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SUPPORTING SPHERE FOR A SPECIAL FAMILY OF COMPACT CONVEX SETS IN THE EUCLIDEAN SPACE

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Abstract: For a family of compact convex sets $A^1, A^2, \dots, A^{n+1}$ in $\mathbb{R}^n$ having empty intersection and such that each n of them have a nonvoid intersection we are proving that there is one and only one supporting sphere in the unique bounded connected component of $\mathbb{R}^n \setminus \cup_{i=1}^{n+1} A^i$. It is constructed a homeomorphism of the mentioned bounded connected component with the open n-dimensional simplex.

1. Introduction and the main result

In the following there will be said that a family $\mathcal{K}$ of sets in the Euclidean space $\mathbb{R}^n$ has a *supporting sphere* , if there exists a sphere S in $\mathbb{R}^n$ having common points with each member of the family $\mathcal{K}$ and the interior of S contains no point of any member of $\mathcal{K}$. The family $\mathcal{K}$ of sets in $\mathbb{R}^n$ will said to be *independent,* if for any $n + 1$ pairwise distinct

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members $K_1, \dots, K_{n+1}$ of $\mathcal{K}$, any set of points $p_1, \dots, p_{n+1}$, where $p_i \in$ $\in K_i, i = 1, \dots, n+1$ determines a simplex of dimension n. In the papers [7, 8, 9] we have used Brouwer's fixed point theorem for the proof of a supporting sphere for an independent family of $n+1$ compact convex sets in $\mathbb{R}^n$ (see also [6]) and respectively in a Minkowski space. The same method was used in [10] for proving the existence of a supporting sphere for a special not independent family of three compact convex sets in the Euclidean plane $\mathbb{R}^2$.

Our terminology used next is in accordance with that in the books [1], [3], [4], [13] and [14].

Let us consider $N = \{1, 2, \dots, n+1\}$ and the family $\mathcal{H} = \{A^1, A^2, ...,$ $A^{n+1}\}$ of convex compact sets in $\mathbb{R}^n$. For $S \subset N$ we denote

$$A^S = \cap_{i \in S} A^i.$$

Suppose that the family $\mathcal{H}$ possesses the following properties:

(i) $A^{N \setminus \{j\}} \neq \emptyset, \ \forall j \in N,$

(ii) $A^N = \emptyset.$

A family of compact convex sets having the above properties (i) and (ii) will be called in the sequel an *$\mathcal{H}$-family*.

Our main result is as follows:

Theorem 1. *Let $\mathcal{H} = \{A^1, A^2, \dots, A^{n+1}\}$ be an $\mathcal{H}$-family. Then the following assertions hold:*

1. *The set $\mathbb{R}^n \setminus \cup_{i \in N} A^i$ possesses exactly two connected components, one of them U (called in the sequel the hole), being bounded.*

2. *The hole U contains a unique equally spaced point from the sets in $\mathcal{H}$, that is, U contains a unique supporting sphere for these sets.*

3. *The hole U is homeomorphic with the open n-dimensional simplex.*

2. Preliminaries

We gather in this section some notions, as well as some well known and easily verifiable results (occasionally with their short proofs) which will play a role in our next proofs.

We shall denote by $\mathbb{R}^n$ the n-dimensional Euclidean vector space. If $M \subset \mathbb{R}^n$ is nonempty, we shall denote by $\operatorname{co} M$ the convex hull and by $\operatorname{aff} M$ the affine hull of M.

Consider the space $\mathbb{R}^n$ to be endowed with the usual scalar product $\langle ., . \rangle$, the norm $\|.\|$ and the topology it induces. The interior, the closure and the boundary of a set $M \subset \mathbb{R}^n$ will be denoted by $\operatorname{int} M$, $\operatorname{cl} M$, and $\operatorname{bd} M$ respectively.

If $C \subset \mathbb{R}^n$ is a nonempty closed convex set, then each $x \in \mathbb{R}^n$ possesses a unique *best approximant* in C, i. e., a unique $y \in C$ with $\|x-y\| = \inf\{\|x-c\| : c \in C\}$. We shall use the notation $d(x,C) = = \inf\{\|x-c\| : c \in C\}$. The function $d(.,C)$ is continuous.

The nonempty subset K in $\mathbb{R}^n$ is called a *convex cone* if it is satisfying the following properties:

1. (k_1) $K + K \subset K$, and
2. (k_2) $\lambda K \subset K$, for every $\lambda \in \mathbb{R}_+$.
3. (k_3) The convex cone K is called *pointed*, if $K \cap (-K) = \{0\}$.

The notions of convex cone and pointed convex cone will be used also for translations of the above defined sets. Then the point corresponding to 0 by the translation will be called the *vertex* of the cone.

The *dual cone* K^* of the convex cone K is the set

$$K^* = \{y \in \mathbb{R}^n : \langle x, y\rangle \geq 0,\ \forall\, x \in K\}.$$

K^* is a closed set satisfying the axioms $(k_1), (k_2)$.

If C is a nonempty convex set in $\mathbb{R}^n$, then the affine functional $f = = \langle h, .\rangle + \alpha$ with $h \in R^n,\ h \neq 0$ and its kernel $H = \{x \in \mathbb{R}^n : f(x) = 0\}$ is called a *supporting hyperplane to C at $c \in C$*, if $C \subset H_+ = \{x \in \in \mathbb{R}^n : f(x) \geq 0\}$ and $c \in H$. In this case H_+ is said the *supporting halfspace*, the vector h the *normal* to the supporting hyperplane. (We consider that the normal of the supporting hyperplane is oriented always towards C, if C has a nonempty interior.) If C is a closed convex set with nonempty interior, then at each point of its boundary it has a supporting hyperplane. We need also the notation $H_- = \{x \in \mathbb{R}^n : f(x) \leq 0\}$ for the other halfspace, determined by the supporting hyperplane to C at c.

If K is a convex cone and does not coincide with the whole space, it possesses a supporting hyperplane at 0.

Lemma 1. *Let us consider the cone given by the intersection $K = = \cap_{i=1}^m H_i^+$ of the halfspaces determined by the hyperplanes $H_1, \ldots, H_m$ through the origin with the normals $h_1, \ldots, h_m$. If $K \neq \{0\}$, then there exists a supporting hyperplane H through 0 to K^* such that $h_i \in H_+$, $i = 1, \ldots, m$.*

Proof. Since K is not the whole space and is not reducing to $\{0\}$, K^* is a convex cone with the same property. Let be H a supporting hyperplane to K^*. Then $h_i \in K^* \subset H_+$, $i = 1, \ldots, m$. $\Diamond$

We say that the boundary of a convex set with nonempty interior is *smooth*, if in each of its points there exists a unique supporting hyperplane to the convex set. An immediate consequence of the above lemma is:

Corollary 1. *If $C_1, \dots, C_m$ are compact convex sets with smooth boundaries in $\mathbb{R}^n$, such that* $\operatorname{int} \cap_{i=1}^m C_i \neq \emptyset$ *and x is a point of the intersection of the boundaries of C_i, $i = 1, \dots, m$, then the normals in x to the supporting hyperplanes of C_i, $i = 1, \dots, m$ are contained in a halfspace determined by some supporting hyperplane in x to $\cap_{i=1}^m C_i$.*

In the following we need also the notion of the *ϵ-neighborhood* of a convex body ([3] p. 2, [14] p. 91), which is also known in the German literature as the *"Parallelkörper"* ([1] p. 48, [4] p. 30, [13] p. 160), and in the English literature *"outer parallel body"* ([11], p. 134). For $\varepsilon > 0$ we denote by $B(x;\varepsilon)$ the *(open) ball centered at x of radius ε*, i.e., the set $B(x;\varepsilon) = \{y \in \mathbb{R}^n : \|y - x\| < \varepsilon\}$. If $M \subset \mathbb{R}^n$ is nonempty, the set $M^\varepsilon = \cup_{x \in M} B(x;\varepsilon)$ is called the *ε-neighborhood of M* (it is called also the *outer parallel body* of M in [11], p. 134) $M_\varepsilon = \operatorname{cl} M^\varepsilon$ will be called the *ε-hull of M*.

If $C \subset \mathbb{R}^n$ is a nonempty convex set, then C^ε and C_ε are booth convex sets. It is immediate that $C_\varepsilon = \{x \in \mathbb{R}^n : d(x, C) \leq \varepsilon\}$.

Lemma 2. *If C is a nonempty compact convex set in $\mathbb{R}^n$, then for any $\varepsilon > 0$, the set C_ε has a smooth boundary.*

Proof. Let $x \in \operatorname{bd} C_\varepsilon$. If y is the best approximant of x in C, then obviously $x \in \operatorname{bd} B(y;\varepsilon)$. Let H be a supporting hyperplane to C_ε in x. Then, since $\operatorname{cl} B(x;\varepsilon) \subset C_\varepsilon$, H will be also a supporting hyperplane to $\operatorname{cl} B(y;\varepsilon)$ at x. Since $\operatorname{bd} B(y;\varepsilon)$ is an Euclidean sphere, it has a unique tangent hyperplane at x. This shows that H is unique. $\Diamond$

3. The proof

We shall carry the proof by verifying a sequence of lemmas.

Lemma 3. [The existence of a bounded connected component.] *Consider the $\mathcal{H}$-family $\mathcal{H} = \{A^1, A^2, \dots, A^{n+1}\}$. Then we have the assertions:*

1. *If $a_i \in A^{N \setminus \{i\}}$, then the points $a_1, a_2, \dots, a_{n+1}$ are in general position (they are affinely independent, respectively are tuples of an n-dimensional simplex).*

2. *If $\Delta^{N \setminus \{i\}} = \operatorname{co}\{a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_{n+1}\}$ then $\Delta^{N \setminus \{i\}} \subset A^i$.*

3. *The simplex Δ^N contains in its interior a bounded connected component of the set*

$$\mathbb{R}^n \setminus \cup_{i \in N} A^i.$$

Proof. 1. It is enough to show that for an arbitrary $k \in N$, $a_k \notin$ $\notin \operatorname{aff}\{a_1, \ldots, a_{k-1}, a_{k+1}, \ldots, a_{n+1}\}$.

Assume the contrary. Denote

$$H = \operatorname{aff}\{a_1, \ldots, a_{k-1}, a_{k+1}, \ldots, a_{n+1}\}.$$

Thus $\dim H \le n-1$. The points a_i are all in the manifold H. Denote

$$B^i = H \cap A^i.$$

Since $a_i \in A^{N\setminus\{i\}}$ and $a_i \in H$ it follows that

$$a_i \in \cap_{j\in N\setminus\{i\}} A^j \cap H = \cap_{j\in N\setminus\{i\}} B^j, \ \forall\, i \in N.$$

This means that the family of convex compact sets $\{B^j : j \in N\}$ in H possesses the property that any n of them have nonempty intersection. Then by Helly's theorem they have a common point. But this would be a point of A^N too, which contradicts (ii).

2. Since $a_i \in \cap_{l\in N\setminus\{i\}} A^l$, it follows that $a_i \in A^j$, $\forall\, i \in N \setminus \{j\}$. Thus

$$\Delta^{N\setminus\{j\}} = \operatorname{co}\{a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_{n+1}\} \subset A^j.$$

3. The assertion follows from an equivalent of Sperner's lemma (see [5]), which asserts that if a collection of closed sets $F^j : j \in N$ possesses the property, that it covers Δ^N and $\Delta^{N\setminus\{j\}} \subset F^j$, then $\cap_{j\in N} F^j \ne \emptyset$. $\Diamond$

Remark 1. The above reasonings have overlappings with the proof of a theorem due to C. Berge [2] (see also [12], Th. 3.7.5) who proved that if a convex compact set in $\mathbb{R}^n$ is covered by a family of $n+1$ convex subsets, each n of them having nonempty intersection, then the whole family has a nonempty intersection. The lemma can be deduced in fact from this theorem. We have supplied the proof for the sake of completeness.

Lemma 4. [The existence in Δ of an equally spaced point.] *Let A^i_ε be the $\varepsilon > 0$-hull of the set A^i, i.e., the set of points with the distance $\le \varepsilon$ from the set A^i. Then:*

1. *There exists an $\varepsilon_0 > 0$ such that:*
   (i) *$\{A^i_\varepsilon : i \in N\}$ is a $\mathcal{H}$-family for $\varepsilon < \varepsilon_0$,*
   (ii) *$B_\varepsilon = \cap_{i\in N}(\Delta \cap A^i_\varepsilon) \ne \emptyset$ for $\varepsilon \ge \varepsilon_0$.*
2. *B_{ε_0} reduces to a single point.*

Here Δ is the simplex Δ^N defined in Lemma 3.

Proof. 1. Assume the contrary: for no $\varepsilon > 0$ is $\mathcal{H}_\varepsilon = \{A^1_\varepsilon, A^2_\varepsilon, \ldots, A^{n+1}_\varepsilon\}$ an $\mathcal{H}$-family. This is equivalent with saying that

$$C_\varepsilon = \cap_{i\in N} A^i_\varepsilon \ne \emptyset, \ \forall \varepsilon > 0.$$

The family $\{C_\varepsilon : \varepsilon > 0\}$ is centered (every finite collection of its members possesses a nonempty intersection). Hence, according the compactness of its sets, the whole family has nonempty intersection. But a direct verification yields that

$$\cap_{\varepsilon>0} C_\varepsilon = \cap_{i\in N} A^i = \emptyset.$$

(ii) Obviously, B_ε is compact and nonempty for ε great enough, and $B_{\varepsilon_1} \subset B_{\varepsilon_2}$ as soon $\varepsilon_1 \leq \varepsilon_2$.

The family of sets $\{B_\varepsilon : B_\varepsilon \neq \emptyset\}$ possesses a nonempty intersection by the compactness of its members. Denote $\varepsilon_0 = \inf\{\varepsilon : B_\varepsilon \neq \emptyset\}$. Then $B_{\varepsilon_0} = \cap\{B_\varepsilon : B_\varepsilon \neq \emptyset\}$.

We shall show first that no point of B_{ε_0} can be an interior point of some $A^i_{\varepsilon_0}$. Assuming the contrary, e.g. that $b \in B_{\varepsilon_0} \cap \operatorname{int} A^i_{\varepsilon_0}$ we have first of all that $d(b, A^i) < \varepsilon_0$ and $d(b, A^j) \leq \varepsilon_0$, $j \in N$. Since $A^{N\setminus\{i\}}$ is nonempty, $\varepsilon_0 > 0$ by the property (i), the set $A^{N\setminus\{i\}}_{\varepsilon_0}$ is convex and has a nonempty interior. Now, $b \in A^{N\setminus\{i\}}_{\varepsilon_0}$ and each of its neighborhoods contains interior points of $A^{N\setminus\{i\}}_{\varepsilon_0}$. Hence so does $\operatorname{int} A^i_{\varepsilon_0}$. Let be x a such point. Then $d(x, A^j) < \varepsilon_0, j \in N$. Denote by $\delta = \sup\{d(x, A^j) : j \in N\}$. It follows that $x \in B_\delta$ with $\delta < \varepsilon_0$, in contradiction with the definition of ε_0.

Thus B_{ε_0} is on the boundary of every $A^i_{\varepsilon_0}$. Hence:

$$d(b, A^j) = \varepsilon_0,\ \forall\, j \in N\ \forall\, b \in B_{\varepsilon_0}.$$

2. If B_{ε_0} would contain two distinct points, b_1 and b_2, the line segment determined by these two points would be in this set too.

The line determined by these points should meet the boundary of Δ^N which is in $\cup_{j\in N} A^j$. Thus the line would meet some set A^i in a point a. Suppose that b_1 is between a and b_2. Let c be the point in A^i at distance ε_0 from b_2. Consider the plane of dimension two determined by the line cb_2 and the line b_1b_2. This plane meets the supporting hyperplane to A^i at c and perpendicular on cb_2 in a line λ which is perpendicular to cb_2. Now, a must be behind the supporting hyperplane, hence the line b_2b_1 meets the line λ in a point d between a and b_2. Thus the triangle dcb_2 is rectangular at c. Since B_{ε_0} is convex, we can suppose without loss of generality that b_1 is on the segment fb_2, where f is the base of the perpendicular from c to b_1b_2. But then the distance from b_1 to c is less then the distance of b_2 to c which is ε_0. This contradiction shows that B_{ε_0} reduces to a point. $\diamondsuit$

Remark 2. In the above lemma it was shown that in Δ there exists a unique equally spaced point of minimal distance from the sets A^i. The proof yields in fact also the existence of such a point for a family of compact convex sets $\{C^1, C^2, \ldots, C^{m+1}\}$ with the property that $C^{i_1} \cap \cap C^{i_2} \cap \ldots \cap C^{i_m} \neq \emptyset \; \forall \, i_j \in \{1, 2, \ldots, m+1\}$ and $\cap_{i=1}^{m+1} C^i = \emptyset$, only the uniqueness needs $m = n$.

Lemma 5. [The uniqueness of the equally spaced point in Δ.] *Suppose that* $U = \Delta \setminus \cup_{i \in N} A^i$. *Then* U *is an open set contained in* $\operatorname{int} \Delta$. *Suppose that* $u \in U$ *and* $b_i, \; i = 1, \ldots, n+1$ *are the best approximants of* u *in* $A^i, \; i = 1, \ldots, n+1$ *respectively. Let be* $\delta_i = \|b_i - u\|, \; i = 1, \ldots, n+1$. *Then*

$$\cap_{i \in N} A^i_{\delta_i} = \{u\}.$$

Here Δ *is the simplex* Δ^N *in Lemma 3. As a consequence of this assertion we shall show that there exists a unique point in* U *which is equally spaced from the sets* $A^i, \; i = 1, \ldots, n+1$.

Proof. We observe first that the vectors $b_i - u, \; i = 1, \ldots, n+1$ are in general position in the sense that they cannot be contained in a half-space determined by some hyperplane through u. Indeed, if H_i is the supporting hyperplane to A^i through b_i with the normal vector $u - b_i$, then $H_i + (u - b_i)$ will be the tangent hyperplane to $A^i_{\delta_i}$ at u. The set $\cap_{i \in N} H_{i-}$ will contain in its interior the point u and will be disjoint from $\cup_{i \in N} A_i$. Hence it must be in U and so in $\operatorname{int} \Delta$. But then it must be an n-dimensional simplex with the vectors $b_i - u, \; i = 1, \ldots, m$ the perpendiculars to the faces of dimension $n-1$ of this simplex whose affine hull contains the point b_i. Hence these vectors are in general position. But $b_i - u$ are in same time normals of the hyperplanes $H_i + (u - b_i)$ which are supporting hyperplanes to $A^i_{\delta_i}$ in the common point u of their boundaries. By Cor. 1 then $\operatorname{int} \cap_{i \in N} A^i_{\delta_i}$ is empty.

The single common point of the boundaries of $A^i_{\delta_i}$ can be u, because if contrary then the common part of these boundaries would contain a segment and we would arrive to a contradiction in the mode it was done earlier in our proof.

Denote $B_{\varepsilon_0} = \{v\}$. We shall show that v is the only point in Δ which is equally spaced from $A^i, \; i = 1, \ldots, n+1$. It was shown above that v is the single equally spaced point of minimal distance ε_0 from $A^i, \; i = 1, \ldots, n+1$. Then if there exists another point w in Δ which is equally spaced from $A^i, \; i = 1, \ldots, n+1$, its distance η must be strictly greater as ε_0. From the definition of ε_0 this would mean that we have $\operatorname{int} \cap_{i \in N} A^i_\eta \neq \emptyset$ and w must be a common point of the boundaries of the sets $A^i_\eta, \; i = 1, \ldots, n+1$. The normals at the point w of the supporting hyperplanes to A^i_η are by the above assertion in general position, but by

Cor. 1 they must be in a halfspace determined by a hyperplane through w. The obtained contradiction shows that w cannot exist. $\Diamond$

Gathering the considerations used in the proofs of Lemmas 4 and 5 we can verify the following assertion:

Corollary 2. *Let us consider the functions ϕ_i, $i \in N$ acting in $[0,\infty)$ having the properties:*

(a) *ϕ_i is continuous and strictly increasing,*

(b) $\phi_i(0)=0$, *(c)* $\lim_{t\to\infty}\phi_i(t)=\infty$, $i\in N$.

Then there exists a unique $t_0>0$ such that:

(i) $\{A^i_{\phi_i(t)} : i \in N\}$ *is an $\mathcal{H}$-family for* $0<t<t_0$,

(ii) $B_t = \cap_{i\in N}(\Delta \cap A^i_{\phi_i(t)}) \neq \emptyset$ *for* $t \geq t_0$.

(iii) B_{t_0} *reduces to a single point.*

Here Δ is the simplex Δ^N considered in Lemma 3.

We shall show that the hole U is homeomorphic with the interior of the standard unit simplex

$$T = \big\{(t_1,t_2,\dots,t_{n+1}) \in \mathbb{R}^{n+1} : \\ t_i \geq 0,\ i=1,2,\dots,n+1, t_1+t_2+\dots+t_{n+1}=1\big\}$$

by constructing effectively the homeomorphism. (This interior is in fact the relative interior of T with respect to the topology of the affine hull of T. We shall denote it by $\operatorname{int} T$.)

Lemma 6. *The mapping*

$$\Phi(x) = \left(\frac{d(x,A^1)}{\sum_{i\in N} d(x,A^i)}, \frac{d(x,A^2)}{\sum_{i\in N} d(x,A^i)}, \dots, \frac{d(x,A^{n+1})}{\sum_{i\in N} d(x,A^i)}\right)$$

is a well defined continuous mapping from U to $\operatorname{int} T$*, which is a bijection, and since U is locally compact, a homeomorphism.*

Proof. Φ *is injective.* Assume that $\Phi(x)=\Phi(y)$ for some $x\neq y$ in U.

Denote

$$\alpha = \frac{1}{\sum_{i\in N} d(x,A^i)}, \qquad \beta = \frac{1}{\sum_{i\in N} d(y,A^i)}.$$

Then $\alpha d(x,A^i) = \beta d(y,A^i),\ i=1,2,\dots,n+1$.

Using the notations $\varepsilon_i = d(x,A^i)$ and $\eta_i = d(y,A^i),\ i\in N$, we have by Lemma 5 that

$$\cap_{i\in N} A^i_{\varepsilon_i} = \{x\} \text{ and } \cap_{i\in N} A^i_{\eta_i} = \{y\}.$$

Assume $\alpha > \beta$. Then $\varepsilon_i = d(x,A^i) < d(y,A^i) = \eta_i,\ i\in N$. Hence $x \in \operatorname{int} A^i_{\eta_i},\ i\in N$ and hence

$$x \in \cap_{i\in N} \operatorname{int} A^i_{\eta_i} \subset \cap_{i\in N} A^i_{\eta_i} = \{y\},$$

which is a contradiction.

Thus we must have $\alpha = \beta$. But then it follows that $d(x, A^i) = = d(y, A^i),\ i \in N$ which by Lemma 5 shows that $x = y$.

Φ *is surjective.* Let $(t_1, t_2, \ldots t_{n+1}) \in \operatorname{int} T$. We shall use Cor. 2 with $\phi_i(t) = t_i t,\ i \in N$ to conclude: *There exist a unique* $\delta > 0$ *and a unique point* $z \in U$, *such that*

$$\cap_{i \in N} \Delta \cap A^i_{\delta t_i} = \{z\}.$$

Then $d(z, A^i) = \delta t_i$ and by substitution in the formula defining Φ we have obviously $\Phi(z) = (t_1, t_2, \ldots, t_{n+1})$. $\Diamond$

Let us denote next the union $\cup_{i \in N} A^i$ by A. We have finally to prove:

Lemma 7. *The set* $\mathbb{R}^n \setminus (A \cup U)$ *is unbounded and connected.*

Proof. Let us consider the points $a_i,\ i \in N$ defined in Lemma 3. Then a_i is outside A^i hence the convex cone C^i with vertex a_i, engendered by the rays issuing from a_i through A^i is pointed.

We show first that the set $D^i = \mathbb{R}^n \setminus (A \cup C^i)$ is arcwise connected.

Since C^i contains the points a_j with $j \neq i$, it will contain Δ and hence the bounded component U.

Consider an arbitrary point $v \in U$. Denote with b_i its best approximant in A^i and let H_i be the hyperplane supporting A^i at b_i with the normal $v - b_i$.

The hyperplane L_i through a_i parallel with H_i will be contained, excepting the point a_i, in the set $B^i = \mathbb{R}^n \setminus C^i$.

The ray d in B^i issuing from a_i meets the set A in a bounded line segment. Indeed, it cannot meet A^i and meets $A^j,\ j \neq i$ in a line segment $a_i c_j$ on d. The union of these segments yield a line segment $a_i c$ on d. Then $d' = d \setminus a_i c$ will be a ray without $A \cup U$.

Consider the points $x,\ y \in D^i$. Then each of them are on some rays of the above type, say d', respectively d''. Now, these rays can be joined by a path in D^i. And thus we can construct a path from x to y in D^i.

Thus D^i is connected.

Since $\mathbb{R}^n \setminus (A \cup U) = \cup_{i \in N} D^i$, to conclude the proof of the lemma it is enough to show that $D^i \cap D^j \neq \emptyset,\ \forall\, i,\ j$.

Observe that the part of the halfspace L_k^+ (where L_k is the hyperplane parallel with H_k through a_k) which is outside the ball containing A, is contained in D^k.

The halfspace L_i^+ through a_i and the halfspace L_j^+ through a_j have an unbounded intersection. This assertion could be false only if L_i and

L_j would be parallel. But this is impossible, since their normals $v - b_i$ and $v - b_j$ by the proof of Lemma 5 cannot be parallel.

The unbounded intersection $L_i^+ \cap L_j^+$ must contain points in $D^i \cap D^j$ and hence the latter set is nonempty. $\Diamond$

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