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ON THE INVERTIBILITY OF A LINEAR FUNCTION ON THE QUATERNIONS

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Abstract: Since multiplication of quaternions is not commutative, the general quadratic function in quaternions reads

$$q(X) = \sum_m F_m X G_m X H_m + \sum_n J_n X K_n + C.$$

Contrary to the complex case, this form cannot be reduced to the one-parameter family $g_Q(X) = X^2 + Q$. Since we expect that some reduction in the number of parameters is possible by using linear conjugation, the inverse of a linear function on the quaternions is considered.

1. Introduction

We are interested in the dynamics of quadratic functions in quaternions. In contrast to the complex case very little is known about iteration of quaternionic functions (see, however [1], [2], [3], [4], [5], [6], [7], [9], [10], [11], [12], [14], [15], [17], [18]).

Since multiplication in quaternions is not commutative, a general quadratic polynomial has the form

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$$q(X) = \sum_m F_m X G_m X H_m + \sum_n J_n X K_n + C$$

and can be rewritten (see [17]) as

$$\begin{aligned} q(X) = {} & A_1X^2 + A_2XiXi + A_3XjXi + A_4XkXi + A_5XiXj + \\ & + A_6XjXj + A_7XkXj + A_8XiXk + A_9XjXk + A_{10}XkXk + \\ & + B_1X + B_2Xi + B_3Xj + B_4Xk + C. \end{aligned}$$

If only one monomial of the highest degree is present in a polynomial p, then the function $p : \mathbb{H} \to \mathbb{H}$ is surjective by the fundamental theorem of algebra for quaternions (see [8]). In quaternions there exist quadratic polynomials that are not surjective. For example, $X^2 + XiXi + + XjXj + XkXk = -2\|X\|^2$. It is well known that in the complex case any quadratic function is linearly conjugate to a member of the one-parametric family $g_Q(X) = X^2 + Q$. (Two quaternionic quadratic functions R and S are linearly conjugate if and only if there is a nonsingular (bijective) linear function $\mathcal{L} : \mathbb{H} \to \mathbb{H}$ such that $S = \mathcal{L}R\mathcal{L}^{-1}$.) But in quaternions even elimination of linear terms is not possible. For example, family $q_\Omega(X) = X^2 + \Omega X$ is not linearly conjugate to any quadratic function without linear terms (see [17]). Anyway, we still expect that some reduction in the number of parameters is possible by using linear conjugation. Hence we focus our attention on nonsingular linear functions.

2. Linear functions in quaternions

The field of quaternions can be represented as a direct sum $\mathbb{H} = = \mathbb{R} \oplus \mathbb{R}^3$.

We take a general quaternion to have the form

$$X = (x_1, \vec{x}), \ \ x_1 \in \mathbb{R}, \ \ \vec{x} = (x_2, x_3, x_4) \in \mathbb{R}^3,$$

where x_1 is the real part and $\vec{x}$ is the vector part of X. We choose an orthonormal, positively oriented basis $\{i, j, k\}$ for $\mathbb{R}^3$ and write a typical quaternion as

$$X = x_1 + x_2 i + x_3 j + x_4 k.$$

Multiplication of quaternions, given by

$$XY = (x_1y_1 - \langle \vec{x}, \vec{y} \rangle, x_1\vec{y} + y_1\vec{x} + \vec{x} \times \vec{y}),$$

is associative and distributive, but not commutative.

Let us define the scalar product of two quaternions by

$$\langle X, Y\rangle = x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4, \quad \|X\|^2 = \langle X, X\rangle\,,$$

and denote

$$\overline{X} = (x_1, -\vec{x}).$$

Then

$$X^{-1} = \frac{\overline{X}}{\|X\|^2} \text{ if } \|X\| \neq 0.$$

Lemma 1. $X\overline{Y} + Y\overline{X} = 2\langle X, Y\rangle$.
Proof.

$$X\overline{Y} + Y\overline{X} = (x_1y_1 - \langle\vec{x}, -\vec{y}\rangle,\, x_1(-\vec{y}) + y_1\vec{x} + \vec{x}\times(-\vec{y})) +$$
$$+ (y_1x_1 - \langle\vec{y}, -\vec{x}\rangle,\, y_1(-\vec{x}) + x_1\vec{y} + \vec{y}\times(-\vec{x})) =$$
$$= 2(x_1y_1 + \langle\vec{x}, \vec{y}\rangle, \vec{0}) = 2\langle X, Y\rangle. \quad \diamond$$

The general linear polynomial has the form

$$\mathcal{L}(X) = \sum_m F_m X G_m + E$$

and can be rewritten as

(1) $$\mathcal{L}(X) = D_1X + D_2Xi + D_3Xj + D_4Xk + E$$

for unique quaternions D_r, $r = 1, 2, 3, 4$.

By straightforward computation we can prove Lemma 2.

Lemma 2. *Let us write*

$$D_i = (d_{i1}, d_{i2}, d_{i3}, d_{i4}),$$
$$E = (e_1, e_2, e_3, e_4),$$
$$X = (x_1, x_2, x_3, x_4),$$

and define

$\tilde{D} =$

$$= \begin{bmatrix} d_{11}-d_{22}-d_{33}-d_{44} & -d_{12}-d_{21}-d_{34}+d_{43} & -d_{13}+d_{24}-d_{31}-d_{42} & -d_{14}-d_{23}+d_{32}-d_{41} \\ d_{12}+d_{21}-d_{34}+d_{43} & +d_{11}-d_{22}+d_{33}+d_{44} & -d_{14}-d_{23}-d_{32}+d_{41} & +d_{13}-d_{24}-d_{31}-d_{42} \\ d_{13}+d_{24}+d_{31}-d_{42} & +d_{14}-d_{23}-d_{32}-d_{41} & +d_{11}+d_{22}-d_{33}+d_{44} & -d_{12}+d_{21}-d_{34}-d_{43} \\ d_{14}-d_{23}+d_{32}+d_{41} & -d_{13}-d_{24}+d_{31}-d_{42} & +d_{12}-d_{21}-d_{34}-d_{43} & +d_{11}+d_{22}+d_{33}-d_{44} \end{bmatrix}.$$

The linear function (1) *can be expressed in the form*

$$\mathcal{L}(X) = (\tilde{D}X^T + E^T)^T,$$

where T means transposition.

In contrast to the complex case, the coordinates x_1, x_2, x_3, x_4 can themselves be written as linear quaternionic polynomials (see [16]). Hence a quaternionic linear function is not always bijective. Note that if a linear function has only one monomial of degree 1, then it is bijective.

Denote the determinant of square matrix A by $|A|$.

Definition 1. A quaternionic linear function is *singular* iff $|\tilde{D}| = 0$.

Let us define

$$D = \begin{bmatrix} d_{11} & d_{12} & d_{13} & d_{14} \\ d_{21} & d_{22} & d_{23} & d_{24} \\ d_{31} & d_{32} & d_{33} & d_{34} \\ d_{41} & d_{42} & d_{43} & d_{44} \end{bmatrix}.$$

A rather cumbersome computation shows the next theorem.

Theorem 1. *For $\tilde{D}$ as in Lemma 2 the next formula is valid:*

$$|\tilde{D}| = \sum_{n=1}^{3} \sum_{m=n+1}^{4} [(\|D_m\|^2 - \|D_n\|^2)^2 + 4\langle D_m, D_n\rangle^2] - 2\sum_{m=1}^{4} \|D_m\|^4 - 8\,|D|.$$

Lemma 3. *Every nonsingular linear function*

$$\mathcal{L}(X) = D_1X + D_2Xi + D_3Xj + D_4Xk + E = \widetilde{\mathcal{L}}(X) + E$$

can be written as

$$\mathcal{L} = \widetilde{\mathcal{L}} \circ \mathcal{T},$$

where $\mathcal{T}$ is a translation

$$\mathcal{T}(X) = X + F$$

and F is the unique solution of the equation $\widetilde{\mathcal{L}}(F) = E$.

Proof.

$$\begin{aligned} \widetilde{\mathcal{L}}(\mathcal{T}(X)) = \widetilde{\mathcal{L}}(X+F) = \\ &= D_1(X+F) + D_2(X+F)i + D_3(X+F)j + D_4(X+F)k = \\ &= D_1X + D_2Xi + D_3Xj + D_4Xk + D_1F + \\ &\qquad + D_2Fi + D_3Fj + D_4Fk = \\ &= \widetilde{\mathcal{L}}(X) + \widetilde{\mathcal{L}}(F) = \mathcal{L}(X). \quad \Diamond \end{aligned}$$

Theorem 2. *Let*

$$\mathcal{L}(X) = D_1X + D_2Xi + D_3Xj + D_4Xk + E = \widetilde{\mathcal{L}}(X) + E$$

be a nonsingular linear function and

$$\mathcal{J}(X) = F_1X + F_2Xi + F_3Xj + F_4Xk + G$$

its inverse function. Let D_{rs} denote the matrix obtained from the matrix D by removing the elements of its r-th row and s-th column. Then

$$F_1 = \frac{1}{|\tilde{D}|}\big[(\|D_1\|^2 - \|D_2\|^2 - \|D_3\|^2 - \|D_4\|^2)\overline{D}_1 + \\ + 2\langle D_1, D_2\rangle \overline{D}_2 + 2\langle D_1, D_3\rangle \overline{D}_3 + 2\langle D_1, D_4\rangle \overline{D}_4 - \\ - 2(|D_{11}|, |D_{12}|, -|D_{13}|, |D_{14}|)\big],$$

$$F_2 = \frac{1}{|\tilde{D}|}\big[(\|D_1\|^2 - \|D_2\|^2 + \|D_3\|^2 + \|D_4\|^2)\overline{D}_2 - \\ - 2\langle D_1, D_2\rangle \overline{D}_1 - 2\langle D_2, D_3\rangle \overline{D}_3 - 2\langle D_2, D_4\rangle \overline{D}_4 + \\ + 2(-|D_{21}|, -|D_{22}|, |D_{23}|, -|D_{24}|)\big],$$

$$F_3 = \frac{1}{|\tilde{D}|}\big[(\|D_1\|^2 + \|D_2\|^2 - \|D_3\|^2 + \|D_4\|^2)\overline{D}_3 - \\ - 2\langle D_1, D_3\rangle \overline{D}_1 - 2\langle D_2, D_3\rangle \overline{D}_2 - 2\langle D_3, D_4\rangle \overline{D}_4 + \\ + 2(|D_{31}|, |D_{32}|, -|D_{33}|, |D_{34}|)\big],$$

$$F_4 = \frac{1}{|\tilde{D}|}\big[(\|D_1\|^2 + \|D_2\|^2 + \|D_3\|^2 - \|D_4\|^2)\overline{D}_4 - \\ - 2\langle D_1, D_4\rangle \overline{D}_1 - 2\langle D_2, D_4\rangle \overline{D}_2 - 2\langle D_3, D_4\rangle \overline{D}_3 + \\ + 2(-|D_{41}|, -|D_{42}|, |D_{43}|, -|D_{44}|)\big],$$

and

$$G = -F_1E - F_2Ei - F_3Ej - F_4Ek.$$

Proof. To prove that $\mathcal{J}$ is the inverse function of $\mathcal{L}$, we have to prove that the next ten equations are valid:

$$\begin{aligned}
F_1D_1 - F_2D_2 - F_3D_3 - F_4D_4 &= (1,0,0,0)\\
F_1D_2 + F_2D_1 - F_3D_4 + F_4D_3 &= (0,0,0,0)\\
F_1D_3 + F_2D_4 + F_3D_1 - F_4D_2 &= (0,0,0,0)\\
F_1D_4 - F_2D_3 + F_3D_2 + F_4D_1 &= (0,0,0,0)\\
F_1E + F_2Ei + F_3Ej + F_4Ek + G &= (0,0,0,0)\\
D_1F_1 - D_2F_2 - D_3F_3 - D_4F_4 &= (1,0,0,0)\\
D_1F_2 + D_2F_1 - D_3F_4 + D_4F_3 &= (0,0,0,0)\\
D_1F_3 + D_2F_4 + D_3F_1 - D_4F_2 &= (0,0,0,0)\\
D_1F_4 - D_2F_3 + D_3F_2 + D_4F_1 &= (0,0,0,0)
\end{aligned}$$

$$D_1G + D_2G\,i + D_3Gj + D_4Gk + E = (0,0,0,0).$$

We will prove only the first equation using Lemma 1, Th. 1 and standard property of the determinant: the value of the determinant $|M|$ of an n-square matrix M is the sum of products obtained by multiplying each element of a row (column) of M by its cofactor $(-1)^{r+s}|M_{rs}|$. In the proof we will also use the fact that the sum of the products formed by multiplying the elements of a column of square matrix M by the corresponding cofactors of another column of M is zero.

Writing $Y = F_1D_1 - F_2D_2 - F_3D_3 - F_4D_4$ we get

$$\begin{aligned}
Y = \frac{1}{|\tilde{D}|}\,[\,&(\|D_1\|^2-\|D_2\|^2-\|D_3\|^2-\|D_4\|^2)\|D_1\|^2+\\
&+2\langle D_1,D_2\rangle \overline{D}_2D_1+2\langle D_1,D_3\rangle \overline{D}_3D_1+2\langle D_1,D_4\rangle \overline{D}_4D_1-\\
&-2(d_{11}|D_{11}|-d_{12}|D_{12}|+d_{13}|D_{13}|-d_{14}|D_{14}|, d_{12}|D_{11}|+d_{11}|D_{12}|-\\
&-d_{14}|D_{13}|-d_{13}|D_{14}|, d_{13}|D_{11}|-d_{14}|D_{12}|-d_{11}|D_{13}|+\\
&+d_{12}|D_{14}|, d_{14}|D_{11}|+d_{13}|D_{12}|+d_{12}|D_{13}|+d_{11}|D_{14}|)-\\
&-(\|D_1\|^2-\|D_2\|^2+\|D_3\|^2+\|D_4\|^2)\|D_2\|^2+\\
&+2\langle D_1,D_2\rangle \overline{D}_1D_2+2\langle D_2,D_3\rangle \overline{D}_3D_2+2\langle D_2,D_4\rangle \overline{D}_4D_2-\\
&-2(-d_{21}|D_{21}|+d_{22}|D_{22}|-d_{23}|D_{23}|+d_{24}|D_{24}|,-d_{22}|D_{21}|-d_{21}|D_{22}|+\\
&+d_{24}|D_{23}|+d_{23}|D_{24}|,-d_{23}|D_{21}|+d_{24}|D_{22}|+d_{21}|D_{23}|-\\
&-d_{22}|D_{24}|,-d_{24}|D_{21}|-d_{23}|D_{22}|-d_{22}|D_{23}|-d_{21}|D_{24}|)-\\
&-(\|D_1\|^2+\|D_2\|^2-\|D_3\|^2+\|D_4\|^2)\|D_3\|^2+\\
&+2\langle D_1,D_3\rangle \overline{D}_1D_3+2\langle D_2,D_3\rangle \overline{D}_2D_3+2\langle D_3,D_4\rangle \overline{D}_4D_3-\\
&-2(d_{31}|D_{31}|-d_{32}|D_{32}|+d_{33}|D_{33}|-d_{34}|D_{34}|, d_{32}|D_{31}|+d_{31}|D_{32}|-\\
&-d_{34}|D_{33}|-d_{33}|D_{34}|, d_{33}|D_{31}|-d_{34}|D_{32}|-d_{31}|D_{33}|+d_{32}|D_{34}|,\\
&d_{34}|D_{31}|+d_{33}|D_{32}|+d_{32}|D_{33}|+d_{31}|D_{34}|)-\\
&-(\|D_1\|^2+\|D_2\|^2+\|D_3\|^2-\|D_4\|^2)\|D_4\|^2+\\
&+2\langle D_1,D_4\rangle \overline{D}_1D_4+2\langle D_2,D_4\rangle \overline{D}_2D_4+2\langle D_3,D_4\rangle \overline{D}_3D_4-\\
&-2(-d_{41}|D_{41}|+d_{42}|D_{42}|-d_{43}|D_{43}|+d_{44}|D_{44}|,-d_{42}|D_{41}|-\\
&-d_{41}|D_{42}|+d_{44}|D_{43}|+d_{43}|D_{44}|,-d_{43}|D_{41}|+d_{44}|D_{42}|+d_{41}|D_{43}|-\\
&-d_{42}|D_{44}|,-d_{44}|D_{41}|-d_{43}|D_{42}|-d_{42}|D_{43}|-d_{41}|D_{44}|)\,] =\\
= \frac{1}{|\tilde{D}|}\,[\,&\|D_1\|^4-2\|D_1\|^2\|D_2\|^2-2\|D_1\|^2\|D_3\|^2-\\
&-2\|D_1\|^2\|D_4\|^2+\|D_2\|^4-2\|D_2\|^2\|D_3\|^2-\\
&-2\|D_2\|^2\|D_4\|^2+\|D_3\|^4-2\|D_3\|^2\|D_4\|^2+\|D_4\|^4+\\
&+4\langle D_1,D_2\rangle^2+4\langle D_1,D_3\rangle^2+4\langle D_1,D_4\rangle^2+4\langle D_2,D_3\rangle^2+
\end{aligned}$$

$$+4\langle D_2, D_4\rangle^2 + 4\langle D_3, D_4\rangle^2 + 8(|D|, 0, 0, 0)\,] = (1,0,0,0).$$

Proof of the sixth equation is analogous, proofs of the others are either more cumbersome or trivial. ◊

Since the formulae for the inverse of a quaternionic linear function are quite complicated, we try to find some other form of linear function. For example, $\mathcal{J}(X) = MXN + PXR + E$. Note that in this case parameters M, N, P and Q are not uniquely determined (for example, we get the same linear function if we multiply M by nonzero real a and divide N by a). The question is, whether every linear function can be written in this form. Although the number of parameters is the same as in the form (1), the answer is no.

Lemma 4. *If* $\mathcal{J}(X) = MXN + PXR + E$ *is rewritten as* $\mathcal{J}(X) = D_1X + D_2Xi + D_3Xj + D_4Xk + E$, *then* $|D| = 0$.

Proof.

$$\begin{aligned}\mathcal{J}(X) &= MXN + PXR + E = \\ &= MX(n_1 + n_2 i + n_3 j + n_4 k) + PX(r_1 + r_2 i + r_3 j + r_4 k) = \\ &= (n_1 M + r_1 P)X + (n_2 M + r_2 P)Xi + (n_3 M + r_3 P)Xj + \\ &\quad + (n_4 M + r_4 P)Xk.\end{aligned}$$

Hence

$$D = \begin{bmatrix} n_1m_1 + r_1p_1 & n_1m_2 + r_1p_2 & n_1m_3 + r_1p_3 & n_1m_4 + r_1p_4 \\ n_2m_1 + r_2p_1 & n_2m_2 + r_2p_2 & n_2m_3 + r_2p_3 & n_2m_4 + r_2p_4 \\ n_3m_1 + r_3p_1 & n_3m_2 + r_3p_2 & n_3m_3 + r_3p_3 & n_3m_4 + r_3p_4 \\ n_4m_1 + r_4p_1 & n_4m_2 + r_4p_2 & n_4m_3 + r_4p_3 & n_4m_4 + r_4p_4 \end{bmatrix} =$$

$$= \begin{bmatrix} n_1 & r_1 & 0 & 0 \\ n_2 & r_2 & 0 & 0 \\ n_3 & r_3 & 0 & 0 \\ n_4 & r_4 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} m_1 & m_2 & m_3 & m_4 \\ p_1 & p_2 & p_3 & p_4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and therefore $|D| = 0$. ◊

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