

# GRASSMANN SPACES OVER HYPERBOLIC AND QUASI HYPERBOLIC SPACES

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**Abstract:** We prove that a bijection of a (finite dimensional) hyperbolic space which preserves an adjacency of its  $k$ -dimensional subspaces (any one from among four distinguished in the paper) is determined by a collineation of the underlying space and, at the same time, every one of these adjacencies can be used to express hyperbolic geometry. The results are obtained, in fact, in the framework of, more general, quasi hyperbolic geometry.

## 1. Introduction

The problem if a particular geometry can be expressed in terms of (some) adjacency<sup>1</sup> of its subspaces has three, mutually equivalent, in fact, aspects:

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<sup>1</sup>We stress on the term some adjacency used here since, as right at the beginning of the paper we show that, an arbitrary geometry admits at least three relations of adjacency of its subspaces, and there are geometries where these three adjacencies are pairwise distinct.

- is every bijection on subspaces preserving this adjacency determined by an automorphism of the underlying geometry? (an algebraic approach),
- can we characterize basic relations of the underlying geometry in terms of the considered adjacency? (a logical approach), and
- can we develop (in a special case: axiomatize) the underlying geometry in terms of this adjacency? (a metamathematical approach).

Problems of this kind were studied and solved for many (more or less classical) geometries in at least one of the above distinguished aspects. As an example we can quote the Chow Theorem (a bijection of  $k$ -subspaces of a projective space  $\mathfrak{P}$  which preserves the adjacency is determined by a collineation and, if  $2k + 1 = \dim(\mathfrak{P})$ , by a duality, see [4]) and, in the particular case  $k = 1$  its metamathematical counterpart consisting in axiomatization of projective geometry in the language of line intersection (see [7], [21], [14]). Analogous (in most cases algebraic) results remain valid for affine geometry (cf. [7], [36], [1]), the geometry of polar spaces (formalizing geometry on quadrics, cf. [25], [10]), the geometry of Segre products of Grassmannians (cf. [2]), and many others, which are related to exotic, not yet axiomatized, geometries (cf. [37], [9], [11], [3], or [32] for example).

In this paper we study the questions stated at the beginning in the context of (classical) hyperbolic geometry and some its extensions. One of the old fundamental results of this theory states that (over Euclidean fields) the (ternary) relation of collinearity of points can be used as a primitive notion (cf. [15]). Therefore, hyperbolic geometry, which in a "full" language with orthogonality, equidistance, betweenness etc. was invented in analogy to Euclidean geometry, can be considered simply as a linear geometry as well. Some axiom systems characterizing hyperbolic geometry in terms of collinearity can be found, e.g. in [33], a survey of results on this subject can be found in [16], [20], and [17]. Relatively early the problem to characterize hyperbolic geometry in terms of relations on lines was solved (see e.g. [29], [30], [23], [12], and the survey [16] together with [20]) and, among others, pencils (proper and hyperparallel) turn out to be sufficient.

As an example of applications of one of our results (cf. 3.14) we get now that line intersection can be used as a primitive notion in at least 3-dimensional hyperbolic geometry; but also e.g. pencils of planes (proper, parallel, and hyperparallel, in dimension at least 4) are sufficient to

characterize hyperbolic geometry. Similarly, results of [22] mostly follow from our 5.10.

In fact, most of the results of the paper are formulated and proved in a more general framework of (some variant of) the quasi hyperbolic geometry. A model  $\mathfrak{H}$  of the quasi hyperbolic geometry can be characterized as an open convex subset  $S$  of an ordered affine space  $\mathfrak{A}$  with lines interpreted as suitable parts of lines of  $\mathfrak{A}$ . A hyperbolic space (the Klein Model) appears in a special case when  $S$  is a projective sphere in the projective completion  $\mathfrak{P}$  of  $\mathfrak{A}$ . Two main reasons forced us to choose this apparatus, generalizing classical hyperbolic geometry. First, the trick (consisting in defining the notion of asymptotic triangle and considering copunctuality defined by means of Desargues configurations) which enables us to define hyperbolic parallelism, betweenness, and the surrounding projective space  $\mathfrak{P}$  in terms of collinearity of  $\mathfrak{H}$ , can be applied in the quasi hyperbolic geometry setting. Consequently, this geometry can also be considered as a pure linear geometry. Second, some standard techniques of projective Grassmann spaces used to define pencils (also stars and tops) of subspaces can be easily adopted here (subspaces of  $\mathfrak{H}$  are restrictions of subspaces of  $\mathfrak{P}$  (of  $\mathfrak{A}$ ) to  $S$ ) and in most cases fundamental characterizations do not rely on the specific type of geometry of  $\mathfrak{H}$ . Only in the last part, when we consider parallelism of subspaces, do we assume that  $\mathfrak{H}$  is a hyperbolic space and use some techniques of Möbius geometry. In the general case the geometry of the boundary  $S^\infty$  of  $S$  becomes too complex to find sufficiently elegant regularities.

A quasi hyperbolic space  $\mathfrak{H}$  resembles some features of affine geometry. In particular, two of its hyperplanes may stay disjoint (being parallel or hyperparallel) and therefore the geometry admits two natural adjacencies: its  $k$ -subspaces  $A$  and  $B$  are adjacent if they have a common  $(k - 1)$ -subspace (in symbols:  $A \sim_- B$ ), or – less restrictively – if they are contained in a  $(k + 1)$ -subspace (then we write  $A \sim^+ B$ ). We prove that (under certain natural dimension assumptions) any one of these two adjacencies suffices to characterize pencils of subspaces and then to characterize points of  $\mathfrak{H}$  and its collinearity relation. At the same time we show that “hyperadjacency” ( $A, B$  are hyperadjacent if they are in one hyperparallel pencil) is also sufficient. This immediately yields that the automorphism groups of the corresponding adjacencies  $\sim_-$  and  $\sim^+$  coincide with the automorphism group of  $\mathfrak{H}$  act-

ing (faithfully) on  $k$ -subspaces of  $\mathfrak{H}$ . Analogous results appear valid for parallelism (of subspaces) of a hyperbolic space. This does not mean, however, that we formulate an axiom system for quasi hyperbolic (or hyperbolic) geometry in terms of an adjacency.

## 2. Quasi hyperbolic spaces and their Grassmannians

Let  $\mathfrak{H} = \langle S, \mathcal{F} \rangle$  be a quasi hyperbolic space. What does it precisely mean? Passing over axiomatic characterizations of such a geometry we start from its analytical representation (cf. [34], [35], [13], [26], [27]). Let  $\mathfrak{A} = \langle P_0, \mathcal{L}_0, \parallel \rangle$  be an ordered affine space (of dimension at least 3), in the standard way completed to the projective space  $\mathfrak{P} = \langle P, \mathcal{L} \rangle$ ; let  $\mathcal{A}^\infty$  be the set of improper points of  $\mathfrak{A}$ .

- (1) We assume that  $S \subset P_0$  is open and convex in  $\mathfrak{A}$ .
- (2) Moreover,  $S$  is regular, i.e. every halfline of  $\mathfrak{A}$  with origin in  $S$  either crosses the boundary  $\text{Fr}(S) =: S^\infty$ , or is entirely contained in  $S$ . The points of  $S^\infty$  can be defined in terms of the geometry of  $\mathfrak{H}$  as the ends of lines (comp. [26], [27]).
- (3) The elements of  $\mathcal{F}$  are suitable parts of lines in  $\mathcal{L}_0$ :  $\mathcal{F} = \{L \cap S : L \in \mathcal{L}_0, L \cap S \neq \emptyset\}$ . From the above, for every  $l \in \mathcal{F}$ , either  $l$  is in  $\mathcal{L}_0$ , or  $l$  is an open segment, or  $l$  is an open halfline of  $\mathfrak{A}$ .

A quasi hyperbolic space  $\mathfrak{H}$  is a hyperbolic space if the coordinate field of  $\mathfrak{A}$  is Euclidean, and  $S^\infty$  is a projective sphere in  $\mathfrak{P}$  (i.e. it is a nonruled quadric determined by a polarity  $\pi$ ).

In fact, condition (1) can be replaced by the following:  $S$  is an open convex subset of an ordered projective space  $\mathfrak{P}$  such that some of hyperplanes of  $\mathfrak{P}$  misses  $S$ . In the sequel we shall use a more general condition:

- (4) If a subspace  $Y$  of  $\mathfrak{P}$  misses  $S$ , then there is a hyperplane  $Y'$  of  $\mathfrak{P}$  missing  $S$  such that  $Y \subset Y'$ .

Note (cf. (2)) that following this projective approach some points of  $\mathcal{A}^\infty$  can be elements of  $S^\infty$  as well (some ambiguity may appear if  $\mathfrak{H}$  admits so called Euclidean directions i.e. when  $\mathcal{L}_0 \cap \mathcal{F} \neq \emptyset$ ). In particular, if  $\mathcal{A}^\infty = S^\infty$ , then  $\mathfrak{H}$  is simply the affine space  $\mathfrak{A}$ . Adjacencies of subspaces of an affine space were already investigated in the literature (cf. [24], [8]) and therefore we pass over this case.

Note, also, that if  $S^\infty$  is (e.g.) an open polytope (compare [6]), then Condition (2) is always valid; if  $S^\infty$  is defined by means of some

nonlinear equations, then (2) may force the underlying coordinate field to have some additional specific properties (like in the case of hyperbolic geometry).

To avoid trivial or already studied cases we assume that

$$(5) \quad \begin{aligned} (\dim(\mathfrak{P}) =) \dim(\mathfrak{H}) =: n > 2 \text{ and } 1 \leq k < n - 1, \\ \text{and} \\ S^\infty \neq \mathcal{A}^\infty, \text{ i.e. } \mathfrak{H} \text{ is not an affine space.} \end{aligned}$$

Every subspace of  $\mathfrak{H}$  is a quasi hyperbolic space as well.

In the sequel we write  $\wp(\mathfrak{P})$  ( $\wp(\mathfrak{H})$ ) for the class of the subspaces of  $\mathfrak{P}$  (of  $\mathfrak{H}$  resp.); if  $m$  is a nonnegative integer we write  $\wp_m(\mathfrak{P})$  ( $\wp_m(\mathfrak{H})$ ) for the  $m$ -dimensional subspaces of  $\mathfrak{P}$  (of  $\mathfrak{H}$  resp.). The term hyperplane is used to denote a maximal proper subspace. For  $Z \subset P$  we write  $\overline{Z}$  for the least subspace of  $\mathfrak{P}$  which contains  $Z$ ; after that for  $Y, Y' \in \wp(\mathfrak{P})$  we put  $Y \sqcup Y' := \overline{Y \cup Y'}$  and we shorten  $Y \sqcup \{a\}$  with  $a \in P$  to  $Y \sqcup a$ .

For every  $X \in \wp(\mathfrak{H})$  there is the unique  $Y \in \wp(\mathfrak{P})$  with  $X = Y \cap S$ ; namely  $Y = \overline{X}$ . We write  $X^\infty = \overline{X} \cap S^\infty$ .

Below we quote without proofs some technical lemmas concerning the geometry of  $\mathfrak{H}$  which will be used in the sequel.

**Lemma 2.1.** *If  $Q$  is a subspace of  $\mathfrak{P}$  such that  $Q \cap S^\infty = \emptyset$ , then  $Q \cap S = \emptyset$ .*

**Lemma 2.2.** *Let  $Q_1, Q_2, Y$  be subspaces of  $\mathfrak{P}$  such that  $Q_i \subsetneq Y$  and  $Q_i \cap S^\infty = \emptyset$  for  $i = 1, 2$ . Then  $Y$  contains a subspace  $Q'$  such that  $\dim(Q') = \dim(Q_1)$ ,  $Q_i \neq Q'$ , and  $Q' \cap S^\infty = \emptyset$ . If, moreover,  $Q_1 \neq Q_2$  and  $\dim(Q_1) = \dim(Q_2)$ , then there are at least two such  $Q'$ , both two containing  $Q_1 \cap Q_2$ .*

We say that a hyperplane  $G$  of a quasi hyperbolic space  $\mathfrak{H}$  separates two subsets  $D'$  and  $D''$  of the points of  $\mathfrak{H}$  if  $D'$  and  $D''$  lie in two distinct open half-spaces of  $\mathfrak{H}$  with boundary  $G$ .

**Lemma 2.3.** *Assume that a hyperplane  $X_0$  of  $\mathfrak{H}$  separates hyperplanes  $X_1$  and  $X_2$  in  $\mathfrak{H}$ . If  $X_1^\infty$  and  $X_2^\infty$  have a common point  $q$ , then  $q \in X_0^\infty$ .*

In accordance with the standards of classical hyperbolic geometry, subspaces of  $\mathfrak{P}$  are classified as follows. Let  $M \in \wp_m(\mathfrak{P})$ .

$M \in \mathcal{H}_m$  ( $M$  is an inner subspace) iff  $M \cap S \neq \emptyset$ ; then  $(M \cap S) \in \wp_m(\mathfrak{H})$ .

$M \in \mathcal{T}_m$  ( $M$  is a tangent subspace) iff  $M \cap S = \emptyset$  and  $M \cap S^\infty \neq \emptyset$  (in hyperbolic geometry this is equivalent to:  $|M \cap S^\infty| = 1$ ).

$M \in \mathcal{E}_m$  ( $M$  is an exterior subspace) iff  $M \cap (S \cup S^\alpha) = \emptyset$  (equivalently:  $M \cap S^\alpha = \emptyset$ , cf. 2.1). We set, finally  $\mathcal{H} := \bigcup_{m=0}^\infty \mathcal{H}_m$ ,  $\mathcal{T} := \bigcup_{m=0}^\infty \mathcal{T}_m$ , and  $\mathcal{E} := \bigcup_{m=0}^\infty \mathcal{E}_m$ .

The following is evident now.

**Fact 2.4.** *The map  $\wp(\mathfrak{H}) \ni X \mapsto \bar{X} \in \mathcal{H}$  is an inclusion- and dimension-preserving bijection. Its inverse is the restriction map  $\mathcal{H} \ni \bar{Y} \mapsto Y \cap S$ .*

Thanks to the maps given in 2.4 we can identify the subspaces of  $\mathfrak{H}$  with some subspaces of  $\mathfrak{P}$  and, after that, use some standard notions related to the geometry of projective Grassmann spaces.

In the family  $\wp_k(\mathfrak{H})$  we introduce two adjacency relations. Let  $X_1, X_2 \in \wp_k(\mathfrak{H})$ , we write:

$$(6) \quad X_1 \sim_- X_2 : \iff X_1 \cap X_2 \in \wp_{k-1}(\mathfrak{H}),$$

$$(7) \quad X_1 \sim^+ X_2 : \iff X_1 \sqcup X_2 \in \wp_{k+1}(\mathfrak{H}).$$

Clearly, if  $X_1 \sim_- X_2$  then  $X_1 \sim^+ X_2$ .

Then we introduce on the set  $\wp_k(\mathfrak{H})$  the block structure  $\mathbf{P}_k(\mathfrak{H}) = \langle \wp_k(\mathfrak{H}), \mathcal{P}_k(\mathfrak{H}) \rangle$  with the blocks (elements of  $\mathcal{P}_k(\mathfrak{H})$ ) defined as pencils

$$(8) \quad \mathbf{p}(A, B) = \{X \in \wp_k(\mathfrak{H}) : A \subset X \subset B\}, \text{ where}$$

$$A \subset B, A \in \wp_{k-1}(\mathfrak{H}), \text{ and } B \in \wp_{k+1}(\mathfrak{H}).$$

The structure  $\mathbf{P}_k(\mathfrak{H})$  will be referred to as the  $k$ -th Grassmann space over  $\mathfrak{H}$ .

In the analogous way we define projective pencils  $\mathbf{p}(Q, R)$  for  $Q \in \wp_{k-1}(\mathfrak{P})$ ,  $R \in \wp_{k+1}(\mathfrak{P})$ , the projective Grassmannian  $\mathbf{P}_k(\mathfrak{P})$  – the  $k$ -th Grassmann space over  $\mathfrak{P}$ , and the adjacencies in the family  $\wp_k(\mathfrak{P})$  (note, however, that over a projective space the two relations  $\sim^+$  and  $\sim_-$  coincide, so we write simply  $\sim = \sim^+$ ).

A block structure  $\mathfrak{B} = \langle Z, \mathcal{B} \rangle$  is called a partial linear space (the elements of  $Z$  are points of  $\mathfrak{B}$  and the elements of  $\mathcal{B}$  – subsets of  $Z$  – are lines of  $\mathfrak{B}$ ) if two distinct lines of  $\mathfrak{B}$  have at most one point in common, every point of  $\mathfrak{B}$  is in at least one line of  $\mathfrak{B}$ , and every line of  $\mathfrak{B}$  contains at least two points. Points  $z_1, \dots, z_l$  of  $\mathfrak{B}$  are collinear if there is a line of  $\mathfrak{B}$  which contains them all. Both structures  $\mathbf{P}_k(\mathfrak{H})$  and  $\mathbf{P}_k(\mathfrak{P})$  are partial linear spaces.

With a slight abuse of language we say that  $\sim_-$  and  $\sim^+$  are defined in  $\mathbf{P}_k(\mathfrak{H})$ . Clearly, if  $X_1, X_2 \in \wp_k(\mathfrak{H})$  are distinct then  $X_1 \sim_- X_2$  if they are collinear in  $\mathbf{P}_k(\mathfrak{H})$ .

**Fact 2.5.** Let  $X_1, X_2 \in \wp_k(\mathfrak{H})$ . Then  $X_1 \sim^+ X_2$  iff  $\overline{X_1} \sim \overline{X_2}$ .

Let  $Q \in \wp_{k-1}(\mathfrak{P})$  and  $R \in \wp_{k+1}(\mathfrak{P})$ ; we define

$S(Q) := \{Y \in \wp_k(\mathfrak{P}) : Q \subset Y\}$  – the star with vertex  $Q$ , and

$T(R) := \{Y \in \wp_k(\mathfrak{P}) : Y \subset R\}$  – the top with base  $R$ .

Write

$$(9) \quad S_0(Q) := \{X \in \wp_k(\mathfrak{H}) : Q \subset \overline{X}\} \text{ and}$$

$$(10) \quad T_0(R) := \{X \in \wp_k(\mathfrak{H}) : \overline{X} \subset R\}$$

for restrictions of suitable stars and tops of projective Grassmannian over  $\mathfrak{P}$  to subspaces of  $\mathfrak{H}$ . After that we put

$$(11) \quad \mathbf{p}_0(Q, R) := S_0(Q) \cap T_0(R) \text{ for } Q \subset R.$$

It is seen that  $S_0(Q) \neq \emptyset$  for every  $Q \in \wp_{k-1}(\mathfrak{P})$ . For  $R \in \wp_{k+1}(\mathfrak{P})$  we have  $T_0(R) \neq \emptyset$  iff  $R = \overline{B}$  for some  $B \in \wp_{k+1}(\mathfrak{H})$  (i.e. iff  $R \in \mathcal{H}$ ); if this is the case then  $T_0(R) = T(B) := \{A \in \wp_k(\mathfrak{H}) : A \subset B\}$ . Similarly, for  $B \in \wp_{k-1}(\mathfrak{H})$  we write  $S(B) := \{A \in \wp_k(\mathfrak{H}) : B \subset A\}$ .

Consequently,  $\mathbf{p}_0(Q, R) \neq \emptyset$  iff  $R \in \mathcal{H}$ ; in this case  $\mathbf{p}_0(Q, R)$  consists simply of the nonempty restrictions to  $S$  of elements of the projective pencil  $\mathbf{p}(Q, R)$ . This yields a natural structure of a partial linear space in  $\mathcal{H}_k$ .

Next, in analogy to the ordinary hyperbolic geometry we classify stars and pencils:

proper:  $S^2 = \{S_0(Q) : Q \in \mathcal{H}_{k-1}\}$ ,  $\mathcal{G}^2 = \{\mathbf{p}_0(Q, R) : Q \in \mathcal{H}_{k-1}, R \in \mathcal{H}_{k+1}, Q \subset R\}$ ,

parallel:  $S^1 = \{S_0(Q) : Q \in \mathcal{T}_{k-1}\}$ ,  $\mathcal{G}^1 = \{\mathbf{p}_0(Q, R) : Q \in \mathcal{T}_{k-1}, R \in \mathcal{H}_{k+1}, Q \subset R\}$ ,

hyper-parallel:  $S^0 = \{S_0(Q) : Q \in \mathcal{E}_{k-1}\}$ ,  $\mathcal{G}^0 = \{\mathbf{p}_0(Q, R) : Q \in \mathcal{E}_{k-1}, R \in \mathcal{H}_{k+1}, Q \subset R\}$ .

We set  $\mathcal{S} := S^2 \cup S^1 \cup S^0$  and  $\mathcal{G} := \mathcal{G}^2 \cup \mathcal{G}^1 \cup \mathcal{G}^0$ . Clearly,  $\mathcal{G}^2$  is the set of the lines of  $\mathbf{P}_k(\mathfrak{H})$ . The structures  $\langle \wp_k(\mathfrak{H}), \mathcal{G} \rangle$  and  $\langle \wp_k(\mathfrak{H}), \mathcal{S} \rangle$  are partial linear spaces. In fact,  $\langle \wp_k(\mathfrak{H}), \mathcal{G} \rangle$  is, up to the bijection defined in 2.4, the restriction of  $\mathbf{P}_k(\mathfrak{P})$  to  $\mathcal{H}_k$ .

Let us fix

$$\mathfrak{H} = \mathbf{P}_k(\mathfrak{H}), \quad \overline{\mathfrak{H}} = \langle \wp_k(\mathfrak{H}), \mathcal{G} \rangle, \quad \text{and} \quad \mathfrak{P} = \mathbf{P}_k(\mathfrak{P}).$$

The classification of geometries on stars and tops of a hyperbolic space is rather easy (cf. 2.7 below). It becomes more complex in the case of a quasi hyperbolic space, but in the sequel we do not need such a detailed classification.

The projective Veblen axiom (formulated in the language of a partial linear space  $\mathfrak{B}$ ) is the formula

V: if two lines  $L_1, L_2$  of  $\mathfrak{B}$  with a common point  $z$  are crossed by two other lines  $M_1, M_2$  missing  $z$ , then  $M_1, M_2$  have a point in common. For our purposes the following classification will suffice.

**Proposition 2.6.** *Let  $\mathfrak{H}$  be a quasi hyperbolic space,  $R \in \mathfrak{P}_{k+1}(\mathfrak{P})$ , and  $Q \in \mathfrak{P}_{k-1}(\mathfrak{P})$ . Set  $\mathcal{X} = T_0(R)$  and  $\mathcal{Y} = S_0(Q)$ . Clearly,  $\mathcal{X}$  and  $\mathcal{Y}$  are both subspaces of  $\overline{\mathfrak{H}}$ . The geometry of  $\mathcal{X}$  (i.e. of the restriction of  $\overline{\mathfrak{H}}$  to  $\mathcal{X}$ ) does not satisfy V, and the geometry of  $\mathcal{Y}$  satisfies V if and only if  $Q \in \mathcal{H}$ .*

**Proof.** It is a folklore that the set  $\mathcal{Q} = S(Q)$  carries the structure of a  $(n - k)$ -dimensional projective space. Consequently, to prove that  $\mathcal{Y}$  satisfies V for  $Q \in \mathcal{H}$  it suffices to use 2.4 and identify the elements of  $\mathcal{Y}$  and of  $\mathcal{Q}$  via the restriction map.

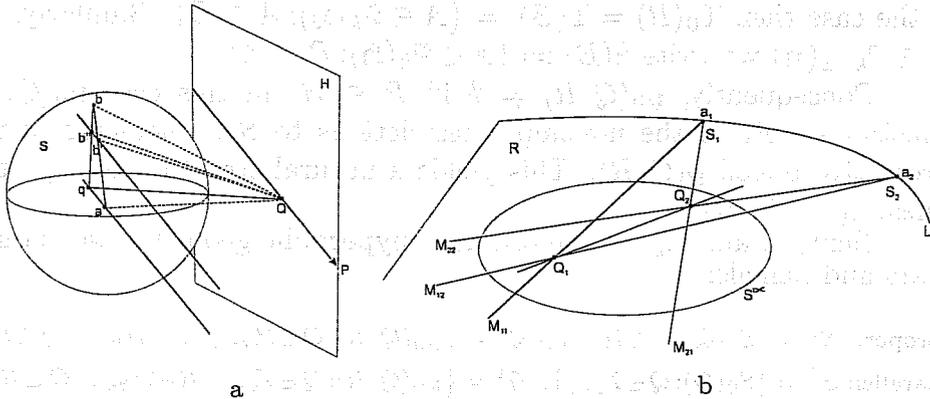


Fig. 1.

Next, let us consider  $\mathcal{Y}$  when  $Q \in \mathcal{E} \cup \mathcal{T}$ . Let  $q \in S$ , then  $\dim(Q \sqcup q) = k$  and  $\dim((Q \sqcup q) \cap \mathcal{A}^\infty) = k - 1$ . Therefore, there exists  $p \in \mathcal{A}^\infty \setminus (Q \sqcup q)$ . Moreover, from (4) there is a hyperplane  $H$  through  $Q$  missing  $S$  and we can choose  $p \in \mathcal{A}^\infty \cap H$ . Next, since  $\dim(Q \sqcup \overline{q, p}) = k + 1$  we take a point  $b \in S \setminus (Q \sqcup \overline{q, p})$ . On the halfline with origin  $q$  passing through  $p$  we take a point  $a \in S$ , and then we take any point  $b'$  between  $a$  and  $b$  and let  $b''$  be the parallel projection of  $b'$  in the direction of  $p$  on  $\overline{q, b}$  (cf. Fig. 1a<sup>2</sup>).

<sup>2</sup>Most of the figures in the paper illustrate the proofs in the case of  $k = 1$  in the Klein model of hyperbolic geometry. It is not too hard to note that in the general case the schema of constructing corresponding points and subspaces remains unchanged.

It is seen that the following four lines of  $\mathcal{Y}$

$\mathbf{p}_0(Q, Q \sqcup \overline{b, a}), \mathbf{p}_0(Q, Q \sqcup \overline{b, q}), \mathbf{p}_0(Q, Q \sqcup \overline{q, a}), \mathbf{p}_0(Q, Q \sqcup \overline{b', b''})$   
 satisfy the premises of  $V$ , but they do not satisfy its conclusion.

Finally, we prove that  $\mathcal{X}$  does not satisfy  $V$ . To this aim we take on  $R$  two subspaces  $Q_1, Q_2 \in \mathcal{H}_{k-1}$  which are affine-parallel; this means  $Q_1 \cap Q_2 \subset \mathcal{A}^\infty$ . Since  $\dim(Q_1 \cap Q_2) = k - 2$  and  $\dim(R \cap \mathcal{A}^\infty) = k$ , there is a projective line  $L \subset \mathcal{A}^\infty \cap R$  skew to  $Q_1 \cap Q_2$ . We take any two distinct points  $a_1, a_2 \in L$ . Let  $M_{i,j} = Q_i \sqcup a_j$ ; we observe that  $S_j := M_{1,j} \cap M_{2,j} = (Q_1 \cap Q_2) \sqcup a_j \subset \mathcal{A}^\infty$  for  $j = 1, 2$  so,  $S_1 \sqcup S_2 \in \mathcal{E} \cup \mathcal{T}$  (cf. Fig. 1b). Thus, finally, the following four lines of  $\mathcal{X}$ :

$$\mathbf{p}_0(Q_1, R), \mathbf{p}_0(Q_2, R), \mathbf{p}_0(S_1, R), \mathbf{p}_0(S_2, R)$$

satisfy the premises of  $V$ ; but they do not satisfy its conclusion.  $\diamond$

**Remark 2.7.** Let  $\mathfrak{H}$  be a hyperbolic space and  $\mathcal{X}, \mathcal{Y}$  be as in 2.6. First,  $\mathcal{X}$  is a dual hyperbolic space; let us symbolize its geometry by  $H^\circ$ . Next, if  $Q \in \mathcal{T}_{k-1}$ , then  $Q \cap S^\infty = \{q\}$  and  $\mathcal{Y}$  is an  $(n - k)$ -dimensional affine space (in symbols, it carries the geometry  $A$ ). Finally, if  $Q \in \mathcal{E}_{k-1}$ , then  $\mathcal{Y}$  is simply a  $(n - k)$ -dimensional hyperbolic space, isomorphic to the restriction of  $\mathfrak{H}$  to  $\pi(Q) \cap S$  (in symbols: its geometry is  $H$ ).

Let  $X_1, X_2 \in \wp_k(\mathfrak{H}), Y_i = \overline{X_i}$  for  $i = 1, 2$ , and let  $X_1 \sim^+ X_2$ ; Clearly, then  $Y_1 \cap Y_2 \in \wp_{k-1}(\mathfrak{P})$ . We write

$$(12) \quad X_1 \sim_j X_2 \text{ iff } \begin{cases} (Y_1 \cap Y_2) \in \mathcal{H}_{k-1} \text{ (i.e. if } X_1 \sim_- X_2) & \text{for } j = 2, \\ (Y_1 \cap Y_2) \in \mathcal{T}_{k-1} & \text{for } j = 1, \\ (Y_1 \cap Y_2) \in \mathcal{E}_{k-1} & \text{for } j = 0. \end{cases}$$

In other words, distinct  $X_1, X_2$  satisfy  $X_1 \sim_j X_2$  iff they are in one pencil in  $\mathcal{G}^j$ , i.e. iff they are collinear in the structure  $\langle \wp_k(\mathfrak{H}), \mathcal{G}^j \rangle$ . Similarly,  $X_1 \sim^+ X_2$  iff  $X_1, X_2$  are collinear in  $\overline{\mathfrak{H}}$ .

In the subsequent sections we shall examine whether it is possible to recover the linear structure of  $\mathfrak{H}$  from any of the adjacencies  $\sim^+, \sim_- = \sim_2, \sim_1$ , and  $\sim_0$ , what are their automorphisms, and whether we can interpret the underlying geometry of  $\mathfrak{H}$  in terms of corresponding adjacencies. The general schema of our considerations goes as follows:

- to characterize the geometry of  $\overline{\mathfrak{H}}$  in terms of  $\sim^+$  (in 3.2),
- to characterize  $\sim_-$  in terms of  $\sim^+$  (in 3.10),
- to characterize the geometry of  $\mathfrak{H}$  in terms of  $\sim_-$  (in 3.9),
- to characterize  $\sim^+$  in terms of  $\sim_0$  (in 4.3),
- to characterize  $\sim^+$  in terms of  $\sim_1$  (in 5.6)

under some additional assumptions, if necessary.

In the most “classical” case of hyperbolic geometry the main results of Sections 3, 4 and 5 (i.e. Thms. 3.13, 3.14, 4.4, and 5.10) give us the following

**Theorem.** *Let  $\mathfrak{H}$  be a hyperbolic space,  $1 \leq k \leq n - 1$ , and  $\sim$  be any one of the following relations:  $\sim^+$ ,  $\sim_-$  (maximal proper intersection of  $k$ -subspaces of  $\mathfrak{H}$ ),  $\sim_0$  (the hyperparallelism of  $k$ -subspaces), and  $\sim_1$  (the parallelism of  $k$ -subspaces). If  $\sim \neq \sim_1$  we assume, additionally, that  $k < n - 1$ , and if  $\sim = \sim_1$  we assume that  $1 < k$ . Then the geometry of  $\mathfrak{H}$  can be expressed in terms of the relation  $\sim$  and the group  $\text{Aut}(\langle \wp_k(\mathfrak{H}), \sim \rangle)$  consists of the automorphism of  $\mathfrak{H}$  acting on its  $k$ -subspaces.*

*Hyperbolic geometry can be also expressed in terms of pencils of  $k$ -subspaces, i.e.  $\mathfrak{H}$  is interpretable in  $\overline{\mathbf{P}_k(\mathfrak{H})}$ , in  $\mathbf{P}_k(\mathfrak{H}) = \langle \wp_k(\mathfrak{H}), \mathcal{G}^2 \rangle$ , and in  $\langle \wp_k(\mathfrak{H}), \mathcal{G}^0 \rangle$  when  $k < n - 1$ , and it is interpretable in  $\langle \wp_k(\mathfrak{H}), \mathcal{G}^1 \rangle$  when  $1 < k$ .*

### 3. $\sim^+$ and $\sim_-$ -adjacencies in quasi hyperbolic spaces and the automorphisms of Grassmann spaces over quasi hyperbolic spaces

In this, main section of the paper we shall examine the possibility to define the geometry of a quasi hyperbolic space  $\mathfrak{H}$  in terms of the adjacencies  $\sim^+$  and  $\sim_-$  and in terms of the Grassmann space  $\mathbf{P}_k(\mathfrak{H})$ . To this aim one more auxiliary notion, related to arbitrary binary relations will be used. For a symmetric relation  $\rho \subset Z \times Z$  and a positive integer  $n$  we introduce two new relations ( $\rho(z_1, \dots, z_n)$  stands for  $\bigwedge_{1 \leq i < j \leq n} [z_i \rho z_j]$ )

$$(13) \quad \begin{aligned} \Delta_n^\rho(z_1, \dots, z_n) &\iff \rho(z_1, \dots, z_n) \wedge \neq(z_1, \dots, z_n) \wedge \\ &\wedge \forall z', z'' \in Z [z', z'' \rho z_1, \dots, z_n \implies z' \rho z'' \vee z' = z'']; \end{aligned}$$

$$(14) \quad \begin{aligned} \not\Delta_n^\rho(z_1, \dots, z_n) &\iff \rho(z_1, \dots, z_n) \wedge \neq(z_1, \dots, z_n) \wedge \\ &\wedge \exists z', z'' \in Z [z', z'' \rho z_1, \dots, z_n \wedge \neg(z' \rho z'') \wedge z' \neq z'']. \end{aligned}$$

For  $z_1, \dots, z_n \in Z$  we write

$$(15) \quad [z_1, \dots, z_n]_\rho := \{z \in Z : z \rho z_1, \dots, z_n\}.$$

A subset  $Z_0 \subset Z$  is a  $\rho$ -clique iff  $z' \rho z''$  holds for any distinct  $z', z'' \in Z_0$ .

Thus  $\Delta_n^\rho(z_1, \dots, z_n)$  states that the set  $\{z_1, \dots, z_n\}$  can be in a unique way extended to a maximal  $\rho$ -clique (which coincides with  $[z_1, \dots, z_n]_\rho$  in this case), while  $\bar{\Delta}_n^\rho(z_1, \dots, z_n)$  states that this set has at least two distinct extensions to a maximal  $\rho$ -clique.

In view of 2.5 and known properties of Grassmannians defined over projective spaces we have

**Fact 3.1.** *Both  $T_0(R)$  with  $R \in \mathcal{H}_{k+1}$  and  $S_0(Q)$  with  $Q \in \wp_{k-1}(\mathfrak{P})$  are  $\sim^+$ -cliques. (Consequently, if  $Q \subset R$  then the intersection of the corresponding cliques is simply  $\mathfrak{p}_0(Q, R)$ .)*

Let  $X_1, X_2, X_3 \in \wp_k(\mathfrak{H})$  be pairwise distinct. Assume that  $\sim^+(X_1, X_2, X_3)$  holds in  $\mathbf{P}_k(\mathfrak{H})$ . Then two possibilities occur:

- (a) There is  $Q \in \wp_{k-1}(\mathfrak{P})$  such that  $X_1, X_2, X_3 \in S_0(Q)$ , or
- (b) there is  $R \in \mathcal{H}_{k+1}$  with  $X_1, X_2, X_3 \in T_0(R)$ .

In the corresponding cases we have

$$(16) \quad [X_1, X_2, X_3]_{\sim^+} = S_0(Q) \quad \text{in the case (a) \& } \neg(b),$$

$$(17) \quad [X_1, X_2, X_3]_{\sim^+} = T_0(R) \quad \text{in the case (b) \& } \neg(a).$$

Fact 3.1 yields immediately, like in projective geometry (and many others, cf. [5]), the following

**Proposition 3.2.** *Let  $X_1, X_2, X_3 \in \wp_k(\mathfrak{H})$ ,  $X_1 \neq X_2$ , and let  $X_1 \sim^+ \sim^+ X_2$  hold in  $\mathfrak{H}$ . The following formulas define the collinearity relation  $L_0$  of  $\mathfrak{P}$  restricted to  $\mathfrak{H}$ , i.e. the collinearity in  $\bar{\mathfrak{H}}$  ( $L_0(X_1, X_2, X_3)$  iff  $\overline{X_1}, \overline{X_2}, \overline{X_3}$  are collinear in  $\mathfrak{P}$ ):*

$$(18) \quad L_0(X_1, X_2, X_3) \iff \sim^+(X_1, X_2, X_3) \wedge \forall X [X \sim^+ X_1, X_2 \implies X \sim^+ X_3]$$

$$(19) \quad \iff \sim^+(X_1, X_2, X_3) \wedge \exists X', X'' [X', X'' \sim^+ X_1, X_2, X_3 \wedge \neg(X' \sim^+ X'')].$$

Loosely speaking, 3.2 states that  $\bar{\mathfrak{H}}$  is definable in terms of  $\sim^+$ .

To define the lines of  $\mathfrak{H}$  in terms of  $\sim_-$  the above trick is insufficient: the lines of  $\mathfrak{H}$  are intersections of stars and tops, but tops are not  $\sim_-$ -cliques of the form  $[X_1, \dots, X_l]_{\sim_-}$ . Thus we need some more complicated methods to handle the relation  $\sim_-$ .

It is immediate from definition that the following holds.

**Fact 3.3.** *Let  $Q \in \wp_{k-1}(\mathfrak{P})$  and let  $S = S_0(Q) \in S^j$  ( $j = 0, 1, 2$ ). The set  $S$  is a maximal clique of  $\sim_j$ .*

Let  $X_1, X_2, X_3 \in \wp_k(\mathfrak{H})$  be pairwise distinct and let  $Y_i = \overline{X_i}$  for  $i = 1, 2, 3$ . Assume that  $\sim_j(X_1, X_2, X_3)$  holds in  $\mathbf{P}_k(\mathfrak{H})$ . In particular, one of (a) and (b) of 3.1 holds as well. In the case (a) we consider  $Y_1 \cap Y_2 = Q$ . Assume that  $j = 2$ ; thus  $A := Q \cap S \in \wp_{k-1}(\mathfrak{H})$ . Then  $S_0(Q) = S(A)$ , which, by 3.3, is a  $\sim_2 = \sim_-$ -clique. Analogous reasoning can be repeated for  $Q \in \mathcal{E}_{k-1}$  and  $Q \in \mathcal{T}_{k-1}$ . Finally, we get

$$(20) \quad [X_1, X_2, X_3]_{\sim_j} = S_0(Q) \text{ in the case } ((a)) \ \& \ \neg(b).$$

The following is nearly evident

**Fact 3.4.** *The set  $T(B)$  with  $B \in \wp_{k+1}(\mathfrak{H})$  is never a  $\sim_-$ -clique, since it contains a pair of nonmeeting subspaces.*

**Fact 3.5.** *Let  $\wp_{k-1}(\mathfrak{P}) \ni Q \subset R \in \mathcal{H}_{k+1}$ . Then  $S_0(Q) \setminus T_0(R) \neq \emptyset$ .*

**Proof.** It suffices to take any  $b \in S \setminus R$  and put  $Y = b \sqcup Q$ ,  $X := Y \cap S$ .  $\diamond$

**Lemma 3.6.** *Let  $X_1, X_2, X_3 \in \mathbf{p}_0(Q, R) \in \mathcal{G}^j$  for some  $j \in \{0, 2\}$ . There is  $X \in T_0(R) \setminus S_0(Q)$  such that  $X \sim_j X_1, X_2, X_3$ .*

**Proof.** Without loss of generality we can assume that  $X_1, X_2, X_3$  are pairwise distinct. It is seen that the required constructions are performed, in fact, in the quasi hyperbolic geometry determined by  $B = S \cap R$  in the projective space  $R$ , and from this point of view the points of  $\mathfrak{H}$  which take part in the reasoning are hyperplanes of  $R$  (of  $B$ , if one prefers this way of thinking).

$j := 2$ : The hyperplane  $X_1$  divides  $B$  into two open halfspaces  $D^+$  and  $D^-$ . Let  $i \in \{2, 3\}$ . Since  $X_i$  crosses  $X_1$  in  $Q$ , there are points  $a^+_i \in X_i \cap D^+$ ,  $a^-_i \in X_i \cap D^-$ ; clearly,  $a^+_i, a^-_i \notin Q$ . Moreover, there is a point  $a_1 \in X_1 \setminus Q$  which lies between  $a^+_2$  and  $a^-_3$ . We take any  $(k - 2)$ -subspace  $D$  of  $R$  missing the line  $L = \overline{a^+_2, a^-_3}$ , we put  $Y = D \sqcup L$ , and then  $X = Y \cap S$  is a required (see Fig. 2a).

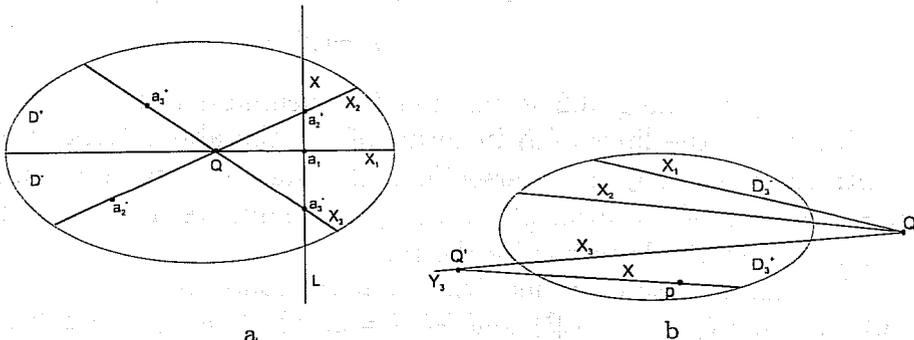


Fig. 2.

$j := 0$ : Every one of the  $X_i$ , as a hyperplane of  $B$ , divides  $B \setminus X_i$  into two open half-spaces  $D^+_i$  and  $D^-_i$ . One can choose  $i_0$  such that  $X_i \subset \subset D^-_{i_0}$  for  $i \neq i_0$ ; without loss of generality we take  $i_0 = 3$ . From 2.2,  $Y_3 = \overline{X_3}$  contains a hyperplane  $Q'$  which is distinct from  $Q$  and misses  $S^\infty$ . We take any point  $p \in D^+_{i_0}$ ; let  $Y$  be the hyperplane of  $R$  through  $p$  and  $Q'$  and  $X = Y \cap S$  (see Fig. 2b). Evidently,  $X \notin S_0(Q)$  (since otherwise we get  $X = X_3$ ), but  $X \in T_0(R)$  and  $X \subset D^+_3$ . Clearly,  $X \sim_0 X_3$ . Since  $X_3$  separates in  $B$  the sets  $X$  and  $X_1 \cup X_2$ , from 2.3 we obtain  $X \sim_0 X_1, X_2$ , as required.  $\diamond$

**Remark 3.7.** The statement 3.6 does not remain valid for  $j = 1$ . It suffices to consider three lines  $X_1, X_2, X_3$  in a (hyperbolic) parallel pencil on a hyperbolic plane. A line  $X$  required in 3.6 yields an asymptotic triangle  $X_1, X_2, X$ , and thus it cannot be parallel to  $X_3$ .

Nevertheless, under assumptions of 3.6 with  $j = 1$  one can find  $X \in T_0(R) \setminus S_0(Q)$  such that  $X \sim_0 X_1, X_2, X_3$  or  $X \sim_1 X_1, X_2, X_3$ .

Slightly generalizing 3.4 we note

**Lemma 3.8.** *Let  $X_1, X_2, X_3 \in \mathcal{P}_k(\mathfrak{H})$  pairwise satisfying  $\sim_j$  belong to  $T(B)$  for some  $B \in \mathcal{P}_{k+1}(\mathfrak{H})$  and  $j \in \{0, 2\}$ . Then  $[X_1, X_2, X_3]_{\sim_j}$  is not a  $\sim_j$ -clique, which means that  $\mathcal{A}_3^{\sim_j}(X_1, X_2, X_3)$  holds.*

**Proof.** Take  $R = \overline{B}$ ,  $Y_i = \overline{X_i}$  for  $i = 1, 2, 3$ , and  $Z = Y_1 \cap Y_2 \cap Y_3$ . Then one of the following holds.

- $Z \in \mathcal{P}_{k-1}(\mathfrak{P})$ ; then  $Y_1, Y_2, Y_3$  lie in one pencil of  $\mathfrak{P}$ , and  $X_1, X_2, X_3 \in \mathfrak{p}_0(Z, B)$ , or
- $Z \in \mathcal{P}_{k-2}(\mathfrak{P})$ .

In the first case, directly from the definition we get  $S_0(Z) \in S^j$ . It suffices to take, in accordance with 3.6 a subspace  $X' \in T_0(R) \setminus S_0(Z)$  such that  $X' \sim_j X_1, X_2, X_3$ , and from 3.5 a subspace  $X'' \in S_0(Z) \setminus T_0(R)$  so,  $X'' \sim_j X_1, X_2, X_3$ . We have  $X' \not\sim^+ X''$ , which closes this part of proof.

In the second case we have to consider two possible values of  $j$ . We write  $Q_i = Y_{j_1} \cap Y_{j_2}$ , where  $\{1, 2, 3\} = \{i, j_1, j_2\}$ . Clearly,  $Q_1 \neq Q_2$ .  $j = 2$ : Let  $X'$  be a subspace of  $\mathfrak{H}$  which: is contained in  $B$ , contains  $X_1 \cap X_2$ , and crosses  $X_3$  in a subspace distinct from  $X_1 \cap X_3, X_2 \cap X_3$ . Put  $Y' = \overline{X'}$  and let  $Y''$  be parallel to  $Y'$  (i.e. formally, let  $Y' \cap Y'' \subset \subset \mathcal{A}^\infty$ ). One can choose  $Y''$  such that  $X'' = Y'' \cap S$  crosses the  $X_i$  (see Fig. 3a). Clearly,  $X' \not\sim_2 X''$ , which is our claim.

$j = 0$ : Every one of the  $X_i$  is a hyperplane in  $B$  (considered as a quasi hyperbolic space) and it divides  $B \setminus X_i$  into two open half-spaces  $D^+_i$  and  $D^-_i$ . One can find  $i_0$  such that  $X_i \subset D^+_{i_0}$  for  $i \neq i_0$ ; without

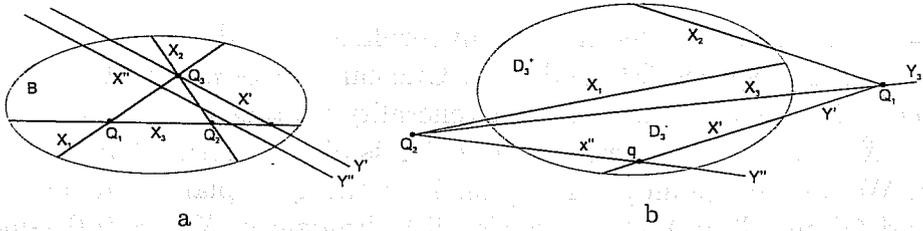


Fig. 3.

loss of generality we can take  $i_0 = 3$ . From the assumptions  $Q_1$  and  $Q_2$  both miss  $S^\alpha$ . We take a point  $q \in D_3^-$ ; let  $Y' = Q_1 \sqcup q$ ,  $Y'' = Q_2 \sqcup q$ , and  $X', X''$  be the respective restrictions of  $Y', Y''$  to  $B$  (cf. Fig. 3b). Then, clearly,  $X', X'' \sim_0 X_3$  and  $X' \not\sim_0 X'', X' \neq X''$ . It is seen that  $X', X''$  are contained in  $D_3^-$  and therefore  $X_3$  separates  $X' \cup X''$  and  $X_1 \cup X_2$ . From 2.3 we conclude with  $X', X'' \sim_0 X_1, X_2$ .  $\diamond$

Thus 3.3, 3.1, and 3.8 yield immediately the following characterizations for  $j = 0, 2$ :

- $\Delta_3^{\sim j}(X_1, X_2, X_3)$  iff  $X_1, X_2, X_3$  are not collinear in  $\overline{\mathfrak{H}}$ , but they are in a star in  $S^j$ ;
- $\Delta_3^{\sim j}(X_1, X_2, X_3)$  iff  $X_1, X_2, X_3$  pairwise satisfy  $\sim_j$  and lie in a top either spanning (projectively) this top, or being collinear;

$$(21) \quad S^j = \{[X_1, X_2, X_3]_{\sim_j} : \neq (X_1, X_2, X_3), \Delta_3^{\sim j}(X_1, X_2, X_3)\}.$$

As a consequence of the above observations we get

**Proposition 3.9.** *Let  $X_1, X_2, X_3 \in \wp_k(\mathfrak{H})$ ,  $X_1 \neq X_2$ , and let  $X_1 \sim_j X_2$  hold in  $\mathfrak{H}$  for  $j \in \{0, 2\}$ . The following formula defines the collinearity relation  $L_0$  of  $\overline{\mathfrak{H}}$*

$$(22) \quad L_0(X_1, X_2, X_3) \iff \sim_j(X_1, X_2, X_3) \wedge \exists X', X'' [X', X'' \sim_j X_1, X_2, X_3 \wedge \neg(X' \sim_j X'') \wedge \Delta_3^{\sim j}(X_1, X_2, X')].$$

**Proof.** Let  $X_1, X_2 \in L$ , where  $L$  is the restriction to  $\mathfrak{H}$  of a projective pencil  $\mathbf{p}(Q, R)$ ,  $Q \in \mathcal{E}_{k-1} \cup \mathcal{H}_{k-1}$ , and  $R \in \mathcal{H}_{k+1}$ . Assume the right-hand side of (22). In view of 3.3 and 3.8,  $X' \in S_0(Q)$ , and thus  $X'' \in T_0(R)$ . Thus, finally,  $X_3 \sim^+ X_1, X_2, X', X''$  gives  $X_3 \in S_0(Q) \cap T_0(R) = L$ .

Next, let  $X_3 \in L$ . From 3.5 we take  $X' \in S_0(Q) \setminus T_0(R)$ ; then  $X' \sim_j X_1, X_2, X_3$  and  $\Delta_3^{\sim j}(X_1, X_2, X')$ . Next, from 3.6 we find

$X'' \in T_0(R) \setminus S_0(Q)$ , such that  $X'' \sim_j X_1, X_2, X_3$ , clearly,  $\neg(X' \sim_j X'')$ .  $\diamond$

Recall that  $\sim_2 = \sim_-$ ; therefore the formula (22) defines (with  $j = 2$ ) simply the collinearity relation  $\mathbf{L}$  of  $\mathfrak{H}$  in terms of its adjacency.

Finally, let us consider the following

**Lemma 3.10.** *Let  $X_1 \neq X_2$  and  $X_1 \sim^+ X_2$ . We have (cf. 2.6)*  
(23)

$$X_1 \sim_2 X_2 \iff \exists X_3 [\Delta_3^{\sim^+}(X_1, X_2, X_3) \wedge \langle [X_1, X_2, X_3]_{\sim^+}, \mathbf{L}_0 \rangle \models \mathbf{V}]$$

(the formula like  $\mathfrak{B} \models \Phi$  states that  $\mathfrak{B}$  is a model of  $\Phi$ ).

Since the Veblen axiom  $\mathbf{V}$  is elementarily expressible<sup>3</sup>, we get that the relation  $\sim_- = \sim_2$  is definable in terms of  $\sim^+$ , and the families  $\mathcal{G}^2$  and  $\mathcal{G}^1 \cup \mathcal{G}^0$  are definable as well.

From 3.2, 3.9 and 3.10 we read that the structure  $\mathfrak{H}$  is definable in terms of  $\sim^+$  and in terms of  $\sim_-$ .

**Remark 3.11.** Assume that  $\mathfrak{H}$  is a hyperbolic space. Let  $X_1 \neq X_2$  and  $X_1 \sim^+ X_2$ . We have (cf. 2.7)  
(24)

$$X_1 \sim_2 X_2 \iff \exists X_3 [\Delta_3^{\sim^+}(X_1, X_2, X_3) \wedge \langle [X_1, X_2, X_3]_{\sim^+}, \mathbf{L}_0 \rangle \models \mathbf{P}],$$

$$X_1 \sim_1 X_2 \iff \exists X_3 [\Delta_3^{\sim^+}(X_1, X_2, X_3) \wedge \langle [X_1, X_2, X_3]_{\sim^+}, \mathbf{L}_0 \rangle \models \mathbf{A}],$$

$$X_1 \sim_0 X_2 \iff \exists X_3 [\Delta_3^{\sim^+}(X_1, X_2, X_3) \wedge \langle [X_1, X_2, X_3]_{\sim^+}, \mathbf{L}_0 \rangle \models \mathbf{H}].$$

Since the geometries  $\mathbf{P}$ ,  $\mathbf{A}$ , and  $\mathbf{H}$  are elementarily distinguishable we get that the relations  $\sim_2$ ,  $\sim_1$  and  $\sim_0$  are definable in terms of  $\sim^+$ , and the families  $\mathcal{G}^2$ ,  $\mathcal{G}^1$ , and  $\mathcal{G}^0$  are definable as well.

Let us quote the known result, fundamental in the context of the considered geometries.

**Fact 3.12** ([35]). *The structures  $\mathfrak{H}$  and  $\langle P, \mathcal{L}, S \rangle$  are mutually definable, where  $\mathfrak{B} = \langle P, \mathcal{L} \rangle$ . Consequently, the automorphism groups of  $\mathfrak{H}$  and of  $\langle P, \mathcal{L}, S \rangle$  are isomorphic. In particular,  $\text{Aut}(\mathfrak{H})$  is the group of the collineations of  $\mathfrak{B}$  which leave the set  $S$  invariant.*

As a nearly immediate consequence we get

**Theorem 3.13.** *The following three groups: of the automorphisms of  $\sim_-$ , of the automorphisms of  $\sim^+$ , and of the automorphisms of  $\mathbf{P}_k(\mathfrak{H})$  coincide. The group in question consists of the automorphisms of  $\mathfrak{H}$  acting on its  $k$ -subspaces.*

<sup>3</sup>In the language of the ternary collinearity relation  $\mathbf{L}$  the axiom  $\mathbf{V}$  can be (equivalently) read as follows:  $\mathbf{L}(p, x_1, x_2) \wedge \mathbf{L}(p, y_1, y_2) \wedge \neg \mathbf{L}(p, x_1, y_1) \wedge \neg \mathbf{L}(p, x_2, y_2) \implies \exists q [\mathbf{L}(q, x_1, y_1) \wedge \mathbf{L}(q, x_2, y_2)]$ .

**Proof.** Let  $F$  be a bijection of  $\wp_k(\mathfrak{H})$ . If  $F$  reserves the relation  $\sim^+$ , from 3.10 we get that  $F$  preserves  $\sim_-$  as well, so we can restrict ourselves to this case.

Let  $F$  preserve  $\sim_-$ ; from 3.9,  $F$  preserves maximal cliques i.e. it preserves the family of the sets  $S_0(Q)$  with  $Q \in \wp_{k-1}(\mathfrak{H})$ . Thus  $F$  determines a bijection  $F'$  of  $\wp_{k-1}(\mathfrak{H})$  such that  $(F', F)$  is an automorphism of the structure  $\langle \wp_{k-1}(\mathfrak{H}), \wp_k(\mathfrak{H}), \subset \rangle$ . Therefore  $F'$  preserves the relation  $\sim^+$  in the structure  $\mathbf{P}_{k-1}(\mathfrak{H})$ . From 3.2 we infer that  $F'$  preserves pencils, and from the classification 3.10 we obtain that  $F' \in \text{Aut}(\mathbf{P}_{k-1}(\mathfrak{H}))$ . In particular,  $F'$  preserves  $\sim_-$  in  $\wp_{k-1}(\mathfrak{H})$ . Continuing, we come to an automorphism  $f$  of  $\mathfrak{H}$ , which proves that  $F$  is determined by  $f$ .  $\diamond$

Similarly, we can characterize the groups  $\text{Aut}(\langle \wp_k(\mathfrak{H}), \mathcal{S} \rangle)$  and  $\text{Aut}(\mathfrak{H})$ .

An elementary counterpart of 3.13 is

**Theorem 3.14.** *Each one of the relations  $\sim^+$  and  $\sim_-$  is sufficient to express the geometry of  $\mathfrak{H}$ . Consequently, the geometry of  $\mathfrak{H}$  can be also formalized in the language of  $\mathbf{P}_{\mathfrak{H}}(k)$  and in the language of  $\mathbf{P}_{\mathfrak{H}}(k)$ .*

**Proof** (a sketch). It suffices to note that  $\mathbf{P}_0(\mathfrak{H}) \cong \mathfrak{H}$  and  $\mathbf{P}_{k-1}(\mathfrak{H})$  is interpretable in  $\mathbf{P}_k(\mathfrak{H})$ . Indeed, the points of  $\mathbf{P}_{k-1}(\mathfrak{H})$  can be identified with equivalence classes of the triples  $(X_1, X_2, X_3) \in (\wp_k(\mathfrak{H}))^3$  such that  $\Delta_3^{\sim-}(X_1, X_2, X_3)$  holds, two such triples  $(X_1, X_2, X_3), (X'_1, X'_2, X'_3)$  being equivalent when  $X_1, X_2, X_3 \sim_- X'_1, X'_2, X'_3$ . Pairs of points which are  $\sim^+$ -adjacent in  $\mathbf{P}_{k-1}(\mathfrak{H})$  correspond to pairs of equivalence classes of triples of the form  $(X_1, X_2, X_3), (X_1, X'_2, X'_3)$ . With 3.2 and 3.10 we recover the collinearity of  $\mathbf{P}_{k-1}(\mathfrak{H})$ .  $\diamond$

Then, the natural question arises how to define (if it is possible) the relation  $\sim^+$  in terms of  $\sim_j$  for  $j \in \{0, 1, 2\}$ . The case of  $\sim_- = \sim_2$  is relatively simple:

**Proposition 3.15.** *Let  $X_1, X_2 \in \mathbf{p}_0(Q, R) \in \mathcal{G}^2$ ,  $X_1 \neq X_2$ , and  $X_3 \in \mathcal{H}_k$ . We have  $R = \overline{X_1 \sqcup X_2}$  and*

$$(25) \quad \begin{aligned} X_3 \in T_0(R) \iff & L'(X_1, X_2, X_3) \vee (\Delta_3^{\sim-}(X_1, X_2, X_3) \wedge \\ & \wedge \exists X', X'' [L_0(X_1, X_2, X') \wedge L_0(X_1, X_2, X'') \wedge \\ & \wedge X' \neq X'' \wedge X', X'' \sim_- X_3]), \end{aligned}$$

where  $L'(X', X'', X''')$  means that  $L_0(X', X'', X''')$  or  $X' = X'''$  or  $X'' = X'''$ . Consequently, the following formula defines  $\sim^+$  in terms of  $\sim_2$ :

$$(26) \quad X'_3 \sim^+ X''_3 \iff \exists X_1, X_2 [X_1 \sim_2 X_2 \wedge X_1 \neq X_2 \wedge X'_3, X''_3 \in \overline{T_0(X_1 \sqcup X_2)}].$$

**Proof.** Clearly, if  $L_0(X_1, X_2, X_3)$ , then  $X_3 \subset R$ . The condition  $\mathcal{A}_3^{\sim-}(X_1, X_2, X_3)$  assures that  $X_3 \notin S_0(Q)$ . Then right-hand side of (25) gives (since  $X_3 \sim^+ X', X''$ ) that  $X_3 \in (S_0(Q) \cup T_0(R)) \setminus S_0(Q)$ . Conversely, if  $Q \not\subset \overline{X_3} \subset R$ , we take any two distinct  $a', a'' \in X_3 \setminus Q$  and put  $X' = Q \sqcup a', X'' = Q \sqcup a''$ .

Finally, the validity of (26) is evident.  $\diamond$

Definition of  $\sim^+$  given in 3.15, though long, has a clear geometrical motivation. One can verify that the following, much shorter formula is also valid in  $\mathfrak{H}$  for distinct  $X_1, X_2$  (see Fig. 4):

$$(27) \quad X_1 \sim^+ X_2 \iff \exists X', X'' [\mathcal{A}_3^{\sim-}(X', X'', X_1) \wedge \mathcal{A}_3^{\sim-}(X', X'', X_2)].$$

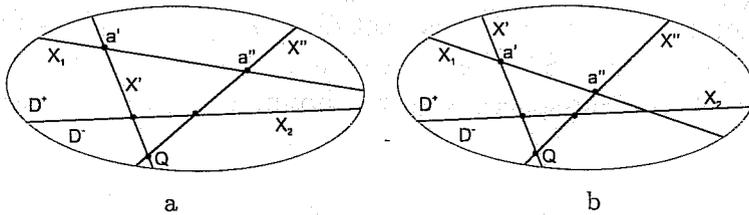


Fig. 4.

#### 4. Remarks on hyperparallelism of subspaces of a quasi hyperbolic space

The problem to characterize the adjacency  $\sim^+$  in terms of the hyperparallelism  $\sim_0$  is, in general, more complex. Note, first of all, that  $\mathfrak{H}$  may contain a subspace  $X$  such that no subspace of  $\mathfrak{H}$  is hyperparallel to  $X$  – this happens when  $X^\infty \subset \mathcal{A}^\infty$ ; let  $\mathcal{K}$  stand for the class of such subspaces. Let  $\Phi$  be a formula in the prenex normal form, formulated in the language of  $\sim_0$ . If the free variables of  $\Phi$  are interpreted as elements of  $\mathcal{K}$  then every one of its atomic subformulas containing these variables is a contradiction and therefore the formula  $\Phi$  is true or false independently of a particular valuation of its free variables in  $\mathcal{K}$ . Consequently, it is impossible to define  $X_1 \sim^+ X_2$  in terms of  $\sim_0$  for  $X_1, X_2 \in \mathcal{K}$ . To avoid this complication we assume the contrary: for

every  $X \in \wp(\mathfrak{H})$  there is  $X'$  with  $X \sim_0 X' \neq X$ . In a more geometrical language this property can be expressed by the phrase  $\mathfrak{H}$  has not a Euclidean direction (cf. [26]) and, analytically, this is assured by the following requirement:

$$(28) \quad S^\alpha \cap \mathcal{A}^\infty = \emptyset \text{ in a suitable projective representation of } \mathfrak{H}.$$

We shall see that (28) suffices to repeat (a slightly more sophisticated variant of) the reasoning of 3.15 for the relation  $\sim_0$  instead of  $\sim_2$ . What we need to do is to prove the fact that for a pair  $X_1, X_2$  of non hyperparallel subspaces contained in a top  $\mathcal{X}$  there is a hyperparallel pencil which contains two pairs of elements, each pair completing corresponding  $X_i$  to a hyperparallel triangle contained in  $\mathcal{X}$ . Let us begin with simple observations.

**Lemma 4.1.** *Let  $Q \in \mathcal{E}_{k-1}$ ,  $Q \subset R \in \mathcal{H}_{k+1}$ , and  $X' \in \mathbf{p}_0(Q, R)$  such that  $X' \sim_0 X$  for some  $X \in \wp_k(\mathfrak{H})$  with  $X \subset R$ . Then there is  $X'' \in \mathbf{p}_0(Q, R)$  such that  $X' \neq X'' \sim_0 X$ .*

**Proof.** Let  $Y = \overline{X}$ ,  $Y' = \overline{X'}$ . We set  $Q' = Y \cap Y'$  and  $U = Q \cap Y$ ; by the assumptions  $Q' \cap S^\alpha = \emptyset$ . In the affine space  $Y \setminus \mathcal{A}^\infty$  the set  $X^\alpha$  lies in one of the half-spaces with the boundary  $Q'$ . It suffices to take

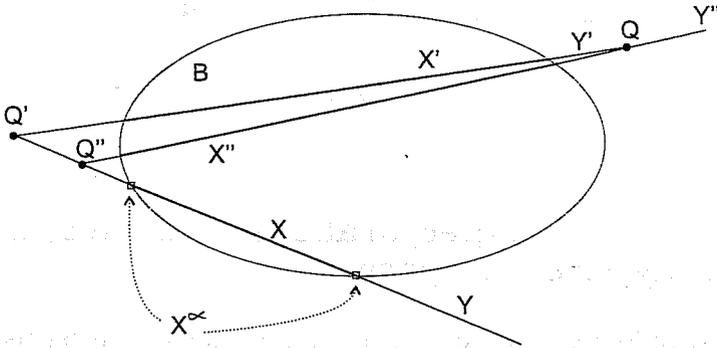


Fig. 5.

a hyperplane  $Q''$  of  $Y$  through  $U$  “between”  $S^\alpha$  and suitable halfspace of  $Q' \setminus U$ ; after that we put  $Y'' = Q'' \sqcup Q$  and  $X'' = Y'' \cap S$  (cf. Fig. 5).  $\diamond$

**Lemma 4.2.** *For every  $B \in \wp_{k+1}(\mathfrak{H})$  and distinct  $X_1, X_2 \in \wp_k(\mathfrak{H})$  contained in  $B$  such that  $X_1 \not\sim_0 X_2$  there are  $X'_1, X'_2 \in \wp_k(\mathfrak{H})$  contained in  $B$  such that  $X'_1 \sim_0 X'_2$  and  $X_i \sim_0 X'_i$  for  $i = 1, 2$ .*

**Proof.** Set  $R = \overline{B}$ ,  $Y_i = \overline{X_i}$ , and  $U = Y_1 \cap Y_2$ . For  $i = 1, 2$  we consider the family  $\mathcal{X}_i = \{Y \in \mathcal{H}_k : Y^\infty = Y_i^\infty\}$  of the hyperplanes of  $R$  which are affine-parallel to  $Y_i$ . If there are  $Y'_1, Y'_2$  such that  $Y'_i \in \mathcal{X}_i$  and  $Y'_1 \cap Y'_2 \cap S^\alpha = \emptyset$  we are done (see Fig. 6a).

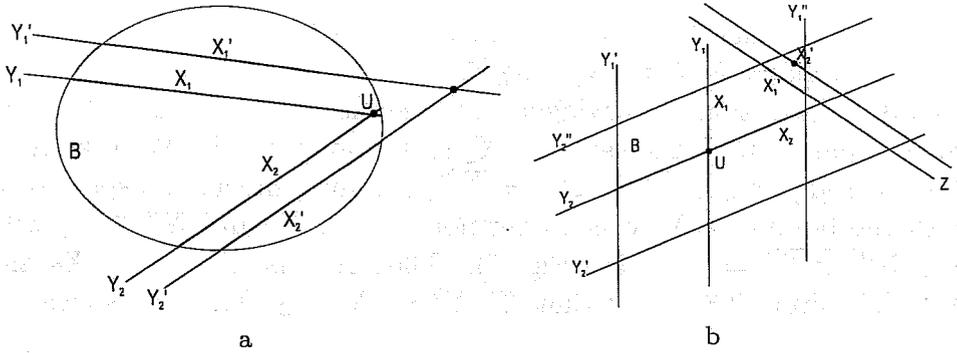


Fig. 6.

Suppose the contrary; then there are in  $R$  two open affine stripes  $W_1, W_2$  such that the boundary hyperplanes of  $W_i$  are  $Y_i', Y_i''$  in  $\mathcal{X}_i$  for  $i = 1, 2$ , and  $S$  is contained in  $W_1 \cap W_2$  in such a way that through every point in  $W_1 \cap W_2$  there passes a subspace parallel to  $U$  which crosses  $S$ . One can find in  $R$  a hyperplane  $Z$  which is affine-parallel to  $U$  such that  $Z \in \mathcal{H}_k$  and  $Z \cap Y_i$  is outside  $W_i$  for  $i = 1, 2$ . Set  $X_1' = Z \cap B$ . Then  $X_1 \cup X_2$  is contained in a halfspace of  $B$  with the boundary  $X_1'$  and the completing halfspace contains  $X_2' \in \mathcal{H}_k$  such that  $Z$  and  $X_2'$  are affine parallel (cf. Fig. 6b). It is seen that  $X_1', X_2'$  satisfy our claim.  $\diamond$

As an immediate consequence of 4.1 and 4.2 we obtain

**Proposition 4.3.** *Under the assumptions of (28) the following formula defines in  $\mathfrak{H}$  the adjacency  $\sim^+$  in terms of  $\sim_0$ :*

$$(29) \quad X_1 \sim^+ X_2 \iff X_1 \sim_0 X_2 \vee \exists X_1', X_1'', X_2', X_2'' [X_1' \neq X_1'' \wedge \wedge X_2' \neq X_2'' \wedge L_0(X_1', X_1'', X_2') \wedge \wedge L_0(X_1', X_1'', X_2'') \wedge \mathfrak{A}_3^{\sim_0}(X_1, X_1', X_1'') \wedge \mathfrak{A}_3^{\sim_0}(X_2, X_2', X_2'')].$$

Finally, from 3.13 and 3.14 we conclude with the following.

**Theorem 4.4.** *Under the assumptions of (28) the group of the automorphisms of  $\sim_0$  consists of the automorphisms of  $\mathfrak{H}$  acting on its  $k$ -subspaces.*

*The hyperparallelism  $\sim_0$  can be used to express the geometry of  $\mathfrak{H}$ .*

One can note that if  $\mathfrak{H}$  is a hyperbolic space, then also (27) remains valid with  $\sim_-$  replaced by  $\sim_0$ . This result can be obtained in a slightly more general case as well. Let us say that a quasi hyperbolic space  $\mathfrak{H}$  is convex if for any two its hyperplanes  $X_1, X_2$  there is a hyperplane  $X$  which separates  $X_1 \cup X_2$  and a point of  $\mathfrak{H}$ .

**Proposition 4.5.** *Assume that every  $(k + 1)$ -subspace of  $\mathfrak{H}$  yields a*

convex quasi hyperbolic space. Then (27) with  $\sim_-$  replaced by  $\sim_0$  is valid for any distinct  $X_1, X_2 \in \wp_k(\mathfrak{H})$ .

**Proof.** Implication  $\Leftarrow$  is evident. Assume that  $X_1, X_2 \subset B \in \wp_{k+1}(\mathfrak{H})$ , let  $X$  separate in  $B$  the set  $X_1 \cup X_2$  and a point  $p \in B$ . We take  $X' \in \mathcal{H}_k$  through  $p$  such that  $\overline{X'}^\infty = \overline{X}^\infty$ , a point  $q$  in this halfspace of  $B$  with the boundary  $X'$  which does not contain  $X$ , and  $X'' \in \mathcal{H}_k$  with  $q \in X''$ ,  $\overline{X''}^\infty = \overline{X}^\infty$  (see Fig. 7). Then  $X$  separates  $X_1 \cup X_2$  and  $X' \cup X''$ . From 2.3 we get that  $X', X'' \sim_0 X_1, X_2$ . The rest is clear.  $\diamond$

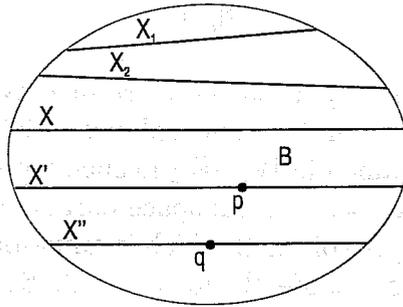


Fig. 7.

However, even if  $\mathfrak{H}$  is relatively regular, e.g. if  $S^\alpha$  is a polytope, this modification of (27) may fail. It suffices to consider a 3-cube and two its diagonal hyperplanes  $X_1, X_2$ : there is no  $X'$  with  $X' \sim_0 X_1, X_2$ .

### 5. Parallelism of subspaces of a hyperbolic space

Next, we pass to the groups  $\text{Aut}(\langle \wp_k(\mathfrak{H}), \sim_1 \rangle)$  and  $\text{Aut}(\langle \wp_k(\mathfrak{H}), \mathcal{G}^1 \rangle)$ . From now on we assume that  $\mathfrak{H}$  is a hyperbolic space, i.e.  $S^\alpha$  is a projective sphere in the space  $\mathfrak{P}$ . While in the preceding sections we have used intensively the projective apparatus, now we shall make use of some standard notions of Möbius geometry.

Let  $\mathfrak{M} = \langle S^\alpha, \mathcal{C} \rangle$  be the Möbius geometry on  $S^\alpha$ , where  $\mathcal{C} = \{Y \cap S^\alpha : Y \in \mathcal{H}_2\}$ . The class of subspaces of  $\mathfrak{M}$  coincides with the set  $\mathcal{M} = \{Y \cap S^\alpha : Y \in \mathcal{H}\}$  so as  $\dim(Y \cap S^\alpha) = \dim(Y) - 1$  and the map

$$\bigcup \{ \wp_m(\mathfrak{H}) : 2 \leq m \leq n \} \ni X \mapsto \overline{X} \cap S^\alpha = X^\alpha$$

is a bijection. Let  $\mathcal{M}_m = \{E \in \mathcal{M} : \dim(E) = m\}$ .

Let us note the following equivalence

$X_1 \sim_- X_2 \iff |X_1^\alpha \cap X_2^\alpha| = 2$ , for every  $X_1, X_2 \in \mathcal{H}_2$ .  
 Therefore, up to the identification  $X \mapsto X^\alpha$ , the adjacency of  $\mathbf{P}_2(\mathfrak{H})$  is definable in  $\mathfrak{M}$ . This observation, together with results of Sect. 3 reproves the known

**Fact 5.1.** *The following structures are mutually definable:*

$$\mathfrak{H}, \mathfrak{M}, \text{ and } \langle P, \mathcal{L}, S^\alpha \rangle.$$

*Consequently, they have isomorphic automorphism groups. In particular,  $\text{Aut}(\mathfrak{H})$  is the group of the collineations of  $\mathfrak{P}$  which leave the sphere  $S^\alpha$  invariant.*

To stress the analogy with ordinary hyperbolic geometry we write from this moment  $X' \parallel X''$  instead of  $X' \sim_1 X''$ . Our goal in this section is to prove that the hyperbolic parallelism  $\parallel$  of subspaces of dimension  $k > 1$  is sufficient to express hyperbolic geometry and to this aim, in view of our previous results, it suffices either to define  $\sim^+$  in terms of  $\parallel$  (for  $k < n - 1$ , cf. global assumption (5)) or directly interpret  $\mathfrak{H}$  in terms of  $\parallel$  (when  $k = n - 1$ ).

Letters  $E, F, G$  stand for elements of  $\mathcal{M}$ ; let  $p \in S^\alpha$ , then  $E \mid_p F$  means that  $E$  and  $F$  are tangent in the point  $p$ . We write  $E \parallel F$  if  $E \mid_p F$  for some  $p$ . This terminology is well founded, since elements  $E, F$  are spheres on  $S^\alpha$ . Let us recall the following well known facts

**Fact 5.2.** *Let  $X_1 \parallel X_2$ ,  $Q = \overline{X_1} \cap \overline{X_2}$ ,  $R = \overline{X_1} \sqcup \overline{X_2}$ , and  $E_i = X_i^\alpha$ . Finally, let  $Q$  be tangent to  $S$  at the point  $p$ . For arbitrary subspace  $X_3$  of  $\mathfrak{H}$  we have*

$$(30) \quad X_3 \in S_0(Q) \iff X_3^\alpha \mid_p E_1,$$

$$(31) \quad X_3 \in p_0(Q, R) \iff X_3^\alpha \subset R \cap S^\alpha \wedge X_3^\alpha \mid_p E_1.$$

**Fact 5.3.** *Let  $p \in S^\alpha$ . The stereographical projection of  $S^\alpha$  (the derived space of  $\mathfrak{M}$  at  $p$ ) yields an Euclidean space  $\mathfrak{M}_p$  such that the elements of the set  $\mathcal{M}$  become the subspaces and the subspheres of  $\mathfrak{M}_p$ , those passing through  $p$  correspond to the subspaces, and the tangency in  $\mathfrak{M}$  corresponds to the union of the parallelism and the tangency in  $\mathfrak{M}_p$ .*

Let us begin with the trivial

**Lemma 5.4.** *The relation  $\parallel$  in the set  $\mathcal{P}_1(\mathfrak{H})$  is insufficient to express hyperbolic geometry.*

**Proof.** Every line  $L$  of  $\mathfrak{H}$  is uniquely determined by a pair  $p_1, p_2$  of points on  $S^\alpha$  (its ends) such that  $L = \overline{p_1, p_2}$ . It suffices to take any bijection  $f$  of  $S^\alpha$  and put  $f'(\overline{p_1, p_2}) = \overline{f(p_1), f(p_2)}$ ; then  $f'$  becomes an automorphism of the relation  $\parallel$ . Clearly,  $f$  needs not to be projective and thus  $f'$  may not preserve copunctuality of lines.  $\diamond$

Note that 3.8 is not valid for  $j = 1$  and  $k = 1$ . Indeed, if the lines  $X_1, X_2, X_3$  form an asymptotic triangle, then  $[X_1, X_2, X_3]_{\parallel} = \{X_1, X_2, X_3\}$  is a  $\sim_1$ -clique. Therefore, some slightly more sophisticated techniques must be used here to determine and distinguish stars and tops. In view of 5.4, in the sequel we assume that

$$(32) \quad 1 < k \leq n - 1.$$

Under assumption of (32) we can prove an analogue of 3.8:

**Lemma 5.5.** *Let  $X_1, X_2, X_3 \in \wp_k(\mathfrak{H})$  lie in some top  $T_0(R)$  and  $E_i = X_i^\alpha$  for  $i = 1, 2, 3$ . Assume that  $X_1, X_2, X_3$  are pairwise parallel and do not belong to one star (equivalently: they are not collinear in  $\mathfrak{P}$ , or: the  $E_i$  are pairwise tangent in pairwise distinct points). Then the set  $[X_1, X_2, X_3]_{\parallel}$  is not a  $\parallel$ -clique and thus  $\mathfrak{A}_{\parallel}^{\#}(X_1, X_2, X_3)$  holds.*

**Proof.** Set  $D = (R \cap S)^\alpha$ ,  $Y_i := \overline{X_i}$ , and  $Q_{i_1} = Y_{i_2} \cap Y_{i_3}$ , where  $\{i_1, i_2, i_3\} = \{1, 2, 3\}$ . Let  $Q_i$  be tangent to  $S^\alpha$  in the point  $q_i$  (see Fig. 8a). From assumptions, the subspaces  $Q_i$  are pairwise distinct. Suppose that  $q_{i_1} = q_{i_2}$ ; then both  $Q_{i_1}$  and  $Q_{i_2}$  are tangent to  $S^\alpha$  in  $q_{i_1}$  and thus  $Y_{6-(i_1+i_2)} = Q_{i_1} \sqcup Q_{i_2}$  lies in the hyperplane of  $\mathfrak{P}$  tangent to  $S^\alpha$ . Since  $X_{6-(i_1+i_2)} \subset Y_{6-(i_1+i_2)}$ , this yields a contradiction. Thus  $q_1, q_2, q_3$  are pairwise distinct as well and, consequently,  $q_{i_1} \notin Q_{i_2}$  for  $i_1 \neq i_2$ . In the Euclidean space  $\mathfrak{E}$  derived from  $D$  at  $q_1$  the (Möbius) subspaces  $E_2$  and  $E_3$  become two parallel hyperplanes, and  $E_1$  becomes a sphere tangent to them both.

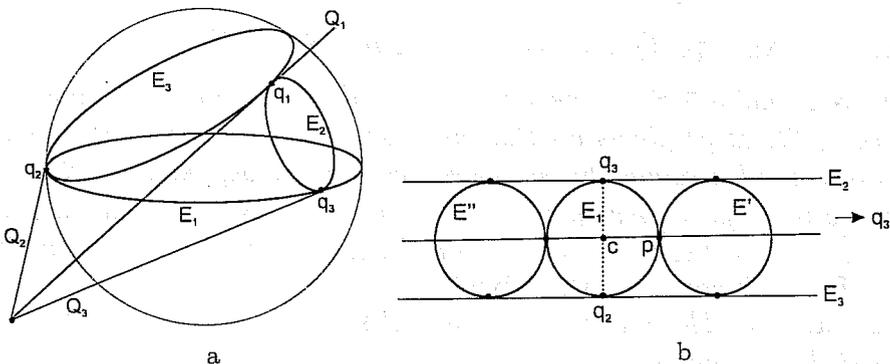


Fig. 8.

We take a point  $p$  of  $\mathfrak{E}$  on the intersection of  $E_1$  and the hyperplane of  $\mathfrak{E}$  parallel to  $E_2$  through the center  $c$  of  $E_1$ . Let  $E'$  be symmetric to  $E_1$  wrt.  $p$ , and  $E''$  be symmetric to  $E'$  wrt.  $c$  (cf. Fig. 8b). It is seen

that  $E_1, E_2, E_3 \mid E', E''$ , but  $\neg E' \mid E''$ . Finally, we take  $X', X'' \in \wp(\mathfrak{H})$  such that  $X'^\alpha = E'$  and  $X''^\alpha = E''$ ; then  $X', X'' \in [X_1, X_2, X_3]_{\parallel}$  and  $X' \not\parallel X''$ .  $\diamond$

**Proposition 5.6.** *The following formula defines the relation  $\sim^+$  in the family  $\wp_k(\mathfrak{M})$  (comp. (27))*

$$(33) \quad X_1 \sim^+ X_2 \iff \exists X', X'' [\Delta_3^{\parallel}(X', X'', X_1) \wedge \Delta_3^{\parallel}(X', X'', X_2)].$$

**Proof.** Assume that there are  $X', X''$  as required in (33). Then  $X', X'' \in \mathbf{p}_0(Q, R)$  for some  $R \in \mathcal{H}_{k+1}$ . Set  $Y_i = \overline{X_i}$ ,  $Y' = \overline{X'}$ , and  $Y'' = \overline{X''}$ . By the assumptions,  $X_i \parallel X', X''$  so,  $Y_i \sim Y', Y''$ . Since  $X', X'', X_i$  do not yield a  $\parallel$ -clique,  $Y', Y'', Y_i$  do not yield a star (cf. 3.3, and (20)) and thus  $Y_i \subset R$ . Thus  $X_i \in T_0(R)$ .

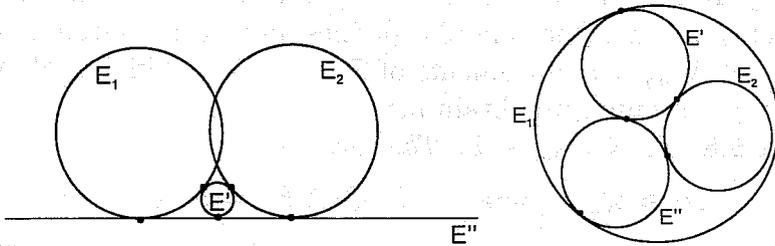


Fig. 9.

Conversely, let  $X_1 \sim^+ X_2$ ; let  $X$  be the  $(k + 1)$ -subspace which contains  $X_1, X_2$ . Set  $E_i = X_i^\alpha$  and  $D = X^\alpha$ ; let  $p \in D \setminus (E_1 \cup E_2)$ . In the derived Euclidean space of  $D$  at  $p$  the  $E_i$  become hyperspheres; clearly there are at least two other hyperspheres  $E', E''$  tangent to the given such that  $E', E''$  are tangent as well. One can choose  $E', E''$  in such a way that the corresponding points of tangency of the spheres  $E_1, E_2, E_3, E', E''$  are pairwise distinct (see Fig. 9). Finally, we take  $X', X''$  such that  $E' = X'^\alpha$  and  $E'' = X''^\alpha$ . From 5.5 we get  $\Delta_3^{\parallel}(X', X'', X_i)$ , as required.  $\diamond$

Now we pass to the case  $k = n - 1$ . Here we follow some ideas of [28], [18], and [22] how to handle the tangency of hyperspheres in metric geometries. In fact, we observe that the notion of a tangent pencil can be expressed elementarily in terms of the tangency in Möbius geometry. In view of 3.3, 3.8, and 5.2, every star  $S_0(Q)$  is determined by a triple  $X_1, X_2, X_3$  of subspaces with  $\Delta_3^{\parallel}(X_1, X_2, X_3)$  which, in this case yield a (parallel) pencil. Moreover,  $S_0(Q)$  determines a point  $p \in S^\alpha$  such that  $Q$  is tangent to  $S$  at  $p$ . Recall:  $n_0 := \dim(\mathfrak{M}) = n - 1$ . If  $p \in S^\alpha$

and  $p \in E \in \mathcal{M}_{n_0-1}$  we write  $\mathbf{p}(p, E)$  for the corresponding tangent pencil of  $\mathfrak{M}$ :  $\mathbf{p}(p, E) = \{F \in \mathcal{M}_{n_0-1} : E \mid_p F\}$ . As already noted (cf. 5.5),  $E_1, E_2, E_3$  are in a tangent pencil if  $\Delta_3 \mid (E_1, E_2, E_3)$  holds.

**Lemma 5.7.** *Let  $p \in S^\alpha$ ,  $p \in E \in \mathcal{M}_{n_0-1}$ , and  $F \in \mathcal{M}_{n_0-1}$ . Then*

$$(34) \quad p \in F \iff F \in \mathbf{p}(p, E) \vee \neg \exists F' \in \mathbf{p}(p, E) [F \mid F'].$$

**Proof.** It suffices to consider the stereographical projection  $\mathfrak{M}_p$ , which makes  $\mathbf{p}(p, E)$  a pencil  $\mathcal{X}$  of parallel hyperplanes. If  $p \notin F$ , then  $F$  corresponds to a hypersphere in  $\mathfrak{M}_p$  and thus there is in  $\mathcal{X}$  a hyperplane tangent to  $F$ . If  $p \in F$ , then  $F$  corresponds to a hyperplane, which either is in  $\mathcal{X}$ , or cannot be parallel to any element of  $\mathcal{X}$ .  $\diamond$

Let  $\mathcal{I}$  be the set of all the tangent pencils  $\mathbf{p}(p, E)$  with  $E \in \mathcal{M}_{n_0-1}$ . Recall that the set  $\mathcal{I}$  corresponds under the map  $X \mapsto X^\alpha$  to the set  $\mathcal{G}_{n-1}^1$ . The formula (34) defines, in fact, a relation  $\Lambda$  between elements of  $\mathcal{M}_{n_0-1}$  and elements of  $\mathcal{I}$ :  $\mathbf{p}(p, E) \Lambda F$  iff  $p \in F$ . With an elementary reasoning we obtain now

**Lemma 5.8.** *Let  $\mathcal{Q}_1, \mathcal{Q}_2 \in \mathcal{I}$ . The formula*

$$(35) \quad \mathcal{Q}_1 \approx \mathcal{Q}_2 \iff \forall F [\mathcal{Q}_1 \Lambda F \iff \mathcal{Q}_2 \Lambda F]$$

*defines an equivalence relation, whose equivalence classes correspond to the points of  $\mathfrak{M}$  (clear:  $\mathbf{p}(p_1, E_1) \approx \mathbf{p}(p_2, E_2)$  iff  $p_1 = p_2$ ). Then the relation  $[\mathcal{Q}]_{\approx} \mid F \iff \mathcal{Q} \Lambda F$  corresponds to the ordinary incidence relation in  $\langle S^\alpha, \mathcal{M}_{n_0-1} \rangle$  i.e. in the Möbius geometry with its hypersubspaces distinguished.*

One more classical result is needed:

**Fact 5.9.** *The structures  $\langle S^\alpha, \mathcal{M}_{n_0-1} \rangle$  and  $\mathfrak{M}$  are mutually definable, and thus they have the same automorphism group.*

Reformulating 5.7 and 5.8 and using 5.1, 5.9, 5.6 and 3.13, 3.14 we conclude with

**Theorem 5.10.** *Assume (32). The relation  $\parallel$  in the set  $\wp_k(\mathfrak{H})$  can be used to express the geometry of  $\mathfrak{H}$ . The automorphism group*

$$\text{Aut}(\langle \wp_k(\mathfrak{H}), \parallel \rangle) = \text{Aut}(\langle \wp_k(\mathfrak{H}), \mathcal{G}^1 \rangle)$$

*consists of the automorphisms of  $\mathfrak{H}$  acting on its  $k$ -subspaces.*

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