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ON THE STANDARD FORM OF THE SOLUTION OF THE TRANSLATION EQUATION IN RINGS OF FORMAL POWER SERIES

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Abstract: The aim of the paper is to find a general form of homomorphisms $\Theta : G \to \Gamma$, $\Theta(t)(X) = \sum_{k=1}^{\infty} c_k(t)X^k$, from an abelian group $(G, +)$ into the group $(\Gamma, \circ)$ of invertible formal power series with coefficients in $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, under the condition that c_1 takes infinitely many values. This is equivalent to determine all the solutions $F(t, X) = \sum_{k=1}^{\infty} c_k(t)X^k$ of the translation equation

$$F(s + t, X) = F(s, F(t, X)) \qquad \text{for} \quad s, t \in G.$$

We will show, using simultaneous conjugation, that in this case the solution of the translation equation in rings of formal power series has the standard form $F(t, X) = S^{-1}(c_1(t)S(X))$ well known for the solutions of the translation equation for real functions. All these results will be proved also in the ring of s-truncated formal power series.

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1. Introduction

By $\mathbb{K}[\![X]\!]$ we denote the ring of all formal power series $\sum_{k=0}^{\infty} c_k X^k$ with coefficients $c_k \in \mathbb{K}$, where $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ is a field of real or complex numbers. For a formal power series $f(X) = \sum_{k=0}^{\infty} c_k X^k$, where $c_k \neq 0$ for some $k \in \mathbb{N} \cup \{0\}$ ($\mathbb{N}$ stands here for the set of all positive integers), we define

$$\operatorname{ord} f(X) := \min\{i \in \mathbb{N} \cup \{0\} : c_i \neq 0\}.$$

assuming additionally $\operatorname{ord}\left(\sum_{i=1}^{\infty} 0X^i\right) = \infty$. It is known that the set $\Gamma = \{f(X) \in \mathbb{K}[X] : \operatorname{ord} f(X) = 1\}$ with the substitution $\circ$ as a binary operation is a group. Moreover, the set $\Gamma_1 = \left\{f(X) = \sum_{k=1}^{\infty} c_k X^k \in \Gamma : c_1 = 1\right\}$ is a subgroup of Γ. A very good reference for this topic is [1].

With every $f(X) = \sum_{i=0}^{\infty} c_i X^i \in \mathbb{K}[\![X]\!]$, we may associate the s-truncation of $f(X)$ defined by

$$f^{[s]}(X) := \sum_{i=0}^{s} c_i X^i \in \mathbb{K}[\![X]\!]_s \subset \mathbb{K}[\![X]\!].$$

In the set $\mathbb{K}[\![X]\!]_s$ of all s-truncated formal power series $f(X) = \sum_{i=0}^{s} c_i X^i$ ($\mathbb{K}[\![X]\!]_s$ may be treated as a set of all polynomials of degree at most s) we introduce, in a natural way, an addition of truncated formal power series. It appears that a multiplication and a substitution must be defined in a specific way that $\mathbb{K}[\![X]\!]_s$ should be closed under them. Let for $f(X), g(X) \in \mathbb{K}[\![X]\!]_s$,

$$(fg)(X) := (fg)^{[s]}(X),$$

and, in the case when $\operatorname{ord} g(X) \geq 1$,

$$(f \circ g)(X) := (f \circ g)^{[s]}(X).$$

Then $(\mathbb{K}[\![X]\!]_s, +, \cdot)$ is a ring, the set $\Gamma^s := \{p(X) \in \mathbb{K}[\![X]\!]_s : \operatorname{ord} p(X) = 1\}$ is a group under substitution and $\Gamma_1^s = \left\{f(X) = \sum_{k=1}^{s} c_k X^k \in \Gamma^s : c_1 = 1\right\}$ is a subgroup. To unify notation, from now on by Γ^∞ and Γ_1^∞ we will mean Γ and Γ_1.

Definition 1. Let s be a positive integer or $s = \infty$. By a one-parameter group of formal power series we understand every homomorphism of a group $(G, +)$ into $(\Gamma^s, \circ)$, i.e. each function $\Theta_G : G \to \Gamma^s$ which satisfies the equation

$$\Theta_G(t_1 + t_2) = \Theta_G(t_1) \circ \Theta_G(t_2) \qquad \text{for } t_1, t_2 \in G. \tag{1}$$

Let $F_t(X) = F(t, X) = \Theta_G(t)(X) \in \Gamma$. In the case when Θ_G is a one-parameter group of formal power series we will also say that the family $(F_t(X))_{t \in G} = (F(t, X))_{t \in G}$ forms a one-parameter group of formal power series. From (1) we then obtain, as an equivalent formulation, the so called translation equation (in the case $s = \infty$)

(2) $$F(t_1 + t_2, X) = F(t_1, F(t_2, X)) \quad \text{for } t_1, t_2 \in G,$$

in a ring of formal power series, and (in the case $s < \infty$),

(3) $$F(t_1 + t_2, X) = F(t_1, F(t_2, X)) \mod X^{s+1} \quad \text{for } t_1, t_2 \in G,$$

in the ring of s-truncated formal power series. Then (2) and (3) may jointly be written in the form

(4) $$F_{t_1+t_2}(X) = (F_{t_1} \circ F_{t_2})(X) \quad \text{for } t_1, t_2 \in G.$$

We recall some basic facts about solutions of the translation equation in $\mathbb{K}[\![X]\!]$, which will be needed in what follows. For integers $k \leq l$, by $|k, l|$ we denote the set of all integers n with $k \leq n \leq l$, whereas by $|k, \infty|$ we will mean the set of all $n \geq k$. If $k > l$, then we assume that $|k, l| = \emptyset$. Moreover, $\sum_{t \in \emptyset} a_0 = 0$ and $\prod_{t \in \emptyset} a_t = 1$.

Let s be a positive integer or $s = \infty$ and let $F(t, X) = \sum_{k=1}^{s} c_k(t) X^k$, where $c_1 : G \to \mathbb{K} \setminus \{0\}$, $c_k : G \to \mathbb{K}$ for $k \in |2, s|$. Then from (2) we get

(5) $$\sum_{k=1}^{\infty} c_k(t_1 + t_2) X^k = \sum_{l=1}^{\infty} c_l(t_1) \left(\sum_{j=1}^{\infty} c_j(t_2) X^j \right)^l, \quad t_1, t_2 \in G.$$

Analogously, from (3) we obtain

(6) $$\sum_{k=1}^{s} c_k(t_1+t_2) X^k = \sum_{l=1}^{s} c_l(t_1) \left(\sum_{j=1}^{s} c_j(t_2) X^j \right)^l \mod X^{s+1}, \ t_1, t_2 \in G.$$

It is known (cf. [3]) that if either

$$\sum_{k=1}^{\infty} a_k \left(\sum_{l=1}^{\infty} b_l X^l \right)^k = \sum_{n=1}^{\infty} d_n X^n,$$

or

$$\sum_{k=1}^{s} a_k \left(\sum_{l=1}^{s} b_l X^l \right)^k = \sum_{n=1}^{s} d_n X^n \mod X^{s+1},$$

then

(7) $$d_n = \sum_{k=1}^{n} a_k \sum_{\overline{u}_n \in U_{n,k}} B_{\overline{u}_n} \prod_{j=1}^{n} b_j^{u_j} \quad \text{for } n \in |1, s|,$$

where

$$U_{n,k} := \left\{ \overline{u}_n := (u_1, \ldots, u_n) \in |0, k|^n : \ \sum_{j=1}^{n} u_j = k \ \wedge \ \sum_{j=1}^{n} j u_j = n \right\},$$

$$B_{\overline{u}_n} := \frac{k!}{\prod_{j=1}^{n} u_j!}.$$

As examples of (7) we quote

$$\begin{aligned} d_1 &= a_1 b_1, \\ d_2 &= a_1 b_2 + a_2 b_1^2, \\ d_3 &= a_1 b_3 + 2a_2 b_1 b_2 + a_3 b_1^3. \end{aligned} \tag{8}$$

Moreover, for every $n \geq 2$, we have (see [3, Cor. 2])

$$d_n = a_1 b_n + \sum_{k=2}^{n-1} a_k \sum_{\overline{u}_n \in U_{n,k}} B_{\overline{u}_n} \prod_{j=1}^{n-k+1} b_j^{u_j} + a_n b_1^n. \tag{9}$$

Then, from (5) and (6), on account of (8) and (9), by comparing coefficients we obtain the system of functional equations

$$\begin{cases} c_1(t_1+t_2) = c_1(t_1)c_1(t_2) \\ c_2(t_1+t_2) = c_1(t_1)c_2(t_2) + c_2(t_1)c_1(t_2)^2 \\ c_3(t_1+t_2) = c_1(t_1)c_3(t_2) + 2c_2(t_1)c_1(t_2)c_2(t_2) + c_3(t_1)c_1(t_2)^3 \\ c_n(t_1+t_2) = c_1(t_1)c_n(t_2) \\ \qquad + \sum_{k=2}^{n-1} c_k(t_1) \sum_{\overline{u}_n \in U_{n,k}} B_{\overline{u}_n} \prod_{j=1}^{n-1} c_j(t_2)^{u_j} + c_n(t_1)c_1(t_2)^n, \ n \in |4, s|, \end{cases} \tag{10}$$

for $t_1, t_2 \in G$. Note that c_1 must be a generalized exponential function.

The main results of our paper are Theorems 3, 4 and 5. In Th. 4 we state for a solution $F(t,X)_{t\in G}$, $F(t,X) = \sum_{i=1}^{s} c_i(t)X^i$ of (4), where s is a positive integer or $s = \infty$ and $(G,+)$ is an abelian group such that the generalized exponential function c_1 takes infinitely many values that there exists a unique $S(X) \in \Gamma_1^s$ for which

$$F(t,X) = \left(S^{-1} \circ L_{c_1(t)} \circ S\right)(X) \quad \text{for all } t \in G$$

holds, the so called standard form of the solution of the translation equation. Here $L_\rho(X) = \rho X$. Th. 4 is based upon Th. 3, where we show the same representation for solutions $F(t,X)_{t\in\mathbb{K}}$, $F(t,X) = \sum_{i=1}^{\infty} c_i(t)X^i$ of (2) with regular (C^∞ or entire) coefficient functions, under the assumption $c_1 \neq 1$. Th. 5 deals with the situation where $F(t,X)_{t\in G}$ is a finite one-parameter group of formal power series (then clearly $\operatorname{im} c_1$ is also finite). We obtain here the standard form for $F(t,X)_{t\in G}$, too. Our method of proof uses certain semicanonical forms of formal power series with respect to conjugation, when the multiplier of the series is a (complex) root of 1.

In the following we use the standard notation

$$\frac{\partial F(t,X)}{\partial X} := \sum_{k=1}^{\infty} k c_k(t) X^{k-1},$$

and, in the case when $G = \mathbb{K}$ and the coefficient functions are differentiable,

$$\frac{\partial F(t,X)}{\partial t} := \sum_{k=1}^{\infty} c_k'(t) X^{k}.$$

For $G = \mathbb{K}$ the following theorem describes the general regular solution of the translation equation (2) in the ring of formal power series, which means that the coefficient functions are analytic when $\mathbb{K} = \mathbb{C}$, or C^∞, when $\mathbb{K} = \mathbb{R}$.

Theorem 1 (cf. [10]). (i) *If a family $(F(t,X))_{t\in\mathbb{K}}$ is a regular one-parameter group of formal power series, then there exists a formal power series $H(X) \in \mathbb{K}[\![X]\!]$ such that*

$$(11) \qquad \begin{cases} \dfrac{\partial F(t,X)}{\partial t} = H(F(t,X)) \quad \textit{for } t \in \mathbb{K}, \\ F(0,X) = X. \end{cases}$$

(ii) *For each formal power series $H(X) \in \mathbb{K}[\![X]\!]$ with* $\operatorname{ord} H \geq 1$, *the family $(F(t,X))_{t\in\mathbb{K}}$ defined by* (11) *is a regular one-parameter group of formal power series.*

(iii) *The series H is uniquely determined by $(F(t,X))_{t\in\mathbb{K}}$. It is given by the formula $H(X) := \frac{\partial F(t,X)}{\partial t}|_{t=0}$. In particular,* $\operatorname{ord} H \geq 1$.

Remark 1. Condition (iii) establishes a $1-1$-correspondence between regular one-parameter groups and formal series H with $\operatorname{ord} H \geq 1$.

The general solution of the system of equations (2) under some assumptions on c_1 is described in the following

Theorem 2 (cf. [5, Th. 6]). *Let s be a positive integer or $s = \infty$. Assume that $(G,+)$ is an abelian group which admits a generalized exponential function from G into $\mathbb{K} \setminus \{0\}$ with infinite image. Then there exists a sequence of polynomials $(P_n)_{n\geq 2}$ defined by*

$$\begin{cases} P_2(X) = 0; \quad R_2(X;\lambda_2) = \lambda_2 X - \lambda_2 \\ P_n(X;\lambda_2,\ldots,\lambda_{n-1}) \\ = \displaystyle\sum_{k=2}^{n-1} ((k-1)\lambda_k \sum_{\overline{u}_n \in U_{n,k}} B_{\overline{u}_n} \int_1^X t^{k-2} \prod_{j=2}^{n-k+1} (R_j(t;\lambda_2,\ldots,\lambda_j))^{u_j}\, dt. \\ R_n(X;\lambda_2,\ldots,\lambda_n) = \lambda_n(X^{n-1}-1) + P_n(X;\lambda_2,\ldots,\lambda_{n-1}), \end{cases}$$

such that for every solution $(c_n)_{n\in|1,s|}$ *of the system of functional equations* (10) *(that is for every solution* $F(t,X)_{g\in G}$, $F(t,X)=\sum_{k=1}^{s}c_k(t)X^k$ *of the translation equation* (4)*) with a generalized exponential function* c_1 *taking infinitely many values, there exists a unique sequence of constants* $(\lambda_n)_{n\in|2,s|}$ *such that*

(12)

$$c_n(t)=\lambda_n(c_1(t)^n-c_1(t))+c_1(t)P_n(c_1(t);\lambda_2,\dots,\lambda_{n-1}),\ t\in G, n\in|2,s|.$$

Conversely, for every exponential function c_1 *and for each sequence* $(\lambda_n)_{n\in|2,s|}$, *the sequence* $(c_n)_{n\in|2,s|}$ *defined by* (12) *is a solution of the system* (10).

2. The standard form of the general regular solution of the translation equation with $c_1\neq 1$

Now we will give, using simultaneous conjugation, another form of the solution $(F(t,x))_{t\in\mathbb{K}}$ of the translation equation in a ring of formal power series, which is familiar for representations of solutions of the translation equation satisfying some regularity conditions (cf. [7] and [8]).

Theorem 3. *Let* $(F(t,X))_{t\in\mathbb{K}}$, $F(t,X)=\sum_{k=1}^{\infty}c_k(t)X^k$, $c_1:\mathbb{K}\to\mathbb{K}\setminus\{0\}$, $c_k:\mathbb{K}\to\mathbb{K}$ *for* $k\geq 2$, *be a regular solution of the translation equation* (2) *with an exponential function* $c_1\neq 1$. *Then there exists a unique formal power series* $S(X)=X+\sum_{k=2}^{\infty}v_kX^k\in\Gamma_1$ *such that*

$$F(t,X)=S^{-1}\left(c_1(t)S(X)\right)\qquad \text{for } t\in\mathbb{K}.$$

Conversely, for every generalized exponential function $c_1:\mathbb{K}\to\mathbb{K}\setminus\{0\}$ *and for an arbitrary* $S(X)=X+\sum_{k=2}^{\infty}v_kX^k\in\Gamma_1$, *the function* $F(t,X)=$ $=S^{-1}\left(c_1(t)S(X)\right)$ *is a solution of the translation equation* (2).

Proof. Let $F(t,X)=\sum_{k=1}^{\infty}c_k(t)X^k$ be a regular solution of the translation equation (2) with an exponential function $c_1\neq 1$. Then, in virtue of Th. 1, there exists a formal power series $H(X)\in\Gamma$ such that

$$\frac{\partial F(t,X)}{\partial t}=H(F(t,X))\quad \text{for } t\in\mathbb{K},$$

and H is given by the formula $H(X)=\frac{\partial F(t,X)}{\partial t}|_{t=0}$. Let $\lambda_1\neq 0$ and put $H(X)=\lambda_1\left(X+\sum_{k=2}^{\infty}(k-1)\lambda_kX^k\right)$. Then $c_1(t)=e^{\lambda_1 t}$ for $t\in\mathbb{K}$.

First, suppose that there is a formal power series $S(X)=X+$ $+\sum_{k=2}^{\infty}v_kX^k\in\Gamma_1$ such that

$$S(F(t,X))=e^{\lambda_1 t}S(X)\quad \text{for } t\in\mathbb{K}. \tag{13}$$

Differentiating (13) with respect to t we get

$$\left.\frac{dS}{dX}\right|_{F(t,X)} \cdot \frac{\partial}{\partial t}F(t,X) = \lambda_1 e^{\lambda_1 t} S(X).$$

Put $t = 0$. Then, since $F(0, X) = X$ and $H(X) = \frac{\partial F(t,X)}{\partial t}|_{t=0}$, we obtain

$$\frac{dS}{dX} \cdot H(X) = \lambda_1 S(X) \tag{14}$$

from which we get

$$\left(1 + \sum_{k=2}^{\infty} k v_k X^{k-1}\right) \lambda_1 \left(X + \sum_{k=2}^{\infty}(k-1)\lambda_k X^k\right) = \lambda_1 \left(X + \sum_{k=2}^{\infty} v_k X^k\right),$$

or, which is the same,

$$\left(1 + \sum_{k=1}^{\infty}(k+1)v_{k+1}X^k\right)\left(1 + \sum_{k=1}^{\infty} k\lambda_{k+1}X^k\right) = 1 + \sum_{k=1}^{\infty} v_{k+1}X^k, \tag{15}$$

Equality (15) is equivalent to the system of equations

$$\begin{cases} \lambda_2 + 2v_2 = v_2, \\ 2\lambda_3 + 2v_2\lambda_2 + 3v_3 = v_3, \\ (n-1)\lambda_n + \sum_{k=2}^{n-1} k(n-k)v_k\lambda_{n+1-k} + nv_n = v_n, \quad n \geq 4, \end{cases}$$

from which one can derive

$$\begin{cases} v_2 = -\lambda_2, \\ v_3 = -\lambda_3 - v_2\lambda_2 = -\lambda_3 + \lambda_2^2, \\ v_n = -\lambda_n - \sum_{k=2}^{n-1} \frac{k(n-k)}{n-1} v_k\lambda_{n+1-k}, \quad n \geq 4. \end{cases} \tag{16}$$

This means that a power series $S(X)$ satisfying (13), if it exists, is determined uniquely.

Now, let us take a power series $S(X) = X + \sum_{k=2}^{\infty} v_k X^k$ satisfying condition (16). Hence also (14) is satisfied. Replace in (14) X by $F(t, X)$. Then we obtain

$$\left.\frac{dS}{dX}\right|_{F(t,X)} \cdot H(F(t,X)) = \lambda_1 S(F(t,X))$$

and, since $H(F(t, X)) = \frac{\partial F}{\partial t}(t, X)$, so we get

$$\frac{dS}{dX}(F(t,X))\frac{\partial}{\partial t}F(t,X) = \lambda_1 S(F(t,X))$$

or, equivalently, $\frac{\partial}{\partial t}S(F(t,X))=\lambda_1 S(F(t,X))$. Put $R(t,X)=S(F(t,X))$. Then

$$\frac{\partial}{\partial t}R(t,X)=\lambda_1 R(t,X) \tag{17}$$

with the initial condition $R(0,X)=S(X)$. Since $e^{\lambda_1 t}S(X)$ is also a solution of (17) satisfying the same initial condition, from the uniqueness theorem for systems of the form (17), we obtain $S(F(t,X))=e^{\lambda_1 t}S(X)$ for every $t\in\mathbb{K}$. Conversely, let $F(t,X)=S^{-1}(e^{\lambda_1 t}S(X))$. This is the standard form of a solution of the translation equation, and hence satisfies (2). ◊

3. The standard form of the general solution of the translation equation with infinite $\operatorname{im} c_1$

Now we are going to generalize the result from the previous section to the general case $(F(t,X))_{t\in G}$ with infinite $\operatorname{im} c_1$. We will show that, in fact, also the same formulas hold as for the general regular solution. This will be done jointly for finite and infinite s. By E_m we denote the set of all roots of 1 of order m in the field $\mathbb{K}$.

We begin with a crucial property of the sequence of polynomials $(P_n)_{n\geq 2}$ from Th. 2. This property we deduce using regular solutions of the translation equation (2). To do this, we need

Lemma 1. *Let $s\geq 2$ be an integer or $s=\infty$. For every $S(X)=X+\sum_{k=2}^{s}v_kX^k\in\Gamma_1^s$ there exist polynomials $\sigma_k(v_2,\ldots,v_k)\in\mathbb{Q}[v_2,\ldots,v_k]$ such that $\Gamma_1^s\ni S^{-1}(X)=X+\sum_{k=2}^{s}\sigma_k(v_2,\ldots,v_k)X^k$.*

Proof. Since Γ_1^s is a group, let $S^{-1}(X)=X+\sum_{k=2}^{s}\sigma_kX^k$. Then $(S^{-1}\circ S)(X)=X$, which is equivalent (cf. (8) and (9)) to the system of equalities

$$\begin{cases} v_2+\sigma_2=0,\\ v_3+2v_2\sigma_2+\sigma_3=0,\\ v_n+\displaystyle\sum_{k=2}^{n-1}\sigma_k\sum_{\overline{u}_n\in U_{n,k}}B_{\overline{u}_n}\prod_{j=2}^{n-k+1}v_j^{u_j}+\sigma_n=0,\quad n\in|4,s|,\end{cases}$$

from which we get

$$\begin{cases} \sigma_2=-v_2=:\sigma_2(v_2),\\ \sigma_3=-v_3-2v_2\sigma_2=-v_3+2v_2^2=:\sigma_3(v_2,v_3),\\ \sigma_n=-v_n-\displaystyle\sum_{k=2}^{n-1}\sigma_k\sum_{\overline{u}_n\in U_{n,k}}B_{\overline{u}_n}\prod_{j=2}^{n-k+1}v_j^{u_j}=:\sigma_n(v_2,\ldots,v_n),\ n\in|4,s|.\end{cases} \tag{18}$$

Conversely, define for $S(X) = X + \sum_{k=2}^{s} v_k X^k$ the polynomials $\sigma_n(v_2, \dots, v_n)$ by (18). Then, for $S'(X) = X + \sum_{k=2}^{s} \sigma_k(v_2, \dots, v_k) X^k$, we get $(S' \circ S)(X) = X$. Since Γ_1^s is a group, so also $(S \circ S')(X) = X$. This means that $S' = S^{-1}$. ◊

Lemma 2. *Let X and Y be independent indeterminates over $\mathbb{K}$. For every $(\lambda_k)_{k\geq 2}$ there exists a unique sequence $(v_k)_{k\geq 2}$ such that*

$$(19)\quad YX + \sum_{k=2}^{\infty} \left(\lambda_k(Y^k - Y) + Y P_k(Y; \lambda_2, ..., \lambda_{k-1}\right) X^k = S^{-1}(Y S(X)),$$

where $S(X) = X + \sum_{k=2}^{\infty} v_k X^k$, and conversely, for each $(v_k)_{k\geq 2}$ there exists a unique sequence $(\lambda_k)_{k\geq 2}$ satisfying (19).

Proof. Assume that $s = \infty$, $(G, +) = (\mathbb{K}, +)$ and let us consider a regular solution $F(t, X) = e^t X + \sum_{k=2}^{\infty} c_k(t) X^k$ of (2). From Th. 2 we know that for every $n \geq 2$ we have $c_n(t) = \lambda_n(e^{nt} - e^t) + e^t P_n(e^t; \lambda_2, \dots, \lambda_{n-1})$, and the sequence $(\lambda_n)_{n\geq 2}$ determines $F(t, X)$ uniquely. On the other hand, by Th. 3, there exists a unique formal power series $S(X) = X + \sum_{k=2}^{\infty} v_k X^k \in \Gamma_1^{\infty}$ such that $F(t, X) = S^{-1}(e^t S(X))$. Then, on account of Lemma 1, we obtain

$$e^t X + \sum_{k=2}^{\infty} \left(\lambda_k(e^{kt} - e^t) + e^t P_k(e^t; \lambda_2, ..., \lambda_{k-1})\right) X^k = F(t, X) = S^{-1}(e^t S(X))$$
$$= e^t \left(X + \sum_{k=2}^{\infty} v_k X^k\right) + \sum_{l=2}^{\infty} \sigma_l(v_2, \dots, v_l) \left(e^t \left(X + \sum_{k=2}^{\infty} v_k X^k\right)\right)^l$$
$$= e^t X + \sum_{k=2}^{\infty} Q_k(e^t; v_2, \dots, v_k) X^k,$$

for every $t \in \mathbb{K}$, where $(Q_k(X; v_2, \dots, v_k))_{k\geq 2}$ is a sequence of polynomials. This implies

$\lambda_k(e^{kt} - e^t) + e^t P_k(e^t; \lambda_2, ..., \lambda_{k-1}) = Q_k(e^t; v_2, ..., v_k)$ for $k \geq 2$ and $t \in \mathbb{K}$. Since e^t runs through infinitely many values, we obtain the polynomial identities

$$\lambda_k(Y^k - Y) + Y P_k(Y; \lambda_2, \dots, \lambda_{k-1}) = Q_k(Y; v_2, \dots, v_k) \quad \text{for } k \geq 2$$

with an indeterminant Y. By the meaning of W_k and Q_k we get (19).

Conversely, it is known that $F(t, X) = S^{-1}(e^t S(X))$ is a regular solution of (2) for every $S(X) = X + \sum_{k=2}^{\infty} v_k X^k \in \Gamma_1^{\infty}$. Then, by Th. 2, there exists a unique sequence $(v_k)_{k\geq 2}$ satisfying

$$S^{-1}(e^t S(X)) = F(t,x) = e^t X + \sum_{k=2}^{\infty} \left(\lambda_k (e^{kt} - e^t) + e^t P_k(e^t; \lambda_2, ..., \lambda_{k-1}) \right) X^k,$$

and similarly as above we obtain (19). ♢

Corollary 1. *Let $s \geq 2$ be an integer. For every sequence $(\lambda_k)_{k \in |2,s|}$ there exists a unique $(v_k)_{k \in |2,s|}$ such that*

$$YX + \sum_{k=2}^{s} \left[\lambda_k (Y^k - Y) + Y P_k(Y; \lambda_2, \ldots, \lambda_{k-1}) \right] X^k = \left(S^{-1} \circ L_Y \circ S \right)(X)$$

and conversely (here $L_Y(X) = YX$).

Proposition 1. *Let $s \geq 2$ be an integer or $s = \infty$. Assume that $(G,+)$ is an abelian group and let $c_1 : G \to \mathbb{K} \setminus \{0\}$ be a generalized exponential function.*

(i) *For every sequence $(\lambda_k)_{k \in |2,s|}$,*

(20)

$$F(t,X) = c_1(t) X + \sum_{k=1}^{s} \left[\lambda_k (c_1(t)^k - c_1(t)) + c_1(t) P_k(c_1(t); \lambda_2, ..., \lambda_{k-1}) \right] X^k$$

is a solution of the translation equation (4).

(ii) *Every solution (20) of the translation equation (4) has a representation*

$$F(t,X) = \left(S^{-1} \circ L_{c_1(t)} \circ S \right)(X) \quad \text{for } t \in G, \tag{21}$$

with some $S(X) = X + \sum_{k=2}^{s} v_k X^k \in \Gamma_1^s$.

(iii) *Conversely, each $F(t,X)$ given by (21) is a solution of (4) and has a representation (20) with some sequence $(\lambda_k)_{k \in |2,s|}$.*

(iv) *If c_1 takes infinitely many values, then (20) and (21) yield the general solution of (4) (with unique sequences $(\lambda_k)_{k \in |2,s|}$ and $(v_k)_{k \in |2,s|}$).*

Proof. (i) is just a part of Th. 2. Let $F(t,X) = c_1(t)X + \sum_{k=2}^{s} c_k(t) X^k$ be a solution of the translation equation (4). Then, by Lemma 2 if $s = \infty$, and from Cor. 1 for $s < \infty$, replacing Y by $c_1(t)$, we get

$$\begin{aligned} F(t,X) &= c_1(t) X + \sum_{k=2}^{s} \left[\lambda_k (c_1(t)^k - c_1(t)) + c_1(t) P_k(c_1(t); \lambda_2, ..., \lambda_{k-1}) \right] X^k \\ &= \left(S^{-1} \circ L_{c_1(t)} \circ S \right)(X). \end{aligned}$$

Further, (21) is a solution of (4), and the representation (20) may be proved as above in (iii). Finally, (iv) is a consequence of Th. 2, conditions (ii) and (iii), and uniqueness in Th. 2, Lemma 2 and Cor. 1. ♢

Remark 2. The formal power series $S(X) = X + \sum_{k=2}^{s} v_k X^k \in \Gamma_1^s$ such that $F(t,X) = \left(S^{-1} \circ L_{c_1(t)} \circ S\right)(X)$, which exists on account of Prop. 1, need not be unique, because we do not assume that a sequence $(\lambda_n)_{n\in|2,s|}$ uniquely determines

$$F(t,X) = c_1(t)X + \sum_{k=1}^{s} \left[\lambda_k(c_1(t)^k - c_1(t)) + c_1(t)P_k(c_1(t);\lambda_2,...,\lambda_{k-1})\right]X^k.$$

If it is the case, then $S(X)$ is unique (cf. Lemma 2 and Cor. 1).

From Th. 2 and Prop. 1 we obtain the main result of the section.

Theorem 4. *Let $s \geq 2$ be an integer or $s = \infty$. Let $(G,+)$ be an abelian group which admits a generalized exponential function from G into $\mathbb{K} \setminus \{0\}$ having infinitely many values. Assume that $(F(t,X))_{t\in G}$, $F(t,X) = \sum_{k=1}^{s} c_k(t)X^k$, $c_1 : G \to \mathbb{K} \setminus \{0\}$, $c_k : G \to \mathbb{K}$ for $k \in |2,s|$, is a solution of the translation equation (4) with a generalized exponential function c_1 taking infinitely many values. Then there exists a unique formal power series $S(X) = X + \sum_{k=2}^{s} v_k X^k \in \Gamma_1^s$ such that*

$$F(t,X) = \left(S^{-1} \circ L_{c_1(t)} \circ S\right)(X) \quad \text{for } t \in G.$$

Conversely, for each generalized exponential function $c_1 : G \to \mathbb{K} \setminus \{0\}$ and for every $S(X) = X + \sum_{k=2}^{s} v_k X^k \in \Gamma_1^s$, the family $F(t,X) = \left(S^{-1} \circ L_{c_1(t)} \circ S\right)(X)$ is a solution of the translation equation (4).

From Th. 4 we obtain nice formulas for coefficients functions of the solution of the translation equation (4) in the considered case.

Corollary 2. *Let $s \geq 2$ be an integer or $s = \infty$. The general solution $(F(t,X))_{t\in G}$, $F(t,X) = \sum_{k=1}^{s} c_k(t)X^k$, $c_1 : G \to \mathbb{K}\setminus\{0\}$, $c_k : G \to \mathbb{K}$ for $k \in |2,s|$, of the translation equation (4) with a generalized exponential function c_1 taking infinitely many values is given by*

$$c_n(t) = v_n(c_1(t)^n - c_1(t)) - \sum_{k=2}^{n-1} c_k(t) \sum_{\overline{u}_n \in U_{n,k}} B_{\overline{u}_n} \prod_{j=2}^{n-k+1} v_j^{u_j} \quad \text{for } t \in G, \tag{22}$$

for $n \in |2,s|$, where $(v_k)_{k\in|2,s|}$ are arbitrary constants.

Proof. Since for every $S(X) = X + \sum_{j=2}^{s} v_j X^j \in \Gamma_1^s$ also $S^{-1}(X) \in \Gamma_1^s$, we derive from Th. 4 that the general solution $F(t,X) = \sum_{k=1}^{s} c_k(t)X^k$ of the translation equation (4) with a generalized exponential function c_1 taking infinitely many values, may be given by the formula

$$F(t,X) = \left(S \circ L_{c_1(t)} \circ S^{-1}\right)(X),$$

where $S(X) = X + \sum_{k=2}^{s} v_k X^k \in \Gamma_1^s$ is an arbitrary formal power series. Thus, substituting $S(X)$ for X, we obtain $(F_t \circ S)(X) = S\,(c_1(t)X)$ for every $t \in G$, which is equivalent to the equality

$$\sum_{k=1}^{s} c_k(t)\left(X+\sum_{l=2}^{s} v_l X^l\right)^k = c_1(t)X+\sum_{l=2}^{s} v_l c_1(t)^l X^l \mod X^{s+1}.$$

Thus, using the formulas (9), for every $n \in |2,s|$ we obtain (put $v_1 = 1$)

$$c_1(t)v_n + \sum_{k=2}^{n-1} c_k(t) \sum_{\overline{u}_n \in U_{n,k}} B_{\overline{u}_n} \prod_{j=2}^{n-k+1} v_j^{u_j} + c_n(t) = s_n c_1(t)^n$$

from which we get (22). ◊

4. The standard form of the solution of the translation equation for finite set $\{F(t,X) : t \in G\}$

We are going to study one-parameter groups of formal power series $F(t,X)_{t\in G}$, $F(t,X) = \sum_{k=1}^{s} c_k(t)X^k$, $c_1 : G \to \mathbb{K} \setminus \{0\}$, $c_k : G \to \mathbb{K}$ for $k \in |2,s|$, where s is a positive integer or $s = \infty$, under the assumption that the set $\{F(t,X) : t \in G\}$ is finite. Note that then also $\operatorname{im} c_1$ must be finite. We will need some properties of (7). In [3] we considered a natural isomorphism between the groups $(\Gamma^\infty, \circ)$ and $(Z_\infty, \cdot) = L^1_\infty$, namely $\Psi : Z_\infty \to \Gamma^\infty$,

$$\Psi(x_1, x_2, \ldots)(X) = \sum_{k=1}^{\infty} \frac{x_k}{k!} X^k.$$

Furthermore, in [2] are proved some properties of the group operation in L^1_s, which are also valid for the group L^1_∞. Using the isomorphism Ψ and these properties one can derive the following lemma.

Lemma 3 (cf. [2, Lemma 2]). *Let p, q be integers such that $1 \le p \le q$. If $a_j = 0$ for all $j \in |2,q|$ and $b_j = 0$ for all $j \in |2,p|$, then d_n given by (7) are of the form*

1) $d_1 = a_1 b_1$,
2) $d_n = 0$ *for* $n \in |2,p|$,
3) $d_n = a_1 b_n$ *for* $n \in |p+1,q|$,
4) $d_n = a_1 b_n + a_n b_1^n$ *for* $n \in |q+1,p+q|$.

From now on, if it will not be another stated, $s \ge 2$ is an integer or $s = \infty$. We begin with

Lemma 4. *If $U(X) = X + \sum_{k=2}^{s} u_k X^k \in \Gamma^s_1$ and $u_l \neq 0$ for some $l \in |2,s|$, then for every $n \in \mathbb{N}$, $n \ge 2$ we have $U^n(X) \neq X$. Moreover, for every $m, n \in \mathbb{N}$, $m \neq n$ we have $U^m(X) \neq U^n(X)$.*

Proof. The proof is by induction on n. Let $n = 2$. Put $l := \min\{k \in \in |2, s| : u_k \neq 0\}$. Then $U(X) = X + u_l X^l + \sum_{k=l+1}^{s} u_k X^k$. Using Lemma 3 ($p = q = l - 1$) we obtain that $U^2(X) = (U \circ U)(X) = X + + 2u_l X^l + \sum_{k=l+1}^{s} u'_k X^k$ with some $u'_{l+1}, \dots, u'_s$, and, since $u_l \neq 0$, so $U^2(X) \neq X$.

Assume now that for some $n \in |3, s|$ we have $U^{n-1}(X) \neq X$, and if

$$U^{n-1}(X) = X + \sum_{k=2}^{s} v_k X^k,$$

then for $m := \min\{k \in |2, s| : v_k \neq 0\}$ we have $m = l$ and $v_l = (n-1)u_l$ (cf. the case $n = 2$). On account of Lemma 3 we get

$$U^n(X) = (U^{n-1} \circ U)(X) = X + nu_l X^l + \sum_{k=l+1}^{s} v'_k X^k$$

with some $v'_{l+1}, \dots, v'_s$. Finally, for $m, n \in \mathbb{N}$, $m \neq n$, we have

$$U^m(X) = X + mu_l X^l + \sum_{k=l+1}^{s} w_k X^k \neq X + nu_l X^l + \sum_{k=l+1}^{s} w'_k X^k = U^n(X),$$

which finishes the proof. ◊

Lemma 5. *Let $F(t, X)_{t \in G}$, $F(t, X) = \sum_{k=1}^{s} c_k(t) X^k$, $c_1 : G \to \mathbb{K} \setminus \{0\}$, $c_k : G \to \mathbb{K}$ for $k \in |2, s|$, be a solution of the translation equation* (4) *(i.e. $\Theta_G^s : G \to \Gamma^s$, $\Theta_G^s(t)(X) = F(t, X)$ is a homomorphism) such that the set $\{F(t, X) : t \in G\} = \Theta_G^s(G)$ is a finite group. Then* $\ker c_1 = \ker \Theta_G^s$.

Proof. Clearly, $\ker \Theta_G \subset \ker c_1$. For the proof by a contradiction let us suppose that for some $t_0 \in \ker c_1$, $t_0 \neq 0$, we have $\Theta_G(t_0)(X) = = \sum_{k=1}^{s} c_k(t_0) X^k = X + \sum_{k=2}^{s} d_k X^k$, where $d_l \neq 0$ for some $l \in |2, s|$. Then $\Theta_G^s(nt_0)(X) = ((\Theta_g^s(t_0))^n)(X)$ for every $n \in \mathbb{N}$, which jointly with Lemma 4 means that the image $\operatorname{im} \Theta_G^s$ is infinite. This contradiction proves $\ker c_1 = \ker \Theta_G^s$. ◊

Lemma 6. *Let $F(X) = d_1 X + \sum_{k=2}^{s} d_k X^k \in \Gamma^s$, where $d_1 \in E_m \setminus \{1\}$ is a primitive root of the order m. Then there exists a formal power series $U(x) = X + \sum_{k=2}^{s} u_k X^k \in \Gamma_1^s$ and a sequence of constante $(\delta_{lm+1})_{l \in |1, r|}$ such that*

$$(U \circ F \circ U^{-1})(X) = d_1 X + \sum_{l=1}^{r} \delta_{lm+1} X^{lm+1} =: N_m(X) \in \Gamma^s,$$

where r is the greatest positive integer such that $rm + 1 \leq s$ if $s < \infty$ and $r = \infty$ otherwise ($N_m(X)$ is called semicanonical form of $F(X)$, cf. [9, 11]).

Proof. Let $F(X)=d_1X+\sum_{k=2}^{s}d_kX^k\in\Gamma^s$, where $d_1\in E_m\setminus\{1\}$ is a primitive root of unit of order m. We find $U(x)=X+\sum_{k=2}^{s}u_kX^k\in\Gamma_1^s$ and $N_m(X)=d_1X+\sum_{l=1}^{r}\delta_{lm+1}X^{lm+1}=d_1X+\sum_{k=2}^{s}\delta_kX^k$, where r is the greatest positive integer such that $rm+1\le s$ if $s<\infty$ and $r=\infty$ otherwise, $\delta_k=0$ for $k\in|2,s|$ with $k\not\equiv 1 \mod m$, such that $(U\circ F)(X)=(N_m\circ U)(X)$, i.e. the system

$$\begin{cases} d_2+u_2d_1^2=d_1u_2, \\ d_n+\sum\limits_{k=2}^{n-1}u_k\sum\limits_{\overline{u}_n\in U_{n,k}}B_{\overline{u}_n}\prod\limits_{j=1}^{n-k+1}d_j^{u_j}+u_nd_1^n=d_1u_n \quad \text{for } n\in|3,m|, \\ d_{m+1}+\sum\limits_{k=2}^{m}u_k\sum\limits_{\overline{u}_{m+1}\in U_{m+1,k}}B_{\overline{u}_{m+1}}\prod\limits_{j=1}^{m-k+2}d_j^{u_j}+u_{m+1}d_1^{m+1} \\ \qquad =d_1u_{m+1}+\delta_{m+1} \quad \text{if } m+1\le s, \\ d_n+\sum\limits_{k=2}^{n-1}u_k\sum\limits_{\overline{u}_n\in U_{n,k}}B_{\overline{u}_n}\prod\limits_{j=1}^{n-k+1}d_j^{u_j}+u_nd_1^n= \\ \qquad d_1u_n+\sum\limits_{k=2}^{n-1}\delta_k\sum\limits_{\overline{u}_n\in U_{n,k}}B_{\overline{u}_n}\prod\limits_{j=2}^{n-k+1}u_j^{u_j}+\delta_n \quad \text{for } n\in|m+2,s| \end{cases}$$

is satisfied with $\delta_k=0$ for $k\ge 2$ with $k\not\equiv 1 \mod m$. This is equivalent to the system of equalities

(23)
$$\begin{cases} u_2(d_1^2-d_1)=-d_2, \\ u_n(d_1^n-d_1)=-d_n-\sum\limits_{k=2}^{n-1}u_k\sum\limits_{\overline{u}_n\in U_{n,k}}B_{\overline{u}_n}\prod\limits_{j=1}^{n-k+1}d_j^{u_j} \quad \text{for } n\in|3,m|, \\ \delta_{m+1}=d_{m+1}+\sum\limits_{k=2}^{m}u_k\sum\limits_{\overline{u}_{m+1}\in U_{m+1,k}}B_{\overline{u}_{m+1}}\prod\limits_{j=1}^{m-k+2}d_j^{u_j} \quad \text{if } m+1\le s, \\ u_n(d_1^n-d_1)-\delta_n=-d_n+\sum\limits_{k=2}^{n-1}\sum\limits_{\overline{u}_n\in U_{n,k}} \\ \qquad B_{\overline{u}_n}\left(\delta_k\prod\limits_{j=2}^{n-k+1}u_j^{u_j}-u_k\prod\limits_{j=1}^{n-k+1}d_j^{u_j}\right) \quad \text{for } n\in|m+2,s|. \end{cases}$$

As it is easy to see, we can find a (not unique) solution $(u_k)_{k\in|2,s|}$, $(\delta_{lm+1})_{l\in|1,r|}$ (here l is the greatest integer such that $rm+1\le s$ if $s<\infty$ and $r=\infty$ otherwise) of this system. Indeed, we find

$$\begin{cases} u_2 = (d_1^2 - d_1)^{-1} - d_2, \\ u_n = (d_1^n - d_1)^{-1} \left(-d_n - \sum_{k=2}^{n-1} u_k \sum_{\overline{u}_n \in U_{n,k}} B_{\overline{u}_n} \prod_{j=1}^{n-k+1} d_j^{u_j} \right) \text{ for } n \in |3, m|, \\ \delta_{m+1} = d_{m+1} + \sum_{k=2}^{m} u_k \sum_{\overline{u}_{m+1} \in U_{m+1,k}} B_{\overline{u}_{m+1}} \prod_{j=1}^{m-k+2} d_j^{u_j}, \end{cases}$$

and we fix u_{m+1} arbitrarily. Finally, consider for some $n \in |m+2, s|$ the equation

$$u_n(d_1^n - d_1) - \delta_n = -d_n + \sum_{k=2}^{n-1} \sum_{\overline{u}_n \in U_{n,k}} B_{\overline{u}_n} \left(\delta_k \prod_{j=2}^{n-k+1} u_j^{u_j} - u_k \prod_{j=1}^{n-k+1} d_j^{u_j} \right).$$

Not that the sum $\sum_{k=2}^{n-1} \sum_{\overline{u}_n \in U_{n,k}} B_{\overline{u}_n} \left(\delta_k \prod_{j=2}^{n-k+1} u_j^{u_j} - u_k \prod_{j=1}^{n-k+1} d_j^{u_j} \right)$ contains only terms with indices less than n. If $n \not\equiv 1 \mod m$, then $\delta_n = 0$ and

$$u_n = (d_1^n - d_1)^{-1} \left(-d_n + \sum_{k=2}^{n-1} \sum_{\overline{u}_n \in U_{n,k}} B_{\overline{u}_n} \left(\delta_k \prod_{j=2}^{n-k+1} u_j^{u_j} - u_k \prod_{j=1}^{n-k+1} d_j^{u_j} \right) \right).$$

Otherwise we fix u_{lm+1} arbitrarily and we find

$$\delta_{lm+1} = d_{lm+1} - \sum_{k=2}^{lm} \sum_{\overline{u}_{lm+1} \in U_{lm+1,k}} B_{\overline{u}_{lm+1}} \left(\delta_k \prod_{j=2}^{lm-k+2} u_j^{u_j} - u_k \prod_{j=1}^{lm-k+2} d_j^{u_j} \right).$$

Thus we find $U(X)$ and $N_m(X)$ satisfying $(U \circ F)(X) = (N_m \circ U)(X)$. $\Diamond$

Lemma 7. *Let $N_m(X) = d_1 X + \sum_{l=1}^{r} \delta_{lm+1} X^{lm+1}$, where $d_1 \in E_m \setminus \{1\}$ is a primitive root of order m, r is the greatest positive integer such that $rm+1 \leq s$ if $s < \infty$ and $r = \infty$ otherwise, $\delta_{lm+1} \neq 0$ for some $l \in |1, r|$. Then, for every $p, q \in \mathbb{N}$, $p \neq q$, we have $N_m^p(X) \neq N_m^q(X)$.*

Proof. Let $\nu := \min\{l \in |1, r| : \delta_{lm+1} \neq 0\}$. Then

$$N_m(X) = d_1 X + \delta_{\nu m+1} X^{\nu m+1} + \sum_{k=\nu+1}^{r} \delta_{km+1} X^{km+1}.$$

We prove by induction on n that

$$N_m^n(X) = d_1^n X + n d_1^{n-1} \delta_{\nu m+1} X^{\nu m+1} + \sum_{k=\nu+1}^{r} \delta'_{km+1} X^{km+1}$$

with some $(\delta_{lm+1})_{l \in |\nu+1, r|}$ Put $n = 2$. Then, on account of Lemma 3, we get

$$N_m^2(X) = d_1^2X + (d_1\delta_{\nu m+1} + \delta_{\nu m+1}d_1^{\nu m+1})X^{\nu m+1} + \sum_{k=\nu+1}^{r} \delta''_{km+1}X^{km+1}$$
$$= d_1^2X + 2d_1\delta_{\nu m+1}X^{\nu m+1} + \sum_{k=\nu+1}^{r} \delta''_{km+1}X^{km+1}.$$

Assuming now that for some $n \geq 3$ we have

$$N_m^{n-1}(X) = d_1^{n-1} + (n-1)d_1^{n-2}\delta_{\nu m+1}X^{\nu m+1} + \sum_{k=\nu+1}^{r} \delta''_{km+1}X^{km+1},$$

we obtain, on account of Lemma 3,

$$N_m^n(X) = (N_m^{n-1} \circ N_m)(X)$$
$$= d_1^nX + (d_1^{n-1}\delta_{\nu m+1} + (n-1)\delta_{\nu m+1}d_1^{n-2}d_1^{\nu m+1})X^{\nu m+1}$$
$$+ \sum_{k=\nu+1}^{r} \delta'_{km+1}X^{km+1} = d_1^nX + nd_1^{n-1}\delta_{\nu m+1}X^{\nu m+1} + \sum_{k=\nu+1}^{r} \delta'_{km+1}X^{km+1}.$$

Since $pd_1^{p-1}\delta_{\nu m+1} \neq qd_1^{q-1}\delta_{\nu m+1}$ for every $p \neq q$, so $N_m^p(X) \neq N_m^q(X)$. ◊

Now we are in a position to prove the main result of this section. We begin with the simple case when $G = E_m$ for some integer $m \geq 2$. We prove

Proposition 2. *A family $F(t,X)_{t\in E_m}$, $F(t,X) = \sum_{k=1}^{s} c_k(t)X^k$, $c_1 : : E_m \to \mathbb{K} \setminus \{0\}$, $c_k : E_m \to \mathbb{K}$ for $k \in |2,s|$, is a solution of the translation equation*

(24) $$F_{z_1\cdot z_2}(X) = (F_{z_1} \circ F_{z_2})(X) \quad \textit{for } z_1, z_2 \in E_m,$$

such that c_1 is a multiplicative function with $\operatorname{im} c_1 = E_m$ if and only if there exists a power series $U(X) \in \Gamma_1^s$ such that

(25) $$F_z(X) = \left(U^{-1} \circ L_{c_1(z)} \circ U\right)(X) \quad \textit{for every } z \in E_m.$$

Proof. Clearly, the family $(F(z,X))_{z\in E_m}$ defined by (25) is a solution of the translation equation (24).

Now, let $F(t,X)_{t\in E_m}$, $F(t,X) = \sum_{k=1}^{s} c_k(t)X^k$, $c_1 : E_m \to \mathbb{K}\setminus\{0\}$, $c_k : E_m \to \mathbb{K}$ for $k \in |2,s|$, be a solution of (24). Clearly $c_1(z_0)$, where $z_0 = e^{\frac{2\pi}{m}i}$, is a primitive root of the unit of order m. Then, from Lemma 6, for $F(z_0,X) = c_1(z_0)X + \sum_{k=2}^{s} c_k(z_0)X^k$ there exists a formal power series $U(x) = X + \sum_{k=2}^{s} u_kX^k \in \Gamma_1^s$ such that

$$(U \circ F_{z_0} \circ U^{-1})(X) = c_1(z_0)X + \sum_{l=1}^{r} \delta_{lm+1}X^{lm+1}$$

with some $(\delta_{lm+1})_{l\in\mathbb{N}}$, where r is the greatest integer such that $rm+1 \leq s$ if $s < \infty$ and $r = \infty$ otherwise. We will show that $\delta_{lm+1} = 0$ for every

$l \in |1, r|$. If not then, on account of Lemma 7, for $p, q \in \mathbb{N}$, $p \neq q$, we obtain

$$(U \circ F_{z_0} \circ U^{-1})^p(X) \neq (U \circ F_{z_0} \circ U^{-1})^q(X),$$

or, equivalently, $(F_{z_0})^p(X) \neq (F_{z_0})^q(X)$. In particular, we have

$$F_{z_0}(X) = F_{z_0^{m+1}}(X) = (F_{z_0})^{m+1}(X) \neq (F_{z_0})^1(X) = F_{z_0}(X).$$

This contradiction proves that $\delta_{lm+1} = 0$ for every $l \in |1, r|$. Thus we have

$$\left(U \circ F_{z_0} \circ U^{-1}\right)(X) = c_1(z_0)X.$$

Then, for arbitrary $E_m \ni z = e^{\frac{2\pi i k}{m}}$ with $0 \leq k \leq m-1$, we have $z = z_0^k$, and

$$\left(U \circ F_z \circ U^{-1}\right)(X) = \left(U \circ F_{z_0^k} \circ U^{-1}\right)(X) = \left(U \circ (F_{z_0})^k \circ U^{-1}\right)(X)$$

$$= \left(U \circ F_{z_0} \circ U^{-1}\right)^k(X) = (c_1(z_0))^k X = c_1(z_0^k)X = c_1(z)X = L_{c_1(z)}(X),$$

which means that (25) holds. $\Diamond$

Theorem 5. *Let $(G, +)$ be an abelian group. To each $F(t, X)_{t \in G}$, $F(t, X) = \sum_{k=1}^{s} c_k(t)X^k$, $c_1 : G \to \mathbb{K} \setminus \{0\}$, $c_k : G \to \mathbb{K}$ for $k \in$ $\in |2, s|$, being a solution of the translation equation* (4) *such that the set $\{F(t, X) : t \in G\}$ is finite, there exists a power series $U(X) \in \Gamma_1^s$ such that*

$$F_t(X) = \left(U^{-1} \circ L_{c_1(t)} \circ U\right)(X) \quad \text{for every } t \in G. \tag{26}$$

Conversely, a family $F(t, X)_{t \in G}$ defined by (26) *is a solution of the translation equation* (4).

Proof. Assume that $F(t, X)_{t \in G}$, $F(t, X) = \sum_{k=1}^{s} c_k(t)X^k$, $c_1 : G \to$ $\to \mathbb{K} \setminus \{0\}$, $c_k : G \to \mathbb{K}$ for $k \in |2, s|$, is a solution of the translation equation (4) such that the set $\{F(t, X) : t \in G\}$ is finite. We know that c_1 is a generalized exponential function, i.e. it is a homomorphism. Then, for the homomorphism $\Theta_G : G \to \Gamma^s$, $\Theta_G(t)(X) = F(t, X)$, on account of Lemma 5, $\ker c_1 = \ker \Theta_G$. Thus, by the first isomorphism theorem (cf. [6, p. 16]), $\{F(t, X) : t \in G\} = \operatorname{im} \Theta_G \cong G/\ker \Theta_G = G/$ $/\ker c_1 \cong \operatorname{im} c_1$ (which means that also $\operatorname{im} c_1$ is finite). So assume that $\operatorname{card}\{F(t, X) : t \in G\} = \operatorname{card} \operatorname{im} \Theta_G = \operatorname{card} \operatorname{im} c_1 =: m$ with a positive integer m. This means that $\operatorname{im} c_1 \cong E_m$ and there exists a canonical homomorphism $\kappa : G \to G/\ker c_1 \cong E_m$. Then the homomorphism Θ_G must be of the form $\Theta_G = \Theta_{E_m} \circ \kappa$, where $\Theta_{E_m} : E_m \to \Gamma^s$ and $\kappa :$ $: G \to E_m$ are homomorphisms such that $\Theta_{E_m}(t)(X) = \sum_{k=1}^{s} \overline{c}_k(t)X^k$, $\overline{c}_1 : E_m \to E_m$ is a multiplicative function such that $\operatorname{im} \overline{c}_1 = E_m$, $\overline{c}_1 \circ \kappa =$ $= c_1$, and $\overline{c}_k : E_m \to \mathbb{K}$ for $k \in |2, s|$. Hence $F(t, X) = \overline{F}(\kappa(t), X)$, where $\overline{F}(z, X) = \Theta_{E_m}(t)(X)$.

If $m = 1$ then clearly $c_1 = 1$, $F(t, X) = X$ for every $t \in G$, so with every $U(X) \in \Gamma_1^s$ we have $(U \circ F_t \circ U^{-1})(X)) = (U \circ U^{-1})(X) = X$ for $t \in G$. Thus assume that $m \geq 2$. Then, from Prop. 2, we get
$(U \circ F_t \circ U^{-1})(X) = (U \circ \overline{F}_{\kappa(t)} \circ U^{-1})(X) = \overline{c}_1(\kappa(t))X = c_1(t)X = L_{c_1(t)}(X)$, which completes the proof. ◊

Remark 3. Note that a power series $U(X) = X + \sum_{k=2}^{s} u_k X^k$, which determines a particular solution $(F(t, X))_{t \in G}$ of the translation equation (4) in the case considered here is not unique. This comes from the fact that the solution $(u_k)_{k \in |2,s|}$ of the system (23) is not unique. Moreover, if a power series $U(X) \in \Gamma_1^s$ determines a solution of the translation equation (4), then also any power series $W(X) \in \Gamma_1^s$, $W(X) = (V \circ U)(X)$ determines the same solution, where $V(X) = X + \sum_{l=1}^{r} v_{lm+1} X^{lm+1}$ with arbitrary sequence $(v_{lm+1})_{l \in |1,r|}$, where $r = \infty$ if $s = \infty$ and r is the greatest integer such that $rm + 1 \leq s$ provided s is finite. Indeed,
$(U^{-1} \circ L_{c_1(t)} \circ U)(X) = \Theta_G(t)(X) = (W^{-1} \circ L_{c_1(t)} \circ W)(X)$ for $t \in G$,
then $c_1(t)X = ((W \circ U^{-1})^{-1} \circ L_{c_1(t)} \circ (W \circ U^{-1}))(X)$, so with $V = W \circ$ $\circ\, U^{-1}$ we have $(V^{-1} \circ L_{c_1(t)} \circ V)(X) = c_1(t)X$, or, which is the same $V(c_1(t)X) = c_1(t)V(X)$ for each $t \in G$. Since $U(X), W(X) \in \Gamma_1^s$, so also $V(X) \in \Gamma_1^s$. Put $V(X) = X + \sum_{k=2}^{s} v_k X^k$. Then we get

$$c_1(t)X + \sum_{k=1}^{s} v_k c_1(t)^k X^k = c_1(t)X + \sum_{k=1}^{s} c_1(t) v_k X^k,$$

and hence $v_k(c_1(t)^k - c_1(t)) = 0$ for $t \in G$ and $k \in |2, s|$. Using the fact that $\operatorname{im} c_1 = E_m$, we get $v_k = 0$ if $k \not\equiv 1 \mod m$ and v_{lm+1} is arbitrary for $l \in |1, r|$. Thus we have $W(X) = (V \circ U)(X)$, where $V(X) = X +$ $+ \sum_{l=1}^{r} v_{lm+1} X^{lm+1}$.

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