Mathematica Pannonica
18/2 (2007), 247–252

A CHARACTERIZATION OF THE BARYCENTRE OF MASSES WITH CONVEX COMPACT SUPPORT

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Dedicated to Professor Maurer on the occasion of his 80th birthday

Received: July 2007

MSC 2000: Primary 52 A 07; Secondary 46 A 55

Keywords: Masses, D-integral, weak convergence of masses, finitely additive Pettis integral, barycentre.

Abstract: In the setting of normal topological real vector spaces with dimension greater than one and dual separating points, we prove that the barycentre is the unique mapping on the set of masses with convex compact support which is associative, internal and weak-continuous.

Let X be a normal topological real vector space with dimension greater than one on which the dual space X^* separates points and $\mathcal{F}$ a field on X including all open sets. We denote by μ (with or without indices) any non-null *mass* (i.e. positive bounded finitely additive measure) on $\mathcal{F}$ and by δ_x the *Dirac mass* at $x \in X$ (i.e. $\delta_x(\{x\}) = \delta_x(X) = 1$).

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Moreover, we call *support* of μ any set F such that $\mu(F) = \mu(X) = \|\mu\|$. Finally, given a net of masses $\{\mu_d; d \in D\}$, the symbol $\mu_d \to {}_w\mu$ means that the net *weakly converges* to μ (i.e. $\mathrm{D}\int_X f d\mu_d \to \mathrm{D}\int_X f d\mu$ for any bounded continuous real function f on X, where letter D denotes Dunford–Schwartz integral as defined in [1; Sec. 4.4]); moreover, $x_d \xrightarrow{w} x$ means that the net $\{x_d; d \in D\}$ in X weakly converges to the point x (in the usual sense).

Now, denoting by S the convex cone of *simple masses* (i.e. non-null positive linear combinations of Dirac masses), we prove in [3] that the unique mapping $\mathrm{m} : \mathrm{S} \mapsto X$ satisfying the following properties:

- A1 (associativity): $\mathrm{m}(\alpha_1\mu_1 + \alpha_2\mu_2) = \mathrm{m}(\alpha_1\nu_1 + \alpha_2\nu_2)$, whenever $\alpha_1 + \alpha_2 > 0$, $\alpha_i \geq 0$ and $\|\mu_i\| = \|\nu_i\|$, $\mathrm{m}(\mu_i) = \mathrm{m}(\nu_i)$ and $\mu_i, \nu_i \in \mathrm{S}$ $(i = 1, 2)$.
- A2 (internality): $\mathrm{m}(\alpha\delta_x + \beta\delta_y) \in [x, y] = \{tx + (1-t)y : t \in [0, 1]\}$, for any α, $\beta > 0$ and $x, y \in X$.
- A3 (continuity): $\mathrm{m}(\mu_n) \to \mathrm{m}(\mu)$, whenever, for some natural number k, we have $\mu_n = \sum_{i=1}^k \alpha_i^{(n)} \delta_{x_i^{(n)}}$ for all n, $\mu = \sum_{i=1}^k \alpha_i \delta_{x_i}$ and $\alpha_i^{(n)} \to \alpha_i$, $x_i^{(n)} \to x_i$ $(i = 1, \dots, k)$ and all $x_i^{(n)}$ belong to a polytope.

is the barycentre of simple masses defined as:

$$\mathrm{b}(\mu) = \frac{1}{\|\mu\|} \sum_{i=1}^n \alpha_i x_i = \sum_{i=1}^n \frac{\alpha_i}{\alpha_1 + \cdots + \alpha_n} x_i$$

for any $\mu = \sum_{i=1}^n \alpha_i \delta_{x_i}$.

Moreover, in the same paper we extend the previous notion by defining the *barycentre* of a mass μ (when it exists) as:

$$\mathrm{b}(\mu) = \frac{1}{\|\mu\|} \int_X x\, \mu(dx),$$

where the integral is a finitely additive version of Pettis integral (that is, given $\Phi : X \to X$ and μ, the integral $\int_X \Phi\, d\mu \equiv \int_X \Phi(x)\, \mu(dx)$ is the unique element of X such that $\Lambda(\int_X \Phi\, d\mu) = \mathrm{D}\int_X (\Lambda \circ \Phi)\, d\mu$ for any $\Lambda \in X^*$).

Finally, for a mapping m from a set of masses including S to X, in [3] we also introduce the following continuity property (extension of A3):

- A3′ (weak-continuity): Let $\{\mu_d; d \in D\}$ be a net in S. Then, $\mathrm{m}(\mu_d) \xrightarrow{w} \mathrm{m}(\mu)$, whenever $\mu_d \to {}_w\mu$ and all μ, μ_d have a common convex compact support.

and claim, without proof, that the following theorem holds.

Theorem. *Let* M *be the set of non-null masses having a convex compact support. Then, for any mass* $\mu \in \mathrm{M}$*, the barycentre is well defined and belongs to any convex compact support of* μ*. Moreover, the barycentre is the unique mapping* $\mathrm{m} : \mathrm{M} \mapsto X$ *satisfying associativity, internality and weak-continuity properties.*

In order to supply a proof of this theorem, we start with the following lemma.

Lemma. *Let* $\Phi : X \mapsto X$ *be a continuous mapping. Then, the following statements hold:*

(i) *Let* K *be a compact set such that* $\overline{co}\Phi(K)$ *is compact. Then, given a probability* μ *having* K *as a support, the mapping* Φ *is integrable w.r.t.* μ *and* $\int_X \Phi\, d\mu \in \overline{co}\Phi(K)$*;*

(ii) *The mapping* $\int_X \Phi\, d\,\cdot$ *is linear on the convex cone of masses with compact support and w.r.t. which* Φ *is integrable;*

(iii) *Given* μ *and a net* $\{\mu_d; d \in D\}$ *such that all* μ, μ_d *have a common compact support, let* $\mu_d \to {}_w\mu$*. Then, if* Φ *is integrable w.r.t.* μ *and all* μ_d*, we have* $\int_X \Phi\, d\mu_d \xrightarrow{w} \int_X \Phi\, d\mu$*.*

Proof. (i) We start proving that any function $f : X \mapsto \mathbb{R}$ continuous on K (in the induced topology) is D-integrable w.r.t. μ and

$$\text{(1)} \qquad \mathrm{D}\int_X f d\mu = \mathrm{D}\int_K f|_K\, d\mu|_{\mathcal{F}\cap K},$$

where $\mathcal{F} \cap K = \{F \cap K : F \in \mathcal{F}\} \subset \mathcal{F}$. Assume f to be not constant on K. Then, given a natural number n, consider real numbers $y_0, \dots, y_{t_n}$ such that $\min f(K) = y_0 < y_1 < \cdots < y_{t_n-1} < y_{t_n} = \max f(K)$ and $y_i - y_{i-1} < \frac{1}{n}$ $(i = 1, \dots, t_n)$. Now, denoting by I_F the indicator function of $F \subset X$, let $f_n = y_0 I_{K^c} + \sum_{i=1}^{t_n} y_{i-1} I_{F_i}$, where $F_i = f^{-1}([y_{i-1}, y_i[) \cap \cap K \in \mathcal{F} \cap K$ $(i = 1, \dots, t_n - 1)$ and $F_{t_n} = f^{-1}([y_{t_n-1}, y_{t_n}]) \cap K \in \mathcal{F} \cap K$. Note that $\{x : |f_n(x) - f(x)| > \frac{1}{n}\} \subset K^c$ and hence

$$\mu^*\left(\left\{x : |f_n(x) - f(x)| > \frac{1}{n}\right\}\right) \le \mu(K^c) = \|\mu\| - \mu(K) = 0.$$

Therefore, the sequence (f_n) converges to f hazily w.r.t. μ; moreover, on noting that $|f_m - f_n| \le |f_m - f| + |f - f_n|$ and $f_m|_{K^c} = f_n|_{K^c}$ we have $|f_m - f_n| \le \frac{1}{m} + \frac{1}{n}$ and hence $\lim\limits_{m,n\to+\infty} \mathrm{D}\int_X |f_m - f_n| d\mu = 0$. Consequently, the sequence (f_n) is a determining sequence of D-integrable simple functions for f so that f is D-integrable w.r.t. μ.

Finally, on noting that the sequence $(f_n|_K)$ is a determining sequence of D-integrable simple functions for $f|_K$ w.r.t. $\mu|_{\mathcal{F}\cap K}$ and $\mathrm{D}\int_X f_n d\mu = \mathrm{D}\int_K f_n|_K\, d\mu|_{\mathcal{F}\cap K}$ for all n, we get (1).

Now we prove (i). First consider the finitely additive probability space $(K, \mathcal{F}\cap K, \mu|_{\mathcal{F}\cap K})$ and note that $\Phi|_K$ is a continuous mapping on the compact Hausdorff space K such that the closed convex hull of $\Phi|_K(K)$ is compact. Then, by the same arguments as in the proof of Th. 3.27 in [4], one can verify that there is $y \in \overline{\mathrm{co}}\Phi|_K(K) = \overline{\mathrm{co}}\Phi(K)$ such that

$$\Lambda(y) = \mathrm{D}\int_K (\Lambda \circ \Phi|_K)\, d\mu|_{\mathcal{F}\cap K}$$

for any $\Lambda \in X^*$. Therefore, from (1) we get

$$\Lambda(y) = \mathrm{D}\int_K (\Lambda \circ \Phi)|_K\, d\mu|_{\mathcal{F}\cap K} = \mathrm{D}\int_X (\Lambda \circ \Phi)\, d\mu$$

for any $\Lambda \in X^*$, so that $y = \int_X \Phi\, d\mu$.

(ii) Let μ_i be a mass having K_i as a compact support and Φ integrable w.r.t. $\mu_i\,(i = 1, 2)$. Given $\Lambda \in X^*$, let $f = (\Lambda \circ \Phi)I_K$, where $K = K_1 \cup K_2$. Then, denoting by $\mathrm{S}\int_X f d\mu$ the Stieltjes type integral defined in [1; Sec. 4.5], observe that this integral exists for any mass μ and is equal to $\mathrm{D}\int_X f d\mu$. Therefore, since K is a support of μ_i $(i = 1,\, 2)$, by (1) and the linearity of the S-integral w.r.t. masses, we have

$$\begin{aligned}\Lambda\left(\sum_{i=1}^{2} \alpha_i \int_X \Phi\, d\mu_i\right) &= \alpha_1 \Lambda\left(\int_X \Phi\, d\mu_1\right) + \alpha_2 \Lambda\left(\int_X \Phi\, d\mu_2\right) = \\ &= \alpha_1\, \mathrm{D}\int_X (\Lambda \circ \Phi)\, d\mu_1 + \alpha_2\, \mathrm{D}\int_X (\Lambda \circ \Phi)\, d\mu_2 = \\ &= \alpha_1\, \mathrm{D}\int_X f\, d\mu_1 + \alpha_2\, \mathrm{D}\int_X f\, d\mu_2 = \\ &= \alpha_1\, \mathrm{S}\int_X f\, d\mu_1 + \alpha_2\, \mathrm{S}\int_X f\, d\mu_2 = \\ &= \mathrm{S}\int_X f\, d(\alpha_1\mu_1 + \alpha_2\mu_2) = \mathrm{D}\int_X f\, d(\alpha_1\mu_1 + \alpha_2\mu_2) = \\ &= \mathrm{D}\int_X (\Lambda \circ \Phi)\, d(\alpha_1\mu_1 + \alpha_2\mu_2).\end{aligned}$$

Since Λ is arbitrarily chosen, we get $\int_X \Phi\, d(\alpha_1\mu_1+\alpha_2\mu_2) = \alpha_1 \int_X \Phi\, d\mu_1 + + \alpha_2 \int_X \Phi\, d\mu_2$.

(iii) Let K be a common compact support for all μ, μ_d. Moreover, let Φ be integrable w.r.t. all μ, μ_d. Given $\Lambda \in X^*$, by Tietze Extension Theorem, there is a bounded continuous real extension f of $(\Lambda \circ \Phi)|_K$ over X. Since $\mu_d \to {}_w\mu$, by (1), we get

$$\Lambda\left(\int_X \Phi\, d\mu_d\right) = \mathrm{D}\int_X (\Lambda\circ\Phi)\, d\mu_d = \mathrm{D}\int_X f\, d\mu_d \to$$

$$\to \mathrm{D}\int_X f\, d\mu = \mathrm{D}\int_X (\Lambda\circ\Phi)\, d\mu = \Lambda\left(\int_X \Phi\, d\mu\right).$$

This completes the proof of the lemma. $\diamondsuit$

Now, we are able to prove the existence of the barycentre of any mass in M and the necessity of properties A1, A2 and A3$'$. Given $\mu \in \mathrm{M}$, let K be a convex compact support of μ and consider, in the lemma, the identity mapping of X as Φ. Then, by (i) and (ii), we get $\mathrm{b}(\mu) = = \mathrm{b}(\frac{1}{\|\mu\|}\mu) \in \overline{\mathrm{co}}(K) = K$. Consequently, the internality property holds and, from(ii), the associativity one follows as well. Finally, from (iii) we get the weak-continuity property. Indeed, let $\{\mu_d; d \in D\}$ be a net in S such that $\mu_d \to {}_w\mu$ and all μ, μ_d have a common convex compact support. Therefore, $\|\mu_d\| = \mathrm{D}\int_X I_X\, d\mu_d \to \mathrm{D}\int_X I_X\, d\mu = \|\mu\|$ and, by (iii), $\int_X x\, \mu_d(dx) \xrightarrow{w} \int_X x\, \mu(dx)$, so that $\mathrm{b}(\mu_d) \xrightarrow{w} \mathrm{b}(\mu)$.

Now, we are going to verify the sufficiency of A1, A2 and A3$'$. Recalling that A3$'$ implies A3 and the characterization theorem proved in [3] (X is Hausdorff!), we have $\mathrm{m}(\mu) = \mathrm{b}(\mu)$ for all $\mu \in \mathrm{S}$. Given $\mu \in \mathrm{M}$, let K be a convex compact support of μ. Then, by Th. 5.3 in [2] (K is a normal subspace of X!), the set of simple masses in $\mathrm{ba}_+(K, \mathcal{F}\cap K)$ is dense in $\mathrm{ba}_+(K, \mathcal{F}\cap K)$ endowed with the Lévy topology. Consequently, there is a net $\{\nu_d; d \in D\}$ of simple masses in $\mathrm{ba}_+(K, \mathcal{F}\cap K)$ such that $\mathrm{D}\int_K f|_K\, d\nu_d \to \mathrm{D}\int_K f|_K\, d\mu|_{\mathcal{F}\cap K}$ for any bounded continuous function f on X. Now, let $\mu_d(F) = \nu_d(F \cap K)$ for any $F \in \mathcal{F}$ and for all $d \in D$. Plainly, K is a support for all simple masses μ_d and hence, by (1), $\mathrm{D}\int_X f\, d\mu_d \to \mathrm{D}\int_X f\, d\mu$ for any bounded continuous function f on X. Consequently, $\mu_d \to {}_w\mu$ and hence, by the weak-continuity of the barycentre, we have $\mathrm{m}(\mu_d) = \mathrm{b}(\mu_d) \xrightarrow{w} \mathrm{b}(\mu)$. On the other hand, by A3$'$, we get $\mathrm{m}(\mu_d) \xrightarrow{w} \mathrm{m}(\mu)$, so that $\mathrm{m}(\mu) = \mathrm{b}(\mu)$. This completes the proof of the theorem. $\diamondsuit$

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