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ON THE QUASI-CONFORMAL CURVATURE TENSOR OF A KENMOTSU MANIFOLD

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Abstract: We consider quasi-conformally flat and quasi-conformally semisymmetric Kenmotsu manifolds. We show that the following three statements are equivalent: **(a)** M is quasi-conformally flat, **(b)** M is quasi-conformally semisymmetric and **(c)** M is locally isometric to the hyperbolic space $H^n(-1)$.

1. Introduction

In [3], B. Y. Chen and K. Yano defined the notion of an n-dimensional Riemannian manifold (M^n, g) of quasi-constant curvature as a conformally flat manifold with the curvature tensor R satisfying the condition

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$$R(X,Y,Z,W) = p[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]+ \tag{1.1}$$
$$+ q[g(X,W)T(Y)T(Z) - g(X,Z)T(Y)T(W)+$$
$$+ g(Y,Z)T(X)T(W) - g(Y,W)T(X)T(Z)]$$

where $R(X,Y,Z,W) = g(\mathcal{R}(X,Y)Z,W)$, $\mathcal{R}$ is the curvature tensor of M, p,q are scalar functions and T is a non-zero 1-form defined by

$$g(X,U) = T(X), \tag{1.2}$$

where U is the unit vector field.

It can be easily seen that if the curvature tensor R is of the form (1.1), then the manifold is conformally flat. On the other hand, in [13], G. Vrănceanu defined the notion of almost constant curvature tensor by the same expression (1.1). Later in [8], A. L. Mocanu pointed out that the manifold introduced by Chen and Yano [3] and G. Vrănceanu [13] are the same. The notion of the quasi-conformal curvature tensor was defined by K. Yano and S. Sawaki (see [9]). According to them a quasi-conformal curvature tensor is defined by

$$\widetilde{C}(X,Y)Z = aR(X,Y)Z+b[S(Y,Z)X-S(X,Z)Y+g(Y,Z)QX- \tag{1.3}$$
$$-g(X,Z)QY]-\frac{r}{n}\left[\frac{a}{n-1}+2b\right][g(Y,Z)X-g(X,Z)Y],$$

where a and b are constants, S is the Ricci tensor, Q is the Ricci operator and r is the scalar curvature of the manifold M^n.

A Riemannian manifold (M^n, g) $(n > 3)$, is called quasi-conformally flat if the quasi-conformal curvature tensor $\widetilde{C} = 0$. If $a = 1$ and $b = -\frac{1}{n-2}$, then the quasi-conformal curvature tensor is reduced to the conformal curvature tensor.

A Riemannian manifold is said to be semi-symmetric (see [12]) if

$$R(X,Y) \cdot R = 0,$$

where R is the Riemannian curvature tensor and $R(X,Y)$ is considered as a derivation of the tensor algebra at each point of the manifold for tangent vector fields X, Y. If a Riemannian manifold satisfies

$$R(X,Y) \cdot \widetilde{C} = 0,$$

where $\tilde{C}$ is the quasi-conformal curvature tensor, then the manifold is said to be quasi-conformally semi-symmetric manifold.

2. Kenmotsu manifolds

Let M be an almost contact metric manifold (see [1]) equipped with an almost contact metric structure (φ, ξ, η, g) consisting of a $(1,1)$ tensor field φ, a vector field ξ, a 1-form η and a compatible Riemannian metric g satisfying

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \varphi\xi = 0, \quad \eta \circ \varphi = 0, \tag{2.1}$$

$$g(X,Y) = g(\varphi X, \varphi Y) + \eta(X)\eta(Y), \tag{2.2}$$

$$g(X, \varphi Y) = -g(\varphi X, Y), \quad g(X, \xi) = \eta(X) \tag{2.3}$$

for all $X, Y \in TM$. An almost contact metric manifold is called a Kenmotsu manifold if it satisfies (see [6])

$$(\nabla_X \varphi) Y = g(\varphi X, Y)\xi - \eta(Y)\varphi X, \qquad X, Y \in TM, \tag{2.4}$$

where ∇ is Levi–Civita connection of the Riemannian metric. From the above equation it follows that

$$\nabla_X \xi = X - \eta(X)\xi, \tag{2.5}$$

$$(\nabla_X \eta) Y = g(X,Y) - \eta(X)\eta(Y). \tag{2.6}$$

Moreover, the curvature tensor R, the Ricci tensor S, and the Ricci operator Q satisfy (see [6])

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.7}$$

$$S(X,\xi) = (1-n)\eta(X), \tag{2.8}$$

$$Q\xi = (1-n)\xi. \tag{2.9}$$

The equation (2.7) is equivalent to

$$R(\xi, X)Y = \eta(Y)X - g(X,Y)\xi, \tag{2.10}$$

which implies that

$$R(\xi, X)\xi = X - \eta(X)\xi. \tag{2.11}$$

From (2.10) we have

$$\eta(R(\xi, X)Y) = \eta(X)\eta(Y) - g(X,Y). \tag{2.12}$$

Kenmotsu manifolds have been studied various authors. For example see [2], [4], [5], [7], [11].

A plane section Π in T_pM of an almost contact metric manifold M is called a φ-section if $\Pi \perp \xi$ and $\varphi(\Pi) = \Pi$. If the sectional curvature $K(\Pi)$ does not depend on the choice of the φ-section Π of T_pM, then M is of pointwise constant φ-sectional curvature. A Kenmotsu manifold of pointwise constant φ-sectional curvature is called a Kenmotsu space form.

A Kenmotsu manifold is normal (that is, the Nijenhuis tensor of φ equals $-2d\eta \otimes \xi$) but not Sasakian. Moreover, it is also not compact since from the equation (2.5) we get $\operatorname{div}\xi = n-1$. In [6], K. Kenmotsu showed (1) that locally a Kenmotsu manifold is a warped product $I \times_f N$ of an interval I and a Kähler manifold N with warping function $f(t) = se^t$, where s is a nonzero constant; and (2) that a Kenmotsu manifold of constant φ-sectional curvature is a space of constant curvature -1, and so it is locally hyperbolic space. Examples of Kenmotsu manifolds of strictly pointwise constant φ-sectional curvature are not known so far and, according to D. Blair, one doubts that there are any, since the warped product structure of a Kenmotsu manifold involves a Kähler structure. Thus, one has to be careful for further study of Kenmotsu space forms with strictly pointwise constant φ-sectional curvature.

An almost contact metric manifold is said to be an η-Einstein if the Ricci tensor S satisfies the condition

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y) \tag{2.13}$$

where a, b are certain scalars. If $b = 0$ then the manifold M is an Einstein manifold.

3. Quasi-conformally flat Kenmotsu manifolds

Assume that M^n is a quasi-conformally flat Kenmotsu manifold. Then from (1.1) we have

$$\begin{aligned} R(X,Y,Z,W) = {} & \frac{b}{a}[S(X,Z)g(Y,W) - S(Y,Z)g(X,W) + \\ & + S(Y,W)g(X,Z) - S(X,W)g(Y,Z)] - \\ & - \frac{r}{na}\left[\frac{a}{n-1} + 2b\right][g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]. \end{aligned} \tag{3.1}$$

Putting $Z = \xi$ in (3.1) and using (2.3), (2.7) and (2.8) we obtain

$$g(Y,W)\eta(X) - g(X,W)\eta(Y) = \frac{b}{a}[(1-n)g(Y,W)\eta(X) - (1-n)g(X,W)\eta(Y) + S(Y,W)\eta(X) - S(X,W)\eta(Y)] + \frac{r}{na}\left[\frac{a}{n-1} + 2b\right][g(X,W)\eta(Y) - g(Y,W)\eta(X)]. \tag{3.2}$$

Now putting $Y = \xi$ in (3.2) and using (2.3), (2.7) and (2.8) it follows that

$$\frac{b}{a}S(X,W) = Ag(X,W) + B\eta(X)\eta(W), \tag{3.3}$$

where

$$A = \left[1 - \frac{b}{a}(1-n) + \frac{r}{na}\left(\frac{a}{n-1} + 2b\right)\right] \tag{3.4}$$

and

$$B = \left[-1 + 2(1-n)\frac{b}{a} + \frac{r}{na}\left(\frac{a}{n-1} + 2b\right)\right]. \tag{3.5}$$

Hence M^n is an η-Einstein manifold. By a contraction of the equation (3.2) we have

$$r = nA + B. \tag{3.6}$$

In view of (3.4) and (3.5) we get

$$\left[\frac{b}{a}(2-n) - 1\right]\left[\frac{1}{n(n-1)}r + 1\right] = 0. \tag{3.7}$$

Hence either

$$b = \frac{a}{2-n} \tag{3.8}$$

or

$$r = n(1-n). \tag{3.9}$$

If $b = \frac{a}{2-n}$ then putting (3.8) into (3.7) we get

$$\widetilde{C}(X,Y)Z = aC(X,Y)Z, \tag{3.10}$$

where $C(X,Y)Z$ denotes the Weyl conformal curvature tensor. So the quasi conformally flatness and conformally flatness are equivalent in this case. But from [5] we know that a Kenmotsu manifold M^n is conformally flat if and only if it is locally isometric to the hyperbolic space $H^n(-1)$. So in this case M^n is is locally isometric to the hyperbolic space $H^n(-1)$.

If $r = n(1-n)$ then putting (3.9) into (3.4) and (3.5) the equation (3.3) turns into the form

$$S(X,W) = (1-n)g(X,W). \tag{3.11}$$

This implies that M^n is an Einstein manifold. So putting (3.11) into (3.1) we obtain

$$R(X,Y,Z,W) = g(X,Z)g(Y,W) - g(Y,Z)g(X,W).$$

Then M^n is of constant curvature -1 and hence it is locally isometric to the hyperbolic space $H^n(-1)$. If M^n is locally isometric to the hyperbolic space $H^n(-1)$ then it is easy to see that M^n is quasi-conformally flat. This leads to the following theorem:

Theorem 3.1. *Let (M^n, g) $(n > 3)$ be a Kenmotsu manifold. Then M^n is quasi-conformally flat if and only if M^n is locally isometric to the hyperbolic space $H^n(-1)$.*

4. Quasi conformally semi-symmetric Kenmotsu manifolds

Let us consider a quasi conformally semi-symmetric Kenmotsu manifold (M^n, g), $(n > 3)$. Then the condition

$$R(X,Y) \cdot \widetilde{C} = 0$$

holds on (M^n, g) for every vector fields X, Y. Hence we have

$$\begin{aligned} 0 &= (R(X,Y)\cdot \widetilde{C})(U,V,W) = \\ &= R(X,Y)\widetilde{C}(U,V)W - \widetilde{C}(R(X,Y)U,V)W - \\ &\quad - \widetilde{C}(U,R(X,Y)V)W - \widetilde{C}(U,V)R(X,Y)W. \end{aligned}$$

So for $X = \xi$ we get

$$\begin{aligned} 0 &= R(\xi,Y)\widetilde{C}(U,V)W - \widetilde{C}(R(\xi,Y)U,V)W - \\ &\quad - \widetilde{C}(U,R(\xi,Y)V)W - \widetilde{C}(U,V)R(\xi,Y)W. \end{aligned} \tag{4.1}$$

In view of (2.10) the equation (4.1) can be written as

$$\begin{aligned} 0 &= \eta(\widetilde{C}(U,V)W)Y - \widetilde{C}(U,V,W,Y)\xi - \eta(U)\widetilde{C}(Y,V)W + \\ &\quad + g(Y,U)\widetilde{C}(\xi,V)W - \eta(V)\widetilde{C}(U,Y)W + g(Y,V)\widetilde{C}(U,\xi)W - \\ &\quad - \eta(W)\widetilde{C}(U,V)Y + g(Y,W)\widetilde{C}(U,V)\xi, \end{aligned} \tag{4.2}$$

where $\widetilde{C}(U,V,W,Y) = g(\widetilde{C}(U,V)W,Y)$. Taking the inner product of

(4.2) with ξ we have

$$
\begin{aligned}
0 = {} & \eta(\widetilde{C}(U,V)W)\eta(Y) - \widetilde{C}(U,V,W,Y) - \eta(U)\eta(\widetilde{C}(Y,V)W) + \\
& + g(Y,U)\eta(\widetilde{C}(\xi,V)W) - \eta(V)\eta(\widetilde{C}(U,Y)W) + \\
& + g(Y,V)\eta(\widetilde{C}(U,\xi)W) - \eta(W)\eta(\widetilde{C}(U,V)Y).
\end{aligned} \tag{4.3}
$$

Putting $Y = U$ the equation (4.3) turns into the form

$$
\begin{aligned}
0 = {} & -\widetilde{C}(U,V,W,U) + g(U,U)\eta(\widetilde{C}(\xi,V)W) + \\
& + g(U,V)\eta(\widetilde{C}(U,\xi)W) - \eta(W)\eta(\widetilde{C}(U,V)U).
\end{aligned} \tag{4.4}
$$

Let $\{e_i\}$, $1 \le i \le n$, be an orthonormal basis of the tangent space at any point. Then in view of the equations (1.3), (2.7), (2.8), (2.10) and (2.12) the sum for $U = e_i$, $1 \le i \le n$, of the relation (4.4) gives us

$$
\begin{aligned}
S(V,W) = {} & \left[\frac{-br - b(n^2-1) + a(1-n)}{a-b}\right] g(V,W) + \\
& + \left[\frac{b[n(n-1)+r]}{a-b}\right] \eta(V)\eta(W).
\end{aligned} \tag{4.5}
$$

So contracting the last equation we find the scalar curvature r of M^n as

$$r = n(1-n). \tag{4.6}$$

Hence putting (4.6) into (4.5) we obtain

$$S(V,W) = (1-n)g(V,W). \tag{4.7}$$

Then M^n is an Einstein manifold. So in view of (4.6), (4.7) and (1.3) the equation (4.2) is reduced to the form

$$R(U,V,W,Y) = \left[\frac{2nb-a}{a}\right](g(V,W)g(U,Y) - g(U,W)g(V,Y)). \tag{4.8}$$

Hence by a suitable contraction of the last equation we find

$$S(V,W) = \left[\frac{2nb-a}{a}\right](n-1)g(V,W). \tag{4.9}$$

Comparing the right-hand sides of the equations (4.7) and (4.9) we obtain $\frac{2nb-a}{a} = -1$, which gives us $b = 0$. So the equation (4.8) turns into the form $R(U,V,W,Y) = g(U,W)g(V,Y) - g(V,W)g(U,Y)$. Then M^n is locally isometric to hyperbolic space $H^n(-1)$. Hence in view of Th. 3.1 we get that M^n is quasi-conformally flat. Then it is trivially quasi-conformally semisymmetric. So we have the following result:

Theorem 4.1. *Let (M^n, g) $(n > 3)$ be a Kenmotsu manifold. Then M^n is quasi conformally semisymmetric if and only if M^n is locally isometric to the hyperbolic space $H^n(-1)$.*

In view of Th. 3.1 and Th. 4.1 we have the following corollary:

Corollary 4.2. *A Kenmotsu manifold (M^n, g) $(n > 3)$ is quasi conformally flat if and only if M^n is quasi conformally semisymmetric.*

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