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THE FIRST LEMOINE CIRCLE OF THE TRIANGLE IN AN ISOTROPIC PLANE

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Abstract: The concept of the first Lemoine circle, Lemoine hexagon and the Brocard angle will be introduced in an isotropic plane. Some statements about relationships between introduced concepts and some other previously studied geometric concepts about triangle will be investigated in an isotropic plane. A number of these statements are new, and some of them are known in Euclidean geometry.

Each triangle ABC in an isotropic plane can be set by a suitable choice of the coordinate system in the *standard position*, in which its

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circumscribed circle has the equation $y = x^2$, and its vertices are the points $A = (a, a^2)$, $B = (b, b^2)$, $C = (c, c^2)$, where $a + b + c = 0$ (see [6]). With the labels $p = abc$, $q = bc + ca + ab$ a number of useful equalities are valid as for example $q = bc - a^2$, $(b - c)^2 = -(q + 3bc)$, $(c - a)(a - b) = 2q - 3bc$, $(b-c)^2 + (c-a)^2 + (a-b)^2 = -6q$, $(b - c)^2(c - a)^2(a - b)^2 = -(27p^2 + 4q^3)$. Really, we get

$$(b - c)^2 = (b + c)^2 - 4bc = a^2 - 4bc = -q - 3bc,$$
$$(c - a)(a - b) = -a^2 - bc + ca + ab = -(bc - q) - 2bc + q = 2q - 3bc,$$
$$(b - c)^2 + (c - a)^2 + (a - b)^2 = -3q - 3(bc + ca + ab) = -6q,$$
$$\begin{aligned}&(b - c)^2(c - a)^2(a - b)^2 =\\ &= -(q + 3bc)(q + 3ca)(q + 3ab) =\\ &= -q^3 - 3q^2(bc + ca + ab) - 9qabc(a + b + c) - 27a^2b^2c^2 =\\ &= -27p^2 - 4q^3.\end{aligned}$$

Further, let the considered triangle be always in the standard position.

Theorem 1. *If the lines through the point $T = (x_o, y_o)$ parallel to the lines BC, CA, AB intersect successively the pairs of the lines CA, AB; AB, BC; BC, CA in the pairs of the points B_a, C_a; C_b, A_b; A_c, B_c, then these six points have successively the abscissas*

$$B_a \ldots \frac{ax_0 + y_0 + ca}{a - b}, \quad C_a \ldots -\frac{ax_o + y_o + ab}{c - a},$$
$$C_b \ldots \frac{bx_0 + y_0 + ab}{b - c}, \quad A_b \ldots -\frac{bx_o + y_o + bc}{a - b},$$
$$A_c \ldots \frac{cx_0 + y_0 + bc}{c - a}, \quad B_c \ldots -\frac{cx_o + y_o + ca}{b - c}.$$

Proof. The lines BC and CA have the equations $y = -ax - bc$ and $y = -bx - ca$, and the line through the point T parallel to the line BC has the equation $y = -a(x - x_o) + y_o$. For the abscissa of the point B_a from the last two equations follows the equation

$$-bx - ca = -ax + ax_o + y_o$$

with the solution

$$x = \frac{ax_o + y_o + ca}{a - b}.$$

The abscissa of the point C_a is got by the substitution $b \leftrightarrow c$, and abscissas of the reminder points are got by the cyclic permutation $a \to b \to c \to a$. $\Diamond$

If T is symmedian center K of the triangle ABC, then according to [5] we get

$$x_o = \frac{3p}{2q}, \quad y_o = -\frac{q}{3}.$$

For that point we get for example

$$\begin{aligned} ax_o + y_o + bc &= \frac{3ap}{2q} - \frac{q}{3} + bc = \frac{1}{6q}(9a^2bc - 2q^2 + 6bcq) = \\ &= \frac{1}{6q}[9bc(bc-q) - 2q^2 + 6bcq] = \frac{1}{6q}(9b^2c^2 - 3bcq - 2q^2) = \\ &= \frac{1}{6q}(3bc+q)(3bc-2q) = \frac{1}{6q}(b-c)^2(c-a)(a-b) \end{aligned}$$

and similarly

$$\begin{aligned} bx_o + y_o + ca &= \frac{1}{6q}(c-a)^2(a-b)(b-c), \\ cx_o + y_o + ab &= \frac{1}{6q}(a-b)^2(b-c)(c-a). \end{aligned}$$

Because of that it follows for example

$$\begin{aligned} ax_o + y_o + ca &= ax_o + y_o + bc + c(a-b) = \\ &= \frac{1}{6q}(b-c)^2(c-a)(a-b) + c(a-b) = \\ &= (a-b)[c + \frac{1}{6q}(b-c)^2(c-a)], \\ ax_o + y_o + ab &= ax_o + y_o + bc - b(c-a) = \\ &= \frac{1}{6q}(b-c)^2(c-a)(a-b) - b(c-a) = \\ &= -(c-a)[b - \frac{1}{6q}(b-c)^2(a-b)], \end{aligned}$$

and in this case the abscissas of the points B_a i C_a from Th. 1 get the values $c + \frac{1}{6q}(b-c)^2(c-a)$ and $b - \frac{1}{6q}(b-c)^2(a-b)$. We have proved:

Theorem 2. *If the lines through the symmedian center K of the triangle ABC parallel to the lines BC, CA, AB intersect successively the pairs of the lines CA, AB; AB, BC; BC, CA in the pairs of the points B_a, C_a; C_b, A_b; A_c, B_c, then these six points have successively the abscissas*

$$B_a \ \ldots \ c + \frac{1}{6q}(b-c)^2(c-a), \quad C_a \ \ldots \ b - \frac{1}{6q}(b-c)^2(a-b),$$

$$C_b \ \dots \ a + \frac{1}{6q}(c-a)^2(a-b), \quad A_b \ \dots \ c - \frac{1}{6q}(c-a)^2(b-c),$$

$$A_c \ \dots \ b + \frac{1}{6q}(a-b)^2(b-c), \quad B_c \ \dots \ a - \frac{1}{6q}(a-b)^2(c-a).$$

Theorem 3. *In the conditions of Th. 2 for the oriented lengths on the lines BC, CA, AB these equalities*

$$BA_c = \frac{1}{6q}(a-b)^2(b-c), \quad A_cA_b = \frac{1}{6q}(b-c)^3, \quad A_bC = \frac{1}{6q}(c-a)^2(b-c),$$

$$CB_a = \frac{1}{6q}(b-c)^2(c-a), \quad B_aB_c = \frac{1}{6q}(c-a)^3, \quad B_cA = \frac{1}{6q}(a-b)^2(c-a),$$

$$AC_b = \frac{1}{6q}(c-a)^2(a-b), \quad C_bC_a = \frac{1}{6q}(a-b)^3, \quad C_aB = \frac{1}{6q}(b-c)^2(a-b)$$

are valid.

Proof. By using the abscissas b and c of the points B and C and the abscissas of the points B_a and C_a from Th. 2 we get the equalities for BA_c and A_bC, and besides that

$$A_cA_b = c - b - \frac{b-c}{6q}[(c-a)^2 + (a-b)^2] =$$
$$= c - b - \frac{b-c}{6q}[-6q - (b-c)^2] =$$
$$= \frac{1}{6q}(b-c)^3 = \frac{BC^3}{BC^2 + CA^2 + AB^2}. \quad \Diamond$$

Corollary 1. *With the labels from Th. 2 the proportions*

$$BA_c : A_cA_b : A_bC = AB^2 : BC^2 : CA^2,$$
$$CB_a : B_aB_c : B_cA = BC^2 : CA^2 : AB^2,$$
$$AC_b : C_bC_a : C_aB = CA^2 : AB^2 : BC^2,$$
$$A_cA_b : B_aB_c : C_bC_a = BC^3 : CA^3 : AB^3$$

and equalities

$$A_cA_b : BC^3 = B_aB_c : CA^3 = C_bC_a : AB^3 = 1 : (BC^2 + CA^2 + AB^2)$$

are valid.

The relationships from Cor. 1 are identical to those ones in the Euclidean geometry.

Because of

$$\begin{aligned} B_cC_b &= \frac{1}{6q}(c-a)^2(a-b) + \frac{1}{6q}(a-b)^2(c-a) = \\ &= -\frac{1}{6q}(b-c)(c-a)(a-b) = \\ &= \frac{(b-c)(c-a)(a-b)}{(b-c)^2+(c-a)^2+(a-b)^2} = -\frac{BC \cdot CA \cdot AB}{BC^2+CA^2+AB^2}, \end{aligned}$$

the following theorem is valid.

Theorem 4. *With the labels from Th. 3 the equalities*

$$B_cC_b = C_aA_c = A_bB_a = -\frac{BC \cdot CA \cdot AB}{BC^2+CA^2+AB^2}$$

are valid.

The equalities from Th. 4 are in accordance to the analogous equalities in the Euclidean geometry.

Theorem 5. *With the labels from Th. 2 the triangles $B_cC_aA_b$ and $C_bA_cB_a$ are directly similar to the triangle ABC and the lengths of their sides are successively $\frac{1}{2}BC$, $\frac{1}{2}CA$, $\frac{1}{2}AB$.*

Proof. By means of the abscissas from Th. 2 we get for example

$$\begin{aligned} C_aA_b &= c-b-\frac{1}{6q}(c-a)^2(b-c) + \frac{1}{6q}(b-c)^2(a-b) = \\ &= \frac{c-b}{6q}[6q+(c-a)^2-(b-c)(a-b)] = \\ &= \frac{c-b}{6q}[6q-(q+3ca)-(2q-3ca)] = \frac{c-b}{6q}\cdot 3q = \frac{1}{2}BC, \\ A_cB_a &= c-b+\frac{1}{6q}(b-c)^2(c-a) - \frac{1}{6q}(a-b)^2(b-c) = \\ &= \frac{c-b}{6q}[6q-(b-c)(c-a)+(a-b)^2] = \\ &= \frac{c-b}{6q}[6q-(2q-3ab)-(q+3ab)] = \frac{c-b}{6q}\cdot 3q = \frac{1}{2}BC. \quad \Diamond \end{aligned}$$

Theorem 6. *The points A_b, A_c, B_c, B_a, C_a, C_b from Th. 2 lie on one circle with the equation*

$$y = 2x^2 - \frac{3p}{q}x + \frac{27p^2-2q^3}{18q^2}. \tag{1}$$

Proof. From the equation (1) and the equation $y = -ax - bc$ of the line BC it follows the equation for the abscissa of the intersection of the circle (1) with the line BC

$$2x^2 + \left(a - \frac{3p}{q}\right)x + bc + \frac{27p^2 - 2q^3}{18q^2} = 0.$$

It is enough to prove that the abscissas of the points A_b i A_c from Th. 2 satisfy this equation. However, for the sum and the product of these abscissas we get the expressions

$$\begin{aligned} &b + c + \frac{b-c}{6q}[(a-b)^2 - (c-a)^2] = \\ &= -a + \frac{b-c}{6q}(c-b)(2a-b-c) = \\ &= -a - \frac{(b-c)^2}{6q} \cdot 3a = -\frac{a}{2q}[2q - (q+3bc)] = \\ &= -\frac{a}{2q}(q - 3bc) = -\frac{1}{2}\left(a - \frac{3p}{q}\right), \end{aligned}$$

$$\begin{aligned} &[b + \tfrac{1}{6q}(a-b)^2(b-c)][c - \tfrac{1}{6q}(c-a)^2(b-c)] = \\ &= bc + \frac{b-c}{6q}[c(a-b)^2 - b(c-a)^2] - \frac{1}{36q^2}(b-c)^2(c-a)^2(a-b)^2 = \\ &= bc + \frac{b-c}{6q}(bc - a^2)(b-c) + \frac{1}{36q^2}(27p^2 + 4q^3) = \\ &= bc + \frac{1}{6}(b-c)^2 + \frac{1}{36q^2}(27p^2 + 4q^3) = \\ &= bc - \frac{1}{6}(q + 3bc) + \frac{1}{36q^2}(27p^2 - 2q^3) + \frac{1}{6}q = \\ &= \frac{1}{2}\left(bc + \frac{27p^2 - 2q^3}{18q^3}\right), \end{aligned}$$

and the statement is proved. ◊

By the analogy with the euclidean case the circle from Th. 6 will be called *the first Lemoine circle* of the triangle ABC (see also [7]). For its segments on the sides BC, CA, AB the relationships from Cor. 1 are valid.

Theorem 7. *The corresponding sides of the similar triangles ABC and $C_bA_cB_a$ from Th. 5, as well as the similar triangles $B_cC_aA_b$ and ABC, form the angles equal to*

$$\omega = -\frac{1}{3q}(b-c)(c-a)(a-b). \tag{2}$$

Proof. The chord A_cB_a with the ends (x_1, y_1) and (x_2, y_2) on the circle (1) has the slope

$$\begin{aligned}\frac{y_1-y_2}{x_1-x_2} &= \frac{2(x_1^2-x_2^2)-\frac{3p}{q}(x_1-x_2)}{x_1-x_2} = 2(x_1+x_2)-\frac{3p}{q} = \\ &= 2(b+c)+\frac{b-c}{3q}[(a-b)^2+(b-c)(c-a)]-\frac{3p}{q} = \\ &= -2a+\frac{b-c}{3q}[-(q+3ab)+2q-3ab]-\frac{3p}{q} = \\ &= -a-\frac{a}{q}(q+3bc)+\frac{b-c}{3q}(q-6ab) = \\ &= -a+\frac{a}{q}(b-c)^2+\frac{b-c}{3q}(q-6ab) = \\ &= -a+\frac{b-c}{3q}[3a(b-c)+q-6ab] = -a+\frac{b-c}{3q}(q-3ca-3ab) = \\ &= -a+\frac{b-c}{3q}(3bc-2q) = -a-\frac{b-c}{3q}(c-a)(a-b) = \omega - a.\end{aligned}$$

As the line BC has the slope $-a$, it follows $\measuredangle(BC, A_cB_a) = \omega$. The analogous calculation gives for the chord C_aA_b

$$\begin{aligned}\frac{y_1-y_2}{x_1-x_2} &= 2(x_1+x_2)-\frac{3p}{q} = \\ &= 2(b+c)-\frac{b-c}{3q}[(b-c)(a-b)+(c-a)^2]-\frac{3p}{q} = \\ &= -2a-\frac{b-c}{3q}[2q-3ca-(q+3ca)]-\frac{3p}{q} = \\ &= -a-\frac{a}{q}(q+3bc)-\frac{b-c}{3q}(q-6ca) = -a+\frac{a}{q}(b-c)^2-\frac{b-c}{3q}(q-6ca) = \\ &= -a+\frac{b-c}{3q}[3a(b-c)-q+6ca] = -a+\frac{b-c}{3q}(3ca+3ab-q) = \\ &= -a+\frac{b-c}{3q}(2q-3bc) = -a+\frac{b-c}{3q}(c-a)(a-b) = -\omega - a\end{aligned}$$

and so $\measuredangle(C_aA_b, BC) = \omega$. Because of the cyclical symmetry of the both obtained formulae the same results follow for the remainder two angles of the corresponding sides of the considered pairs of the triangles. ◊

The triangle ABC with the vertices $A = (a, a^2)$, $B = (b, b^2)$, $C = (c, c^2)$ has the area $\triangle$ given by formula

$$2\triangle = \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} = (b-c)(c-a)(a-b) = -BC \cdot CA \cdot AB,$$

so because of the equality $-6q = BC^2 + CA^2 + AB^2$ from (2) follows

$$\omega = -\frac{4\triangle}{BC^2 + CA^2 + AB^2}.$$

Because of the analogy of this formula with the formula in the Euclidean geometry, the angle ω from Th. 7 will be called *Brocard's angle* of the triangle ABC (see [9], [10], [11] and [13]).

The hexagon, whose vertices are the points A_c, A_b, B_a, B_c, C_b, C_a will be called (like in the Euclidean geometry) *Lemoine hexagon* of the triangle ABC.

The four points A_b, A_c, B_c, C_b lie on the first Lemoine circle, and as $A_bC_b \cap A_cB_c = K$, it follows that intersections $A_bA_c \cap B_cC_b = BC \cap B_cC_b$ and $C_bA_c \cap A_bB_c$ lie on the polar line of the point K for the considered circle. The same thing is valid for the pairs of the intersections $CA \cap C_aA_c$, $A_cB_a \cap B_cC_a$ and $AB \cap A_bB_a$, $B_aC_b \cap C_aA_b$. We have proved:

Theorem 8. *The intersections of the opposite sides of the Lemoine hexagon of the triangle ABC and the intersections of the corresponding sides of the triangle $B_aC_bA_c$ and $C_aA_bB_c$ from Th. 5 lie on the polar line $\mathcal{K}$ of symmedian center K of the triangle ABC with respect to its first Lemoine circle.*

Let us find the equation of the polar line $\mathcal{K}$ from Th. 8. The point (x_o, y_o) has the polar line with the equation

$$y + y_o = 4x_ox - \frac{3p}{q}(x + x_o) + \frac{27p^2 - 2q^3}{9q^2}$$

with respect to the circle (1). With $x_o = \frac{3p}{2q}$, $y_o = -\frac{q}{3}$ we get

$$4x_o - \frac{3p}{q} = \frac{3p}{q},$$

$$-y_o - \frac{3p}{q}x_o + \frac{27p^2 - 2q^3}{9q^2} = \frac{q}{3} - \frac{9p^2}{2q^2} + \frac{27p^2 - 2q^3}{9q^2} = \frac{1}{18q^2}(2q^3 - 27p^2),$$

so the equation we were looking for is

$$y = \frac{3p}{q}x + \frac{1}{18q^2}(2q^3 - 27p^2). \tag{3}$$

The same equation will be got if the equation (1) is subtracted from the equation $2y = 2x^2$ of the circumscribed circle of the triangle ABC, so (3) is the equation of the potential line of these two circles, i.e. it is valid

Theorem 9. *The polar line of the symmedian center of the triangle with respect to its first Lemoine circle is the potential line of that circle and the circumscribed circle of the considered triangle.*

In [6] it is shown that the curve with the equation $\mathcal{K}(x, y) = 0$ in the standard triangle ABC is complementary to the curve with the equation $\mathcal{K}(-2x, -2y - 2q) = 0$. Because of that the Euler circle, as the complementary circle to its circumscribed circle, has the equation $-2y - 2q = (-2x)^2$, i.e. $y = -2x^2 - q$. If we add this equation to the equation (1) of the first Lemoine circle, after dividing by 2 we get this equation

$$y = -\frac{3p}{2q}x + \frac{27p^2 - 20q^3}{36q^2} \tag{4}$$

of the potential line of these two circles. This potential line can be got in the geometrical way too, because the following is valid.

Theorem 10. *The potential axis of the Euler circle and the first Lemoine circle of the triangle passes through the intersections of the sides of its orthic triangle with the lines through its symmedian center parallel to its corresponding sides.*

Proof. The standard triangle ABC has the orthic triangle $A_hB_hC_h$ where the line B_hC_h has the equation

$$y = 2ax + 2bc - q \tag{5}$$

(see [6]). The line with the equation

$$y = -ax + \frac{3ap}{2q} - \frac{q}{3} \tag{6}$$

is parallel to the line BC and passes through the point $K = (\frac{3p}{2q}, -\frac{q}{3})$. It is necessary to prove that there is the point, which lie on all three lines (5), (6) and (4). This is the point (x, y), given by the formula

$$x = \frac{1}{18aq}(4q^2 - 12bcq + 9ap), \quad y = \frac{1}{9q}(6bcq + 9ap - 5q^2).$$

Really, for that point we get successively

$$\begin{aligned}
y - 2ax &= \frac{1}{9q}(6bcq + 9ap - 5q^2 - 4q^2 + 12bcq - 9ap) = \\
&= \frac{1}{9q}(18bcq - 9q^2) = 2bc - q, \\
y + ax &= \frac{1}{18q}(12bcq + 18ap - 10q^2 + 4q^2 - 12bcq + 9ap) = \\
&= \frac{1}{18q}(27ap - 6q^2) = \frac{3ap}{2q} - \frac{q}{3}, \\
y + \frac{3p}{2q}x &= \frac{1}{36aq^2}(24pq^2 + 36a^2pq - 20aq^3 + 12pq^2 - 36bcpq + 27ap^2) = \\
&= \frac{1}{36aq^2}[36pq(q + a^2 - bc) + 27ap^2 - 20aq^3] = \\
&= \frac{1}{36aq^2}(27ap^2 - 20aq^3) = \frac{27p^2 - 20q^3}{36q^2}. \quad \Diamond
\end{aligned}$$

The results about the first Lemoine circle in an isotropic plane are analogous to those in Euclidean plane, and there is very reach bibliography about it. In the bibliography of this article only the most important articles and books are quoted [1]–[4] and [8]. The analogous statements for Cor. 1 can be found in [8], [2] and [3], for Th. 4 in [4], for Th. 8 in [2], [3, p. 49], [1, p. 159], for Th. 9 in [3, p. 49–50] and [1, p. 297].

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