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ON THE CONCIRCULAR CURVATURE TENSOR OF AN $N(k)$-QUASI EINSTEIN MANIFOLD

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Abstract: It is proved that an $N(k)$-quasi Einstein manifold cannot satisfy the derivation conditions $\mathcal{Z}(\xi, X) \cdot \mathcal{Z} = 0$, $\mathcal{Z}(\xi, X) \cdot \mathcal{R} = 0$ or $\mathcal{Z}(\xi, X) \cdot S = 0$, where $\mathcal{Z}, \mathcal{R}$ and S denote the concircular curvature tensor, Riemannian curvature tensor and Ricci tensor, respectively. A necessary and sufficient condition for an $N(k)$-quasi Einstein manifold to satisfy $\mathcal{R}(\xi, X) \cdot \mathcal{Z} = 0$ is also obtained.

1. Introduction

A non-flat n-dimensional Riemannian manifold (M, g) is said to be a quasi Einstein manifold [2] if its Ricci tensor S satisfies

$$S(X,Y) = a\, g(X,Y) + b\, \eta(X)\, \eta(Y), \qquad X, Y \in TM$$

for some smooth functions a and $b \neq 0$, where η is a nonzero 1-form such that

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$$g(X,\xi) = \eta(X), \qquad g(\xi,\xi) = \eta(\xi) = 1$$

for the associated vector field ξ. The 1-form η is called the associated 1-form and the unit vector field ξ is called the generator of the manifold. If the generator ξ belongs to some k-nullity distribution $N(k)$ then the quasi Einstein manifold is called an $N(k)$-quasi Einstein manifold [5]. In [5], it is shown that an n-dimensional conformally flat quasi Einstein manifold is an $N\left(\frac{a+b}{n-1}\right)$-quasi Einstein manifold and in particular a 3-dimensional quasi Einstein manifold is an $N(\frac{a+b}{2})$-quasi Einstein manifold. The derivation conditions $\mathcal{R}(\xi,X)\cdot\mathcal{R} = 0$ and $\mathcal{R}(\xi,X)\cdot S = 0$ are also studied [5], where $\mathcal{R}$ is the curvature tensor.

On the other hand in [1], the derivation conditions $\mathcal{Z}(\xi,X)\cdot\mathcal{Z} = 0$, $\mathcal{Z}(\xi,X)\cdot\mathcal{R} = 0$ and $\mathcal{R}(\xi,X)\cdot\mathcal{Z} = 0$ on contact metric manifolds are studied, where $\mathcal{Z}$ is the concircular curvature tensor. In [4], the condition $\mathcal{Z}(\xi,X)\cdot S = 0$ is studied. In this paper, we study the derivation conditions $\mathcal{Z}(\xi,X)\cdot\mathcal{Z} = 0$, $\mathcal{Z}(\xi,X)\cdot\mathcal{R} = 0$, $\mathcal{R}(\xi,X)\cdot\mathcal{Z} = 0$ and $\mathcal{Z}(\xi,X)\cdot S = 0$ on an $N(k)$-quasi Einstein manifold. The paper is organized as follows. Sec. 2 contains necessary details about $N(k)$-quasi Einstein manifolds and the concircular curvature tensor. It is proved that in an n-dimensional $N(k)$-quasi Einstein manifold $k = \frac{a+b}{n-1}$. In Sec. 3, it is shown that in an $N(k)$-quasi Einstein manifold the conditions $\mathcal{Z}(\xi,X)\cdot S = 0$, $\mathcal{Z}(\xi,X)\cdot\mathcal{Z} = 0$ or $\mathcal{Z}(\xi,X)\cdot\mathcal{R} = 0$ are not possible. In the last, it is proved that an $N(k)$-quasi Einstein manifold satisfies the condition $\mathcal{R}(\xi,X)\cdot\mathcal{Z} = 0$ if and only if $a+b = 0$.

2. $N(k)$-quasi Einstein manifolds

A non-flat n-dimensional Riemannian manifold (M,g) is said to be a quasi Einstein manifold [2] if its Ricci tensor S satisfies

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y), \qquad X,Y \in TM \tag{2.1}$$

or equivalently, its Ricci operator Q satisfies

$$Q = aI + b\eta\otimes\xi \tag{2.2}$$

for some smooth functions a and $b \neq 0$, where η is a nonzero 1-form such that

$$g(X,\xi) = \eta(X), \qquad g(\xi,\xi) = \eta(\xi) = 1 \tag{2.3}$$

for the associated vector field ξ. The 1-form η is called the associated 1-form and the unit vector field ξ is called the generator of the manifold.

From (2.2) and (2.3) it follows that

$$S(X,\xi) = (a+b)\,\eta(X), \tag{2.4}$$

$$r = na + b, \tag{2.5}$$

where r is the scalar curvature of M.

The k-nullity distribution $N(k)$ [3] of a Riemannian manifold M is defined by

$$N(k): p \rightarrow N_p(k) = \{U \in T_pM \mid \mathcal{R}(X,Y)U = k(g(Y,U)X - g(X,U)Y)\}$$

for all $X, Y \in TM$, where k is some smooth function. In a quasi Einstein manifold M if the generator ξ belongs to some k-nullity distribution $N(k)$, then M is said to be an $N(k)$-quasi Einstein manifold [5]. In fact, k is not arbitrary as we see in the following

Lemma 2.1. *In an n-dimensional $N(k)$-quasi Einstein manifold it follows that*

$$k = \frac{a+b}{n-1}. \tag{2.6}$$

Proof. Since

$$\mathcal{R}(X,Y)\,\xi = k\,\{\eta(Y)\,X - \eta(X)\,Y\},$$

therefore we have

$$\mathcal{R}(X,Y,\xi,W) = k\,\{\eta(Y)\,g(X,W) - \eta(X)\,g(Y,W)\}.$$

From the above equation we obtain

$$S(Y,\xi) = k\,(n-1)\,\eta(Y),$$

which in view of (2.4) gives (2.6). $\Diamond$

Now, it is immediate to note that in an n-dimensional $N(k)$-quasi Einstein manifold

$$\mathcal{R}(X,Y)\,\xi = \frac{a+b}{n-1}\,\{\eta(Y)\,X - \eta(X)\,Y\}, \tag{2.7}$$

which is equivalent to

$$\mathcal{R}(X,\xi)\,Y = \frac{a+b}{n-1}\,\{\eta(Y)\,X - g(X,Y)\,\xi\} = -\mathcal{R}(\xi,X)\,Y. \tag{2.8}$$

From (2.7) we get

$$\mathcal{R}(\xi,X)\,\xi = \frac{a+b}{n-1}\,\{\eta(X)\,\xi - X\}. \tag{2.9}$$

The *concircular curvature tensor* $\mathcal{Z}$ in an n-dimensional Riemannian manifold (M,g) is defined by ([6], [7])

$$(2.10)\quad \mathcal{Z}(X,Y)W=\mathcal{R}(X,Y)W-\frac{r}{n(n-1)}\{g(Y,W)X-g(X,W)Y\}$$

for all $X,Y,W\in TM$, where r is the scalar curvature of M. The equation (2.10) implies that a Riemannian manifold with vanishing concircular curvature tensor is of constant curvature. Thus the concircular curvature tensor can be thought as a measure of the failure of a Riemannian manifold to be of constant curvature. Also a necessary and sufficient condition that a Riemannian manifold be reducible to a Euclidean space by a suitable concircular transformation is that its concircular curvature tensor vanishes.

Now, we prove the following proposition for later use.

Proposition 2.2. *In an n-dimensional $N(k)$-quasi Einstein manifold, the concircular curvature tensor $\mathcal{Z}$ satisfies*

$$(2.11)\qquad \mathcal{Z}(X,Y)\xi=\frac{b}{n}\{\eta(Y)X-\eta(X)Y\},$$

$$(2.12)\qquad \mathcal{Z}(\xi,X)Y=\frac{b}{n}\{g(X,Y)\xi-\eta(Y)X\}.$$

Consequently, we have

$$(2.13)\qquad \mathcal{Z}(\xi,X)\xi=\frac{b}{n}\{\eta(X)\xi-X\},$$

$$(2.14)\qquad \eta(\mathcal{Z}(X,Y)\xi)=0,$$

$$(2.15)\qquad \eta(\mathcal{Z}(\xi,X)Y)=\frac{b}{n}\{g(X,Y)-\eta(X)\eta\ (Y)\}.$$

Proof. From (2.5), (2.10), (2.7) and (2.8) the equations (2.11) and (2.12) follow easily. ◊

3. Main results

In [5], it is proved that an $N(k)$-quasi Einstein manifold satisfies $\mathcal{R}(\xi,X)\cdot S=0$ if and only if $k=0$. Here, we prove the following theorem.

Theorem 3.1. *There is no $N(k)$-quasi Einstein manifold satisfying $\mathcal{Z}\cdot S=0$.*

Proof. Let M be an n-dimensional $N(k)$-quasi Einstein manifold. From (2.4) and (2.15) we get

$$S(\mathcal{Z}(\xi, X) Y, \xi) = \frac{(a+b) b}{n} \{g(X, Y) - \eta(X) \eta(Y)\}. \tag{3.1}$$

Similarly from (2.13) and (2.4) we obtain

$$S(\mathcal{Z}(\xi, X) \xi, Y) = \frac{(a+b) b}{n} \eta(X) \eta(Y) - \frac{b}{n} S(X, Y). \tag{3.2}$$

If $\mathcal{Z}(\xi, X) \cdot S = 0$ then

$$S(\mathcal{Z}(\xi, X) Y, \xi) + S(Y, \mathcal{Z}(\xi, X) \xi) = 0,$$

which in view of (3.1) and (3.2) gives

$$0 = \frac{b}{n} \{S - (a+b) g\}. \tag{3.3}$$

Since M is not Einstein, the above equation gives a contradiction. ◊

Next, we have the following theorem.

Theorem 3.2. *There is no $N(k)$-quasi Einstein manifold M satisfying*

$$\mathcal{Z}(\xi, X) \cdot \mathcal{Z} = 0.$$

Proof. From the condition $\mathcal{Z}(\xi, U) \cdot \mathcal{Z} = 0$, we get

$$0 = [\mathcal{Z}(\xi, U), \mathcal{Z}(X, Y)] \xi - \mathcal{Z}(\mathcal{Z}(\xi, U) X, Y)\xi - \mathcal{Z}(X, \mathcal{Z}(\xi, U) Y)\xi,$$

which in view of (2.12) gives

$$\begin{aligned} 0 = \frac{b}{n}\big\{ & g(U, \mathcal{Z}(X, Y)\xi)\xi - \eta(\mathcal{Z}(X, Y)\xi)U - \\ & - g(U, X)\mathcal{Z}(\xi, Y)\xi + \eta(X) \mathcal{Z}(U, Y)\xi - g(U, Y)\mathcal{Z}(X, \xi)\xi + \\ & + \eta(Y) \mathcal{Z}(X, U)\xi - \eta(U)\mathcal{Z}(X, Y)\xi + \mathcal{Z}(X, Y)U\big\}. \end{aligned}$$

Equation (2.11) then gives

$$\frac{b}{n}\left(\mathcal{Z}(X, Y) U - \frac{b}{n} \{g(Y, U) X - g(X, U) Y\}\right) = 0.$$

Since $b \neq 0$, therefore from the above equation it follows that M is of constant curvature. But in this case M is an Einstein manifold, which is a contradiction. ◊

Using the fact that $\mathcal{Z}(\xi, X) \cdot \mathcal{R}$ denotes $\mathcal{Z}(\xi, X)$ acting on $\mathcal{R}$ as a derivation, we have the following theorem as a corollary of Th. 3.2.

Corollary 3.3. *There is no $N(k)$-quasi Einstein manifold M satisfying*

$$\mathcal{Z}(\xi, X) \cdot \mathcal{R} = 0.$$

On the other hand reversing the order of $\mathcal{Z}$ and $\mathcal{R}$ gives the following result.

Theorem 3.4. *An n-dimensional $N(k)$-quasi Einstein manifold M satisfies*

$$\mathcal{R}(\xi, X) \cdot \mathcal{Z} = 0$$

if and only if $a + b = 0$.

Proof. The condition $\mathcal{R}(\xi, U) \cdot \mathcal{Z} = 0$ implies that

$$0 = [\mathcal{R}(\xi, U), \mathcal{Z}(X, Y)]\,\xi - \mathcal{Z}(\mathcal{R}(\xi, U)X, Y)\,\xi - \mathcal{Z}(X, \mathcal{R}(\xi, U)Y)\,\xi,$$

which in view of (2.8) gives

$$\begin{aligned} 0 = \frac{a+b}{n-1}\{&g(U, \mathcal{Z}(X,Y)\xi)\xi - \eta(\mathcal{Z}(X,Y)\xi)U - \\ &- g(U,X)\mathcal{Z}(\xi,Y)\xi + \eta(X)\mathcal{Z}(U,Y)\xi - g(U,Y)\mathcal{Z}(X,\xi)\xi + \\ &+ \eta(Y)\mathcal{Z}(X,U)\xi - \eta(U)\mathcal{Z}(X,Y)\xi + \mathcal{Z}(X,Y)U\}. \end{aligned}$$

In view of (2.11) the previous equation yields

$$\frac{a+b}{n-1}\left(\mathcal{Z}(X,Y)U - \frac{b}{n}\{g(Y,U)X - g(X,U)Y\}\right) = 0.$$

Since M cannot be of constant curvature therefore $a + b = 0$. The converse statement is trivial. $\Diamond$

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