^{A+1}-1, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$S_{i,j}^{\kappa}(f; x^1, x^2) = $$
$$= \begin{cases} (D_i^{\kappa}(x^1) - D_{2^A}(x^1))(D_j^{\kappa}(x^2) - D_{2^A}(x^2)), & \text{if } i,j = 2^A+1, ..., 2^{A+1}-1, \\ f_A(x^1,x^2), & \text{if } i,j \geq 2^{A+1}, \\ 0, & \text{otherwise.} \end{cases}$$

We can write the nth Dirichlet kernel with respect to the Walsh–Kaczmarz system in the following form:

$$D_n^{\kappa}(x) = D_{2^{|n|}}(x) + \sum_{k=2^{|n|}}^{n-1} r_{|k|}(x) w_{k-2^{|n|}}(\tau_{|k|}(x)) =$$
$$= D_{2^{|n|}}(x) + r_{|n|}(x) D_{n-2^{|n|}}^{w}(\tau_{|n|}(x)).$$

Thus, we have

$$\mathcal{M}^{\kappa *} f_A(x^1, x^2) =$$
$$= \sup_{n \in \mathbf{N}} \left| \mathcal{M}_n^{\kappa}(f_A; x^1, x^2) \right| \geq \max_{1 \leq N \leq 2^A} \left| \mathcal{M}_{2^A+N}^{\kappa}(f_A; x^1, x^2) \right| =$$

$$= \max_{1\le N\le 2^A} \frac{1}{2^A+N}\left|\sum_{k=0}^{2^A+N-1} S^{\kappa}_{k,k}(f_A;x^1,x^2)\right| \ge$$

$$\ge \max_{1\le N\le 2^A} \frac{1}{2^{A+1}}\left|\sum_{k=2^A+1}^{2^A+N-1} (D^{\kappa}_k(x^1)-D_{2^A}(x^1))(D^{\kappa}_k(x^2)-D_{2^A}(x^2))\right| =$$

$$= \max_{1\le N\le 2^A} \frac{1}{2^{A+1}}\left|\sum_{k=2^A+1}^{2^A+N-1} r_A(x^1)D^w_{k-2^A}(\tau_A(x^1))r_A(x^2)D^w_{k-2^A}(\tau_A(x^2))\right| =$$

$$= \max_{1\le N\le 2^A} \frac{1}{2^{A+1}}\left|\sum_{l=1}^{N-1} D^w_l(\tau_A(x^1))D^w_l(\tau_A(x^2))\right| =$$

$$= \frac{1}{2^{A+1}} \max_{1\le N\le 2^A} N\left|\mathcal{K}^w_N(\tau_A(x^1),\tau_A(x^2))\right|.$$

Since, we have

$$f^*_A(x^1,x^2) = \sup_{n\in\mathbf{N}}\left|S_{2^n,2^n}(f_A;x^1,x^2)\right| = \left|f_A(x^1,x^2)\right|$$

and

$$\|f_A\|_{H_p} = \|f^*_A\|_p = \|D_{2^A}\|_p^2 = 2^{2A(1-1/p)}.$$

We obtain

$$\frac{\|\mathcal{M}^{\kappa*}f_A\|_{2/3}}{\|f_A\|_{H_{2/3}}} \ge$$

$$\ge \frac{1}{2^{A+1}2^{-A}}\left(\int_{G^2} \max_{1\le N\le 2^A}(N|\mathcal{K}^w_n(\tau_A(x^1),\tau_A(x^2))|)^{2/3}d\mu(x^1,x^2)\right)^{3/2}.$$

To investigate the integral $\int_{G^2} \max_{1\le N\le 2^A}(N|\mathcal{K}^w_N(\tau_A(x^1),\tau_A(x^2))|)^{2/3}d\mu(x^1,x^2)$, we decompose the set G as the following disjoint union

$$G = I_A \cup \bigcup_{t=0}^{A-1} J^A_t,$$

where $A > t \ge 1$ and $J^A_t := \{x \in G : x_{A-1} = \cdots = x_{A-t} = 0, x_{A-t-1} = 1\}$, $J^A_0 := \{x \in G : x_{A-1} = 1\}$. Notice that, by the definition of τ_A we have $\tau_A(J^A_t) = I_t \backslash I_{t+1}$. By Cor. 2.4 in [4], for $(x^1,x^2) \in I_A \times I_A$

$$\mathcal{K}^w_{2^A}(x^1,x^2) = \frac{(2^A+1)(2^{A+1}+1)}{6}.$$

Therefore,

$$\int\limits_{G\times G} \max_{1\le N\le 2^A} \left(N\left|\mathcal{K}_N^w(\tau_A(x^1),\tau_A(x^2))\right|\right)^{2/3} d\mu(x^1,x^2) \ge$$

$$\ge \sum_{t=1}^{A-1} \int\limits_{J_t^A\times J_t^A} \max_{1\le N\le 2^A} \left(N\left|\mathcal{K}_N^w(\tau_A(x^1),\tau_A(x^2))\right|\right)^{2/3} d\mu(x^1,x^2) \ge$$

$$\ge \sum_{t=1}^{A-1} \int\limits_{J_t^A\times J_t^A} \left(2^t\left|\mathcal{K}_{2^t}^w(\tau_A(x^1),\tau_A(x^2))\right|\right)^{2/3} d\mu(x^1,x^2) =$$

$$= \sum_{t=1}^{A-1} \int\limits_{(I_t\setminus I_{t+1})\times(I_t\setminus I_{t+1})} \left(2^t\left|\mathcal{K}_{2^t}^w(x^1,x^2)\right|\right)^{2/3} d\mu(x^1,x^2) =$$

$$= \sum_{t=1}^{A-1} \int\limits_{(I_t\setminus I_{t+1})\times(I_t\setminus I_{t+1})} \left(2^t\frac{(2^t+1)(2^{t+1}+1)}{6}\right)^{2/3} d\mu(x^1,x^2) \ge$$

$$\ge \sum_{t=1}^{A-1} \int\limits_{(I_t\setminus I_{t+1})\times(I_t\setminus I_{t+1})} \left(\frac{2^{3t}}{6}\right)^{2/3} d\mu(x^1,x^2) \ge$$

$$\ge c(A-1).$$

This completes the proof of the main theorem. $\Diamond$

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