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ON THE MOMENTS OF SUMS OF INDEPENDENT IDENTICALLY DISTRIBUTED RANDOM VARIABLES

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Abstract: A general theorem is obtained for the moments of sums of independent identically distributed Banach space valued random variables. Then it is applied to prove an almost sure limit theorem for variables being in the domain of attraction of a stable law.

1. Introduction

In [3] an almost sure limit theorem is presented for random variables from the domain of geometric partial attraction of semistable laws (Th. 1 of [3]). The proof is partially based on a lemma concerning the moments of sums of independent identically distributed random variables (Lemma 1 of [3]). On the other hand, in [5] an almost sure limit theorem is obtained for a stochastic process converging to a stable law (Prop. 3.1 in [5]). However, in [5] the proof is based on an other method. In this paper we shall show that an appropriate version of Lemma 1 of [3] can be used in the proof of Prop. 3.1 of [5].

The main result of this paper is Th. 2.1. It provides sufficient conditions for the boundedness of moments of normalized sums of inde-

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pendent identically distributed Banach space valued random variables. It is a generalization of Lemma 1 of [3]. The main steps of proof are included in the proof of Th. 6.1 in [1]. (Actually, Th. 6.1 in [1] concerns variables being in the domain of attraction of a stable law.) Our Th. 3.1 is the same as Prop. 3.1 in [5]. Here we present a new proof based on Th. 2.1. In the proof a part of our calculation is similar to the one given in Lemma 6.1 of [1]. In this paper we use some basic facts from the theory of Banach space valued random variables (we refer to the papers [1], [6] and [9]). Recently several papers are devoted to the study of almost sure limit theorems (see [2], [3], [5], [7], [10] and the references therein).

2. The main result

Let B be a real separable Banach space with norm $\|.\|$. We suppose that B is equipped with its Borel σ-field $\mathcal{B}$. Our main result is the following theorem.

Theorem 2.1. *Let $\xi_1, \xi_2, \ldots$ be independent identically distributed B-valued random variables, $S_n = \xi_1 + \cdots + \xi_n$, $n = 1, 2, \ldots$. Let $a_1, a_2, \ldots$ be an increasing sequence of positive real numbers. Let $\alpha \in (0, 2]$ be fixed. Assume that*

$$\frac{a_{nm}}{a_n} \le C m^{1/\alpha + \tau(n)}, \quad n, m = 1, 2, \ldots \tag{2.1}$$

where $\tau(n)$ is a sequence of nonnegative numbers with $\lim_{n\to\infty} \tau(n) = 0$. Assume that for any $\beta \in (0, \alpha)$

$$\mathbb{E}\|\xi_n\|^\beta < \infty. \tag{2.2}$$

Let $\{a_{l_n}\}$ be a subsequence of $\{a_n\}$ so that for some $c < \infty$, $a_{l_n} \le c a_{l_{n-1}}$, $n = 1, 2, \ldots$. Let $b_1, b_2, \ldots$ be a B-valued sequence. Assume that

$$\left\{ \frac{S_{l_n}}{a_{l_n}} - b_{l_n}, \quad n = 1, 2, \ldots \right\} \quad \text{is stochastically bounded.} \tag{2.3}$$

Then, for any $\beta \in (0, \alpha)$,

$$\sup_n \mathbb{E} \left\| \frac{S_{l_n}}{a_{l_n}} - b_{l_n} \right\|^\beta < \infty. \tag{2.4}$$

Proof. Let $S_0 = 0$. Let $\xi'_1, \xi'_2, \ldots$ be an independent copy of the sequence $\xi_1, \xi_2, \ldots$. Then $\tilde{\xi}_n = \xi_n - \xi'_n, n = 1, 2, \ldots$ is the symmetrization of $\xi_n, n=1,2,\ldots$. Moreover, let $S'_n = \xi'_1 + \cdots + \xi'_n, n=1,2,\ldots$ ($S'_0=0$). Then $\tilde{S}_n = S_n - S'_n, n=1,2,\ldots$ is the symmetrization $S_n, n=1,2,\ldots$.

Let r be an arbitrary positive integer, $l_{n-1} < r \le l_n$. Then, for any $d>0$,

$$\mathbb{P}\left(\frac{\|\tilde{S}_r\|}{a_r} > d\right) \le \mathbb{P}\left(\max_{1\le i\le l_n} \|\tilde{S}_i\| > da_r\right) \le \mathbb{P}\left(\max_{1\le i\le l_n} \|\tilde{S}_i\| > \frac{d}{c}a_{l_n}\right) \le$$

$$\le 2\mathbb{P}\left(\|\tilde{S}_{l_n}\| > \frac{d}{c}a_{l_n}\right) \le 4\mathbb{P}\left(\|S_{l_n} - b_{l_n}a_{l_n}\| > \frac{1}{2}\frac{d}{c}a_{l_n}\right) =$$

$$= 4\mathbb{P}\left(\left\|\frac{S_{l_n}}{a_{l_n}} - b_{l_n}\right\| > \frac{1}{2}\frac{d}{c}\right).$$

Here we used the Lévy inequality (see Hoffmann-Jørgensen [9]), the symmetrization inequality, and the properties of $\{a_{l_n}\}$. So we obtain that (2.3) implies that the sequence $\frac{\tilde{S}_n}{a_n}$ is stochastically bounded. That is for any $\varepsilon>0$ there exists a $d>0$ such that for all $r\in\mathbb{N}$ we have

$$\mathbb{P}\left(\frac{\|\tilde{S}_r\|}{a_r} \ge \frac{d}{2}\right) \le \varepsilon. \tag{2.5}$$

The random variables

$$\tilde{S}_{nk} - \tilde{S}_{n(k-1)} = \tilde{\xi}_{n(k-1)+1} + \cdots + \tilde{\xi}_{nk}, \quad n=1,2,\ldots,m,$$

are independent and identically distributed thus

$$\left[\mathbb{P}\left(\frac{\|\tilde{S}_n\|}{a_{nm}} < d\right)\right]^m = \mathbb{P}\left(\frac{\max_{1\le k\le m}\|\tilde{S}_{nk} - \tilde{S}_{n(k-1)}\|}{a_{nm}} < d\right) =$$

$$= 1 - \mathbb{P}\left(\frac{\max_{1\le k\le m}\|\tilde{S}_{nk} - \tilde{S}_{n(k-1)}\|}{a_{nm}} \ge d\right) \ge$$

$$\ge 1 - \mathbb{P}\left(\frac{\max_{1\le k\le m}\|\tilde{S}_{nk}\|}{a_{nm}} \ge \frac{d}{2}\right) \ge 1 - 2\mathbb{P}\left(\frac{\|\tilde{S}_{nm}\|}{a_{nm}} \ge \frac{d}{2}\right) \ge 1-2\varepsilon.$$

Here we applied the Lévy inequality. Using the mean value theorem, we see that $1-(1-2\varepsilon)^{\frac{1}{m}} \le H(\varepsilon)\frac{1}{m}$, where $H(\varepsilon)\to 0$ as $\varepsilon\to 0$ ($\varepsilon>0$).

Consequently, the above inequality gives

$$\mathbb{P}\left(\frac{\|\tilde{S}_n\|}{a_{mn}} \ge d\right) \le 1-(1-2\varepsilon)^{\frac{1}{m}} \le H(\varepsilon)\frac{1}{m}$$

where $H(\varepsilon)$ depends only on ε. So

$$H(\varepsilon)\frac{1}{m} \geq \mathbb{P}\left(\frac{\|\tilde{S}_n\|}{a_n} \geq d\frac{a_{mn}}{a_n}\right).$$

Then (2.1) implies

$$H(\varepsilon)\frac{1}{m} \geq \mathbb{P}\left(\frac{\|\tilde{S}_n\|}{a_n} \geq dCm^{\frac{1}{\alpha}+\tau(n)}\right).$$

Substitute $dCm^{\frac{1}{\alpha}+\tau(n)}$ by t. Then we get that $m = (\frac{t}{dC})^{\frac{1}{\frac{1}{\alpha}+\tau(n)}} = (\frac{1}{dC})^{\frac{1}{\frac{1}{\alpha}+\tau(n)}} t^{\alpha-\delta}$, where $\delta > 0, \delta \to 0$, if $n \to \infty$. Then we can write

$$H(\varepsilon)(dC)^{\frac{1}{\frac{1}{\alpha}+\tau(n)}} \geq t^{\alpha-\delta}\mathbb{P}\left(\frac{\|\tilde{S}_n\|}{a_n} \geq t\right). \tag{2.6}$$

We explain relation (2.6). Here C and α are fixed, $\tau(n) > 0$, $\tau(n) \to 0$. For $\varepsilon > 0$ the value of d is chosen so that (2.6) is satisfied. Then $H(\varepsilon)$ depends on ε and $\lim_{\varepsilon\to 0} H(\varepsilon) = 0$. So the left side is bounded. As $t = dCm^{\frac{1}{\alpha}+\tau(n)}$, we see that (2.6) is valid for $t > t_0$, where $t_0 > 0$ is large enough. (To this end we have to choose m to be large.)

So for all $t > t_0$ the following is valid: for each $\delta > 0$ there exists an n_δ so that if $n > n_\delta$ then

$$A \geq t^{\alpha-\delta}\mathbb{P}\left(\frac{\|\tilde{S}_n\|}{a_n} \geq t\right). \tag{2.7}$$

Therefore for each fixed (small) $\delta > 0$ for $n > n_\delta$ we have

$$\begin{aligned}
\mathbb{E}\left(\frac{\|\tilde{S}_n\|}{a_n}\right)^{\alpha-2\delta} &= \int_0^\infty (\alpha-2\delta)t^{\alpha-2\delta-1}\mathbb{P}\left(\frac{\|\tilde{S}_n\|}{a_n} \geq t\right) dt \leq \\
&\leq (\alpha-2\delta)\int_0^{t_0} t^{\alpha-2\delta-1} dt + (\alpha-2\delta)\int_{t_0}^\infty t^{\alpha-2\delta-1}\mathbb{P}\left(\frac{\|\tilde{S}_n\|}{a_n} \geq t\right) dt \leq \\
&\leq (\alpha-2\delta)\int_0^{t_0} t^{\alpha-2\delta-1} dt + (\alpha-2\delta)\int_{t_0}^\infty At^{-\delta-1} dt = \\
&= (\alpha-2\delta)\frac{t_0^{\alpha-2\delta}}{\alpha-2\delta} + (\alpha-2\delta)A\frac{t_0^{-\delta}}{\delta} = t_0^{\alpha-2\delta} + (\alpha-2\delta)A\frac{t_0^{-\delta}}{\delta}.
\end{aligned} \tag{2.8}$$

By (2.2) we have

$$\max_{n \le n_\delta} \mathbb{E}\left(\frac{\|\tilde{S}_n\|}{a_n}\right)^{\alpha-2\delta} < \infty.$$

This and (2.8) give that for $\delta > 0$ (δ is small)

$$\sup_{n\in\mathbb{N}} \mathbb{E}\left(\frac{\|\tilde{S}_n\|}{a_n}\right)^{\alpha-2\delta} < \infty. \tag{2.9}$$

For the desymmetrization procedure we use the known inequality

$$\mathbb{P}\left(\|\tilde{X}\| > \frac{t}{2}\right) \ge \mathbb{P}(\|X-a\| > t)\,\mathbb{P}\left(\|X'-a\| > \frac{t}{2}\right)$$

for all $t \ge 0$. Now we have

$$\mathbb{P}\left(\frac{\|\tilde{S}_{l_n}\|}{a_{l_n}} \ge \frac{t}{2}\right) \ge \mathbb{P}\left(\left\|\frac{S_{l_n}}{a_{l_n}} - b_{l_n}\right\| \ge t\right)\mathbb{P}\left(\left\|\frac{S'_{l_n}}{a_{l_n}} - b_{l_n}\right\| \le \frac{t}{2}\right).$$

(2.3) implies that $\mathbb{P}\left(\left\|\frac{S'_{l_n}}{a_{l_n}} - b_{l_n}\right\| \le \frac{t}{2}\right) > \frac{1}{2}$ for all n if t is large enough. Applying again the formula $\mathbb{E}\|X_n\|^s = \int_0^\infty s u^{s-1}\mathbb{P}(\|X\| \ge u)du$ and (2.9), we obtain (2.4). ◊

3. An application in the almost sure limit theory

Here we present an application of Th. 2.1 for proving an almost sure limit theorem. We give a new proof of Prop. 3.1 of [5]. The result states convergence to p-stable limit.

Let $V(t)$, $t \ge 0$, be a random process with independent stationary increments. Assume that $V(0) = 0$, $\{V(t,\omega) : t \ge 0, \omega \in \Omega\}$ is measurable, and the trajectories of $V(t)$ are right continuous and have left limit. For each infinitely divisible distribution F, there exists such a process $V(t)$ so that $V(1)$ has distribution F (see [12]). Therefore $V(t)$ has the following characteristic function

$$\varphi_{V(t)}(x) = \mathbb{E}\left(e^{ixV(t)}\right) = \psi\left(t, x, b, \sigma^2, L(y), R(y)\right) =$$

$$= \exp\left(t\left\{ibx - \frac{\sigma^2}{2}x^2 + \int_{-\infty}^0 \left(e^{ixy} - 1 - \frac{ixy}{1+y^2}\right)dL(y) + \right.\right.$$
$$\left.\left. + \int_0^\infty \left(e^{ixy} - 1 - \frac{ixy}{1+y^2}\right)dR(y)\right\}\right), \tag{3.1}$$

$x \in \mathbb{R}$ (Lévy's formula, see [8], Sect. 18). Here $L(y)$ is (left-continuous and) non-decreasing on $(-\infty, 0)$ with $L(-\infty) = 0$, $R(y)$ is (right-continuous and) non-decreasing on $(0, \infty)$ with $R(\infty) = 0$ and they satisfy $\int_{-\varepsilon}^{0} y^2 dL(y) + \int_{0}^{\varepsilon} y^2 dR(y) < \infty$ for all $\varepsilon > 0$.

We will consider a random process having the form

$$X(t) = \frac{V(f(t))}{A(t)} - B(t), \quad 0 < t < \infty, \tag{3.2}$$

where $f : [0, \infty) \to [0, \infty)$ is a fixed strictly increasing function and $A : [0, \infty) \to (0, \infty)$ is a fixed positive function. Moreover, we will consider for $l < k$ the processes

$$X_{lk}(t) = \frac{V(f(t)) - V(f(l+1))}{A(t)} - B_l(t), \quad k \leq t < k+1. \tag{3.3}$$

Then, for $l < k$, $\{X(t) : l \leq t < l+1\}$ and $\{X_{lk}(t) : k \leq t < k+1\}$ are independent families.

We shall consider the process $V(t)$ with $b = 0$ and $\sigma = 0$, fix the functions f and $A(t)$, then choose the function $B(t)$ such that the characteristic function of $X(t)$ has the form

$$\begin{aligned} \varphi_{X(t)}(x) &= \psi\left(1, x, 0, 0, f(t)L(A(t)y), f(t)R(A(t)y)\right) = \\ &= \overline{\psi}\left(x, f(t)L(A(t)y), f(t)R(A(t)y)\right). \end{aligned} \tag{3.4}$$

Such choice is possible:

$$B(t) = \int_{-\infty}^{0} g(t,y) dL(y) + \int_{0}^{\infty} g(t,y) dR(y),$$

where $g(t,y) = \frac{f(t)}{A(t)} \frac{y^3}{(1+y^2)(1+y^2/A^2(t))} \left(1 - \frac{1}{A^2(t)}\right)$. Then we shall choose $B_l(t)$ such that the characteristic function of $\frac{A(t)}{A(l+1)}[X(t) - X_{lk}(t)]$ is

$$\begin{aligned} &\varphi_{\frac{A(t)}{A(l+1)}[X(t)-X_{lk}(t)]}(x) = \\ &= \overline{\psi}\left(x, f(l+1)L(A(l+1)y), f(l+1)R(A(l+1)y)\right). \end{aligned} \tag{3.5}$$

Such choice is possible:

$$B_l(t) = B(t) + \frac{A(l+1)}{A(t)} \left[\int_{-\infty}^{0} g(l,y) dL(y) + \int_{0}^{\infty} g(l,y) dR(y) \right].$$

In [5] a.s. limit theorems for some important classes of the above processes are proved.

Here we study distributions belonging to the domain of attraction of a stable law. We shall consider the case when $f(x) \equiv x$, so the characteristic function has the form

$$\varphi_{X(t)}(x) = \overline{\psi}\left(x, tL(A(t)y), tR(A(t)y)\right). \tag{3.6}$$

Moreover, the characteristic function of $\frac{A(t)}{A(l+1)}[X(t) - X_{lk}(t)]$ is

(3.7)

$$\varphi_{\frac{A(t)}{A(l+1)}[X(t)-X_{lk}(t)]}(x) = \overline{\psi}\left(x, (l+1)L(A(l+1)y), (l+1)R(A(l+1)y)\right).$$

Following [5], we shall study a process converging to a p-stable law and for this process we shall consider an integral analogue of the almost sure limit theorem.

First let $0 < p < 2$. Let $V(t)$ be a process with Lévy's representation (3.1) and with $L(y)$ and $R(y)$ satisfying

$$\frac{L(-t)}{|R(t)|} \to \frac{c_1}{c_2}, \quad \text{as } t \to \infty, \tag{3.8}$$

and

$$\frac{L(-t) + |R(t)|}{L(-tx) + |R(tx)|} \to x^p, \quad \text{as } t \to \infty, \tag{3.9}$$

for all $x > 0$, and for $c_1, c_2 \geq 0$ such that $c_1 + c_2 > 0$. We mention that by [8], Sect. 35, we have the following. If $F(x)$ is a distribution function such that for some $x_0 > 0$ we have $F(x) = L(x)$, $x < -x_0$ and $F(x) - 1 = R(x)$, $x > x_0$, then F belongs to the domain of attraction of the p-stable law having Lévy's representation $L_p(t) = c_1/|t|^p$, $R_p(t) = c_2/(-t^p)$, if and only if (3.8) and (3.9) are valid.

We mention that (3.8) and (3.9) imply that $1/L(-t)$ and $1/|R(t)|$ are regularly varying with exponent p, if $c_1 \neq 0$ and $c_2 \neq 0$, respectively. Here we shall consider the case $c_1 \neq 0$ (in the case $c_1 = 0$ but $c_2 \neq 0$ we should impose condition on R instead of L).

Let $A(t)$ be a positive increasing function such that

$$tL(-A(t)) \to c_1 > 0, \quad \text{as} \quad t \to \infty. \tag{3.10}$$

Relation (3.10) implies that $A(t)$ is the (asymptotic) inverse of $1/L(-t)$, therefore $A(t)$ is regularly varying with exponent $1/p$ (see [4], Th. 1.5.12).

In [5] it is shown that (3.8), (3.9) and (3.10) imply that $X(t) \xrightarrow{d} V$, as $t \to \infty$, where V is a (p-stable) random variable with characteristic function

$$\varphi_V(x) = \overline{\psi}\left(x, \frac{c_1}{|y|^p}, -\frac{c_2}{y^p}\right). \tag{3.11}$$

Now let $p = 2$. Consider a process $V(t)$ with Lévy's representation (3.1) and with $L(y)$ and $R(y)$ satisfying

$$\frac{t^2\left(L(-t) - R(t)\right)}{\int_{-t}^{0} x^2\,dL(x) + \int_0^t x^2\,dR(x)} \to 0, \text{ as } t \to \infty. \tag{3.12}$$

When $F(x)$ is a distribution function such that for some $x_0 > 0$ we have $F(x) = L(x)$ for $x < -x_0$, while $F(x) - 1 = R(x)$ for $x > x_0$, then F belongs to the domain of attraction of a Gaussian law if and only if (3.12) is valid (see [8], Sect. 35). Relation (3.12) implies that the function

$$G(t) = \int_{-t}^{0} x^2\,dL(x) + \int_0^t x^2\,dR(x) \tag{3.13}$$

is slowly varying (apply Th. 8.3.1 of [4]). Let $A(t)$ be a positive increasing function such that

$$t\left(\int_{-A(t)}^{0}\left(\frac{x}{A(t)}\right)^2 dL(x) + \int_0^{A(t)}\left(\frac{x}{A(t)}\right)^2 dR(x)\right) \to 1, \text{ as } t \to \infty. \tag{3.14}$$

Relation (3.14) implies that $A(t)$ is regularly varying with exponent $1/2$ (see [4], Th. 1.5.12).

It was pointed out in [5] that (3.12) and (3.14) imply the conditions of Th. 2 of [8], Sect. 19. Therefore $X(t)$ converges to the standard normal law as $t \to \infty$.

Let δ_x denote the unit mass at point x, μ_ξ the distribution of ξ, $\xrightarrow{w}$ the convergence in distribution.

Theorem 3.1. (Prop. 3.1 in [5].) *Let $X(t)$ be a process with characteristic function* (3.6). *If* $0 < p < 2$ *assume* (3.8), (3.9) *and* (3.10). *If* $p = 2$, *assume* (3.12) *and* (3.14). *Then*

$$\frac{1}{\log(T)}\int_1^T \delta_{X(t,\omega)}\frac{dt}{t} \xrightarrow{w} \mu_Z, \quad as\ T \to \infty,$$

for almost all $\omega \in \Omega$, where Z is p-stable, more precisely $Z \stackrel{d}{=} \gamma$ (γ denotes the standard normal random variable) for $p = 2$ and $Z \stackrel{d}{=} V$ for $p < 2$ (here V has characteristic function (3.11)).

Proof. We have to check the assumptions of the general a.s. limit theorem of [5] (see Prop. 4.1 in the Appendix). In [5] the validity of (4.3) is proved by using the tools of [8]. Moreover, characteristic functions are applied to check conditions (4.1). Here we shall show (4.1) with the help of Th. 2.1.

Let $k > l$ and let l be large enough. We shall show that for all $0 < \beta < p$

$$(3.15)\quad \mathbb{E}|X(t) - X_{lk}(t)|^\beta \le \left(C\frac{l}{t}\right)^{\beta/p'} \le \left(C\frac{l}{k}\right)^{\beta/p'}, \quad k \le t \le k+1$$

where p' is an arbitrary number with $p' > p$ and C does not depend on l, k and t. The details are the following. The characteristic function of the process $\frac{A(t)}{A(l+1)}[X(t) - X_{lk}(t)]$, $k \le t < k+1$, is of the form

$$\overline{\psi}\,(x, (l+1)L(A(l+1)y), (l+1)R(A(l+1)y)).$$

Therefore the distribution of $\frac{A(t)}{A(l+1)}[X(t) - X_{lk}(t)]$ is the same as that of $X(l+1) = S_{l+1}/A(l+1) - B_{l+1}$, where $S_{l+1} = \xi_1 + \cdots + \xi_{l+1}$, and $\xi_1, \xi_2, \ldots$ are i.i.d. with common characteristic function $\psi\,(1, x, b, 0, L(y), R(y))$. So it converges to a p-stable law. Therefore it is bounded in probability, so (2.3) is satisfied.

To prove (2.1), we can use that $A(t)$ is regularly varying with exponent $1/p$. So there exists a number $B > 0$ such that for all $x \ge B$ we have

$$A(t) = t^{\frac{1}{p}} \exp\left(\eta(t) + \int_B^t \frac{\varepsilon(x)}{x}dx\right)$$

where η is a bounded measurable function on $[B, \infty[$ such that $\eta(x) \to c$, $(|c| < \infty)$, $\varepsilon(x)$is a continuous function on $[B, \infty[$ such that $\varepsilon(x) \to 0$, as $x \to \infty$ (see Th. 1.2 of [13]). Then

$$\frac{A(mn)}{A(n)} = m^{\frac{1}{p}} \exp\left(\eta(mn) - \eta(n) + \int_n^{mn} \frac{\varepsilon(x)}{x}dx\right) \le$$
$$\le Cm^{\frac{1}{p}} \exp\left(\tau(n)\int_n^{mn} \frac{1}{x}dx\right) = Cm^{\frac{1}{p}+\tau(n)}$$

where $\tau(n) \to 0$, as $n \to \infty$.

Finally, to prove (2.2), we use that the infinitely divisible distribution F has finite moment of order p if and only if $\int_{]\infty,-1[} |x|^p dL(x) + \int_{]1,\infty[} x^p dR(x) < \infty$ (see Th. 8 of [11]).

Consider first the case $0 < p < 2$. As $c_1 \neq 0$, we know that $L(t)$ is regularly varying with exponent $-p$. Then

$$\int_{-\infty}^{-1} |x|^\beta dL(u) = \left[(|x|^\beta L(x)\right]_{-\infty}^{-1} - \int_{-\infty}^{-1} L(x) d|x|^\beta < \infty$$

if $\beta < p$. Therefore we see that $\mathbb{E}\|\xi_i\|^\beta < \infty$.

When $p = 2$, the function $G(t)$ in (3.13) is slowly varying. Then

$$\int_{-\infty}^{-1} |x|^\beta dL(x) + \int_1^\infty x^\beta dR(x) = \int_1^\infty x^\beta d(R(x) - L(-x)) =$$
$$= \int_1^\infty x^\beta \frac{1}{x^2} dG(x) < \infty$$

if $\beta < 2$.

Therefore, if $l < k \le t \le k+1$,

$$\mathbb{E}|X(t) - X_{lk}(t)|^\beta = \mathbb{E}\left|\frac{A(l+1)}{A(t)}\left(\frac{S_{l+1}}{A(l+1} - B(l+1)\right)\right|^\beta \le$$
$$\le C\left(\frac{A(l+1)}{A(t)}\right)^\beta \le C\left(\left(\frac{l+1}{t}\right)^{\frac{1}{p'}}\right)^\beta$$

for l large enough, where $p' > p$. Here we applied that A is regularly varying with exponent $1/p$. This implies (3.15). $\Diamond$

4. Appendix

For the sake of completeness we quote Th. 2.1 of [5].

Let $(\boldsymbol{B}, \rho)$ be a complete separable metric space, denote by $\mathcal{B}(\boldsymbol{B})$ the σ-algebra of the Borel sets of $\boldsymbol{B}$. Let $X(t), t \ge 0$, be a measurable random process with values in $\boldsymbol{B}$.

Proposition 4.1. *Assume that there exist $C < \infty$, $\beta > 0$, an increasing sequence of positive numbers c_n with $\lim_{n\to\infty} c_n = \infty$, $c_{n+1}/c_n = O(1)$, moreover, there exists a strictly increasing unbounded sequence of nonnegative numbers v_n such that for each pair (l, k), with $l < k, l, k \in \mathbb{N}$, there exists a $\boldsymbol{B}$-valued random process $X_{lk}(t), v_k \le \le t < v_{k+1}$, with the following properties. For $l < k$ $\{X(t) : v_l \le \le t < v_{l+1}\}$ and $\{X_{lk}(t) : v_k \le t < v_{k+1}\}$ are independent families of random variables, moreover, for all t with $v_k \le t < v_{k+1}$*

$$\mathbb{E}\varrho\,(X(t), X_{lk}(t)) \le C\left(\frac{c_l}{c_k}\right)^{\beta}. \tag{4.1}$$

Suppose that there exists a decreasing positive function $d(t)$, $v_1 \le t$, *with* $\int_{v_k}^{v_{k+1}} d(t)\,dt \le \log(c_{k+1}/c_k)$ *for each* k, *and* $\int_{v_1}^{\infty} d(t)\,dt = \infty$. *Set* $D(T) = \int_{v_1}^{T} d(t)\,dt$ *and*

$$Q_T^I(\omega)(A) = \frac{1}{D(T)} \int_{v_1}^{T} \delta_{X(t,\omega)}(A) d(t)\,dt, \quad A \in \mathcal{B}(\boldsymbol{B}).$$

Then for any probability distribution μ *on the Borel* σ*-algebra* $\mathcal{B}(\boldsymbol{B})$ *the following two statements are equivalent*

$$Q_T^I(\omega) \xrightarrow{w} \mu, \quad \text{as } T \to \infty, \text{ for almost all } \omega \in \Omega; \tag{4.2}$$

$$\frac{1}{D(T)} \int_{v_1}^{T} \mu_{X(t)} d(t)\,dt \xrightarrow{w} \mu, \quad \text{as } T \to \infty. \tag{4.3}$$

References

[1] DE ACOSTA, A. and GINÉ, E.: Convergence of moments and related functionals in the general central limit theorem in Banach spaces, *Z. Wahrscheinlichkeitstheorie verw. Gebiete* **48** (1979), 213–231.

[2] BERKES, I. and CSÁKI, E.: A universal result in almost sure central limit theory, *Stoch. Proc. Appl.* **94** (1) (2001), 105–134.

[3] BERKES, I., CSÁKI, E., CSÖRGŐ, S. and MEGYESI, Z.: Almost sure limit theorems for sums and maxima from the domain of geometric partial attraction of semistable laws, in: Limit Theorems in Probability and Statistics, Vol. I., pp. 133–157, János Bolyai Math. Soc., Budapest, 2002.

[4] BINGHAM, N. H., GOLDIE, C. M. and TEUGELS, J. L.: Regular Variation, Cambridge University Press, Cambridge, 1987.

[5] CHUPRUNOV, A. and FAZEKAS, I.: Integral analogues of almost sure limit theorems, *Periodica Math. Hungar.* **50** (1-2) (2005), 61–78.

[6] FAZEKAS, I.: Convergence rates in the law of large numbers for arrays, *Publ. Math. Debrecen* **41** (1–2) (1992), 53–71.

[7] FAZEKAS, I. and RYCHLIK, Z.: Almost sure functional limit theorems, *Ann. Univ. Marie Curie-Skłodowska, Sect. A* **56** (2002), 1–18.

[8] GNEDENKO, B. V. and KOLMOGOROV, A. N.: Limit Distributions for Sums of Independent Random Variables, Addison-Wesley, Reading, Massachusetts, 1954.

[9] HOFFMANN-JØRGENSEN, J.: Sums of independent Banach space valued random variables, *Studia Math.* **52** (1974), 159–186.

[10] MAJOR, P.: Almost sure functional limit theorems. Part I. The general case, *Studia Sci. Math. Hungar.* **34** (1998), 273–304.

[11] RAMACHANDRAN, B.: On characteristic functions and moments, *Sankhyā Ser. A* **31** (1969), 1–12.

[12] SATO, KEN-ITI: Lévy Processes and Infinitely Divisible Distributions, Cambridge University Press, Cambridge, 1999.

[13] SENETA, E.: Regularly Varying Functions, Lecture Notes in Mathematics **508**, Springer, Berlin, 1976.