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AFFINE SURFACES WITH PLANAR AFFINE NORMALS IN 3-DIMENSIONAL MINKOWSKI SPACE $\mathbb{R}^3_1$

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Abstract: We give a classification of affine surfaces with planar affine normals which admit a one parameter subgroup of Lorentzian motions.

In three-dimensional euclidean space *cylindrical surfaces* can be characterized by the fact that the normals are parallel to a plane. The corresponding surfaces in affine geometry are called *surfaces with planar affine normals*. This class of surfaces has been investigated among others by B. Opozda in [9] and B. Opozda, T. Sasaki in [10]. Denote by S and ∇ the affine shape operator and the induced connection respectively (for definitions see Sec. 1 or [8]). A surface (which is not an improper affine sphere) has planar affine normals if and only if S is of constant rank 1 and $\operatorname{im} S$ is parallel with respect to ∇ [9, p. 79]. In [9] there is given a method how to construct these surfaces using the solutions of certain differential equations (see also [8, p. 220]).

In the present paper we classify surfaces with planar affine normals admitting a 1-parameter subgroup of Lorentzian motions in three-

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dimensional Minkowski space. In case of Lorentzian rotations with timelike or spacelike axis (see Sec. 2 the surfaces with planar affine normals are well known ([4], [5], [15]). They also appear as examples of surfaces with locally symmetric induced connection ∇ in [8, p. 220] and [3, p. 210].

1. Affine surfaces

Concerning the following basic facts of affine differential geometry we refer to [8]. Let $f : M \to \mathbb{R}^3$ be an immersion of a 2-dimensional, smooth, orientable, connected, differentiable manifold M in the standard affine space $\mathbb{R}^3$. Denoting by $\overline{\nabla}$ the standard flat connection in $\mathbb{R}^3$ and by ξ a vector field transversal to f, we have the equations of Gauss and Weingarten

$$\overline{\nabla}_X f_*(Y) = f_*(\nabla_X Y) + G(X,Y)\xi \tag{1}$$

and

$$\overline{\nabla}_X \xi = -f_*(SX) + \tau(X)\xi \tag{2}$$

for arbitrary vector fields X, Y tangent to M. This gives an affine connection ∇ (the induced connection) and a symmetric bilinear map G, a $(1,1)$-tensorfield S (the shape operator) and a 1-form τ on M. The immersion f (the surface $\Phi = f(M)$) is called *nondegenerate*, if G has rank two, a fact that is independent of the transversal vector field ξ. In the following we restrict our considerations to nondegenerate surfaces.

Let $\nu_\xi(X,Y) := \det(f_*X, f_*Y, \xi)$, where det is the standard volumeform in $\mathbb{R}^3$ (parallel with respect to $\overline{\nabla}$). Then the connection ∇ is equiaffine with respect to ν_ξ, iff ξ is a *relative normal*, that means $\tau = 0$ on M. Denoting by ν_G the volumeform of G the *affine normal* is (up to sign) the unique transversal vectorfield ξ with the property $\nu_\xi = \nu_G$. In this case G is called the *Blaschke metric* and S the *affine shape operator*, which is selfadjoined with respect to G. The *affine curvature* K and the *affine mean curvature* H are defined by

$$K := \det(S), H := (1/2)\mathrm{tr}(S). \tag{3}$$

Denoting I the identity map a surface is called a *proper* or *improper affine sphere* if $S = \rho I$ where $\rho \neq 0$ or $\rho = 0$. In case of an improper affine sphere the affine normal vector is constant hence $H = 0$, $K = 0$.

The induced connection ∇ is *locally symmetric*, iff $\nabla R = 0$ or equivalently $\nabla \mathrm{Ric} = 0$. Here R and Ric denote the curvature tensor of ∇ and the Ricci tensor respectively. Further ∇ is *projectively flat* iff it is projectively equivalent to a flat connection. It is well known, that ∇ is projectively flat iff the Ricci tensor Ric satisfies the Codazzi equation

$$(\nabla_X \mathrm{Ric})(Y, Z) = (\nabla_Y \mathrm{Ric})(X, Z).$$

Denoting by (u, v) or $(u^1 := u, u^2 := v)$ local coordinates, we write partial derivatives of a vector function q with respect to the local coordinates

$$q_{,j} := \frac{\partial}{\partial u^j} q,\ q_{,jk} := \frac{\partial}{\partial u^j \partial u^k} q.$$

If we put

$$D_{jk} := \det(f_{,1}, f_{,2}, f_{,jk}),\ D := \det(D_{jk}), \tag{4}$$

then the components of the affine metric G are

$$G_{jk} := |D|^{-1/4} D_{jk}. \tag{5}$$

That means that the surface $\Phi = f(M)$ is nondegenerate, iff $D \neq 0$. In case of $D > 0$ and $D < 0$ the surfaces are locally strongly convex and non-convex, respectively. The affine normal vector ξ can be calculated by

$$\xi := (1/2)\Delta x, \tag{6}$$

where Δ is the laplacian with respect to G.

If ξ has no critical point then $\xi(M)$ is a curve iff rank S=1 ([10]). A special case occurs when the affine normals of a surface $f(M)$ are parallel to a plane; we say $f(M)$ has *planar affine normals*. That means $\xi(M)$ is a curve in a plane ε, which contains the origin.

2. Motions in Minkowski space $\mathbb{R}^3_1$

In the following

$$ds^2 = -dx^2 + dy^2 + dz^2 \tag{7}$$

is the indefinite metric in Minkowski 3-space $\mathbb{R}^3_1$. Affine transformations respecting (7) are called *Lorentzian motions.* The possible non trivial 1-parameter subgroups can be written in the following way ([2, p. 310])

$$(8)\qquad \begin{pmatrix} x\\ y\\ z\end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0\\ 0 & \cos v & -\sin v\\ 0 & \sin v & \cos v\end{pmatrix}\begin{pmatrix} x\\ y\\ z\end{pmatrix} + p\begin{pmatrix} v\\ 0\\ 0\end{pmatrix},$$

$$(9)\qquad \begin{pmatrix} x\\ y\\ z\end{pmatrix} \mapsto \begin{pmatrix} \cosh v & \sinh v & 0\\ \sinh v & \cosh v & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix} x\\ y\\ z\end{pmatrix} + p\begin{pmatrix} 0\\ 0\\ v\end{pmatrix},$$

$$(10)\qquad \begin{pmatrix} x\\ y\\ z\end{pmatrix} \mapsto \begin{pmatrix} 1+\frac{v^2}{2} & -\frac{v^2}{2} & v\\ \frac{v^2}{2} & 1-\frac{v^2}{2} & v\\ v & -v & 1\end{pmatrix}\begin{pmatrix} x\\ y\\ z\end{pmatrix} + p\begin{pmatrix} \frac{v^3}{3}+v\\ \frac{v^3}{3}-v\\ v^2\end{pmatrix}.$$

In case of $p = 0$ we have *Lorentzian rotations* fixing the points of an axis g which is timelike (x-axis), spacelike (z-axis) and isotropic ($x = y, z = 0$) in case of transformations (8), (9) and (10), respectively.

In case of $p \neq 0$ the transformations are called *Lorentzian screw motions.* Transformations (8) and (9) are compositions of a Lorentzian rotation around a nonisotropic axis g with a translation parallel g. In case of transformation (10) the orbits are cubic parabolas carried by parabolic cylinders which are equal by translation. A map of type (10) is called a *cubic screw motion* (*"kubische Schraubung"* in K. Strubecker [12], [13, p. 58] or *"Grenzschraubung"*) in W. Wunderlich [17]). A cubic screw motion has no proper fixed point.

Remark 1. In case of (8), (9) and (10) exactly the pencil of planes $x =$ const., $z =$ const. and $x = y$, respectively, are invariant.

3. Rotational surfaces with planar affine normals

3.1. Rotational surfaces with timelike and spacelike axis

Surfaces admitting Lorentzian rotations with timelike and spacelike axis coincide with the *proper affine surfaces of rotation of elliptic and hyperbolic type*, respectively, in the sense of W. Süss [16] and P. A. Schirokow [15]. So we have the following result (see [4, p. 168], [5]):

Theorem 1. *Let $\Phi \subset \mathbb{R}^3_1$ be a Lorentzian surface of rotation with timelike axis and planar affine normals which is not an improper affine sphere. Then Φ admits a representation in local coordinates*

$$f(u,v) = (u,\, r(u)\, \cos v,\, r(u)\, \sin v)^T \text{ and}$$
$$r(u) = \alpha\, \sin u + \beta\, \cos u, \text{ or}$$
$$r(u) = \alpha\, \sinh u + \beta\, \cosh u$$

with $(\alpha, \beta) \in \mathbb{R}^2 \backslash \{0,0\}$.

In case of spacelike axis

$$f(u,v) = (r(u) \cosh v,\, r(u) \sinh v,\, u)^T$$

with the same possibilities for $r(u)$ *as above.*

Remark 2. These surfaces are well known, as they appear as examples for surfaces with locally symmetric induced connection ∇ for instance in [3, p. 210] and [8, p. 220]. Because the affine shape operator is diagonalizable for the surfaces of Th. 1, ∇ is projectively flat, too ([9, Th. 3.2]).

Remark 3. For improper affine spheres admitting Lorentzian rotations with timelike or spacelike axis see [5, p. 168] or [14].

3.2. Rotational surfaces with isotropic axis

Theorem 2. *Let* $\Phi \subset \mathbb{R}^3_1$ *be a Lorentzian surface of rotation with isotropic axis and planar affine normals. Then* Φ *is an improper affine sphere and in pseudoisothermal coordinates (with respect to the affine metric)* Φ *admits the representation*

(11)

$$f(u,v) = \left(\frac{v^2}{2}(u+\alpha) + w(u), \frac{v^2}{2}(u+\alpha) + w(u) - u - \alpha, v(u+\alpha)\right)^T$$

where

$$w(u) = \varepsilon \left(\frac{u^3}{6} + \alpha\frac{u^2}{2}\right) + \beta u \quad (\alpha \in \mathbb{R}\backslash\{0\},\ \beta \in \mathbb{R}).$$

In a suitable coordinate system Φ *solves*

$$(12) \quad 6xz = 3y^2 + 2\varepsilon x(x-\alpha)(x^2+\alpha x - 2\alpha^2) + 12\beta x(x-\alpha) - 6x^2,$$

where $\varepsilon = +1$ *and* $\varepsilon = -1$ *refers to locally convex and locally non convex surfaces respectively. So the surfaces in consideration are algebraic of order four. The affine normals are parallel to the lines* $x = y = 0$.

Proof. (a) The Lorentzian rotation (10) ($p = 0$), σ say, has the axis $x = y, z = 0$. Applying σ to the curve $(c_1(u), c_2(u), 0)^T$ gives the local parametrization

(13)
$$f(u,v) = \Big(\frac{v^2}{2}(c_1(u) - c_2(u)) + \\ + c_1(u), \frac{v^2}{2}(c_1(u) - c_2(u)) + c_2(u), v(c_1(u) - c_2(u))\Big)^T.$$

From Rem. 1 the planes $x = y, z = 0$ are invariant, so we have planar affine normals iff the affine normal $\xi = (\xi^1, \xi^2, \xi^3)$ fulfills

(14) $$\xi^1(u,v) = \xi^2(u,v).$$

Denoting by a *dot* the derivative with respect to u, we calculate D_{jk} according to (4) in case of surfaces (13)

(15) $$D_{11} = (\ddot{c}_1\dot{c}_2 - \dot{c}_1\ddot{c}_2)(c_1 - c_2), D_{12} = 0, D_{22} = -(\dot{c}_1 - \dot{c}_2)(c_1 - c_2)^2.$$

So from (5) we have $G_{jj} = G_{jj}(u)(j = 1,2), G_{12} = 0$. In this case it is easy to see that the affine normal is

(16) $$\xi = \frac{1}{2\,G_{11}} f_{,11} + \frac{1}{2\,G_{22}} f_{,22} + \frac{1}{4\,G_{22}} \left(\frac{d}{du}\left(\frac{G_{22}}{G_{11}}\right)\right) f_{,1}.$$

With (13) and (16) condition (14) reads

(17) $$\frac{1}{2\,G_{11}}(\ddot{c}_1 - \ddot{c}_2) + \frac{1}{4\,G_{22}}\left(\frac{d}{du}\left(\frac{G_{22}}{G_{11}}\right)\right)(\dot{c}_1 - \dot{c}_2) = 0.$$

Calculating ξ^3 using (17) gives $\xi^3 = 0$, so we have an improper affine sphere. This fact could be seen without calculation: Every point of the surface, P say (not on the axis g) determines on the one hand a *meridian plane* μ_P connecting P with g and on the other hand an invariant plane φ_P, containing his orbit (a parabola with isotropic diameters). Clearly the affine normal ξ_P is the intersection $\xi_P = \varphi_P \cap \mu_p$. Because all invariant planes are parallel to the axis g, all affine normals are parallel g.

(b) To get a parametrization of the discussed surfaces we have to determine $c_1(u), c_2(u)$ solving (17). Instead of solving this differential equation directly, we introduce (pseudo-) isothermal coordinates with respect to the affine metric. A change of the local coordinates in a way that $u = \gamma(u'), v = v'$, that means a reparametrization of the meridian curve gives

(18) $$G_{1'1'} = (G_{11} \circ \gamma) \left(\frac{d\gamma}{du'} \right)^2, \; G_{1'2'} = 0, \; G_{2'2'} = (G_{22} \circ \gamma).$$

We can take γ in a way that

(19) $$G_{1'1'} = \sigma(u'), \; G_{1'2'} = 0, \; G_{2'2'} = \varepsilon\sigma(u').$$

where $\varepsilon = +1$ and $\varepsilon = -1$ refers to locally convex and locally non convex surfaces respectively. Because of

$$G_{j'k'} := |\det(D_{j'k'}|^{-1/4} D_{j'k'}.$$

the requirement (19) gives with (15) the relation of the functions c_1, c_2

(20) $$\ddot{c}_1\dot{c}_2 - \dot{c}_1\ddot{c}_2 = -\varepsilon(c_1 - c_2)(\dot{c}_1 - \dot{c}_2),$$

where the *dot* now denotes the derivative with respect to u'. In the following we write the previous notation u, v again. In case of $G_{11} = \sigma(u)$, $G_{12} = 0$, $G_{22} = \varepsilon\sigma(u)$ we calculate from (16)

(21) $$\xi = \frac{1}{2\sigma}(f_{,11} + \varepsilon f_{,22}).$$

Using (13) we get

$$\xi = \frac{1}{2\sigma}\left(\frac{v^2}{2}(\ddot{c}_1 - \ddot{c}_2) + \ddot{c}_1 + \varepsilon(c_1 - c_2), \frac{v^2}{2}(\ddot{c}_1 - \ddot{c}_2) + \ddot{c}_2 + \varepsilon(c_1 - c_2), v(\ddot{c}_1 - \ddot{c}_2)\right)^T.$$

Thus the condition (14) becomes

(22) $$\ddot{c}_1 = \ddot{c}_2 \Leftrightarrow c_1(u) = c_2(u) + d_1 u + d_2, \; d_1, \, d_2 \in \mathbb{R}, \; d_1 \neq 0.$$

The affine normal is

(23) $$\xi = \frac{1}{2\sigma}(\ddot{c}_1 + \varepsilon(c_1 - c_2))(1, 1, 0)^T.$$

Inserting (22) into (20) we get

(24) $$c_1(u) = \varepsilon\left(d_1 \frac{u^3}{3} + d_2 \frac{u^2}{2} \right) + d_3 u + d_4$$

and $c_2(u)$ follows from (22). Rescaling (13) by ${d_1}^{-1}$ and $\alpha := d_2 {d_1}^{-1}$, $\beta := d_3 {d_1}^{-1}$ and $d_4 = 0$ (without loss of generality) we get (11).

(c) Applying the coordinate transformation

(25) $$x_1 = x - y, \; x_2 = \sqrt{2}\, z, \; x_3 = x + y$$

to the surfaces (11) gives a representation solving (in the previous notation $x := x_1, y := x_2, z := x_3$) equation (12). $\Diamond$

4. Screw surfaces with planar affine normals

4.1. Ruled screw surfaces

In case of a ruled surface the affine normals along a generator are contained in a plane. So the affine normals are planar iff the generators are planar, that means they are parallel to a plane (*"konoidale Regelfläche"*) (see [10, Th. 3.3]). So in case of transformations (8) and (9) it yields

Theorem 3. *Let $\Phi \subset \mathbb{R}^3_1$ be a ruled surface which is not an improper affine sphere. If Φ has planar affine normals and Φ is a screw surface with respect to transformations* (8) *(timelike axis), then Φ admits a local representation*

(26) $$f(u,v)=(pv,\ u\cos v - a\sin v,\ u\sin v + a\cos v),\quad a\in\mathbb{R},\ p\in\mathbb{R}\backslash\{0\}.$$

In case of a screw transformation (9) (spacelike axis) we get

(27) $$f(u,v) = (a\cosh v + u\sinh v,\ a\sinh v + u\cosh v,\ p\,v)^T, \text{ or}$$

(28) $$f(u,v) = (u\cosh v + a\sinh v,\ u\sinh v + a\cosh v,\ p\,v)^T.$$

4.2. Screw surfaces with timelike axis

Theorem 4. *Let $\Phi \subset \mathbb{R}^3_1$ be a non ruled surface which is not an improper affine sphere. If Φ has planar affine normals and Φ is a screw surface with respect to transformations* (8) *(timelike axis), then Φ admits a local representation*

(29) $$f(u,v) = \big(p\,v,\ \dot g(u)\sin(u-v) - g(u)\cos(u-v), -\dot g(u)\cos(u-v) - g(u)\sin(u-v)\big)^T,$$

where

(30) $$g(u) = \Big(\alpha\sinh\big(\sqrt{k}\,u\big) + \beta\cosh\big(\sqrt{k}\,u\big)\Big)\quad (k>0),$$

(31) $$g(u) = \Big(\alpha\sin\big(\sqrt{-k}\,u\big) + \beta\cos\big(\sqrt{-k}\,u\big)\Big)\quad (k<0, k\neq -1)$$

with $(\alpha,\beta)\in\mathbb{R}^2\backslash\{0,0\}$. Solutions with $-1<k<0$ are locally strongly convex. In case of $k\in\mathbb{R}\backslash(-1,0)$ the surfaces are locally not strongly convex. The induced connection ∇ of surfaces (29) *is locally symmetric and projectively flat.*

Proof. We generate a screw surface with timelike axis by applying transformation (8) to a cross section $q = c(I)$ in the plane $x = 0$, where

$c : I \subset \mathbb{R} \to \mathbb{R}$. Because q is not a straight line, we can put $c(u) = (0, Y(u), Z(u))$, where

(32) $Y(u) = -g(u)\cos u + \dot{g}(u)\sin u,\ Z(u) = -g(u)\sin u - \dot{g}(u)\cos u,$

where $g : I \subset \mathbb{R} \to \mathbb{R}$ is the support function with

(33) $$\dot{g} \neq 0,\ g + \ddot{g} \neq 0.$$

Applying (8) to $(0, Y(u), Z(u))$ gives (29). From this representation we calculate

(34) $$D_{11} = -p(g+\ddot{g})^2,\ D_{12} = p(g+\ddot{g})^2,\ D_{22} = -p(g+\ddot{g}).$$

If the surface has planar affine normals they are necessarily parallel to the invariant planes $x =$ const., that means the affine normal is

(35) $$\xi = (\xi^1 = 0, \xi^2, \xi^3).$$

Calculating ξ from (6) we get

(36) $$\xi^1 = 0 \Leftrightarrow \ddot{g} = k\,g,\ k \in \mathbb{R}\backslash\{0, -1\}.$$

This gives the solutions (30) and (31) for g.

The affine normal of surfaces (29) are

$$\xi = \mu\,(0, \cos(u-v), \sin(u-v))^T,$$

where $\mu \neq 0$ is a constant. Hence ∇ is locally symmetric (see [10, Th. 2.3], [8, p. 219]). Because the affine shape operator is diagonalizable (applying the reparametrization $u = u' + v'$, $v = v'$), ∇ is projectively flat too ([9, Th. 3.2]). $\Diamond$

Remark 4. If Φ is an improper affine sphere admitting a screw transformation (8), the affine normals are parallel to the axis. For representations see [6], [14].

Remark 5. If $k < 0$ the cross sections $(0, X(u), Y(u))$ of surfaces (29) are *cusped cycloids*, which are closed curves if $\sqrt{-k} \in \mathbb{Q}$. In case of $-1 < k < 0$ the cross sections are *epicycloids* (for instance $k = -1/4$ ... *Nephroide*, $k = -1/9$ ... *Cardioide*) while $k < -1$ gives *hypocycloids* (for instance $k = -4$ ... *Astroide*, $k = -9$ ... *Steiner threecusp*).

Remark 6. If $k < 0$ the surfaces of Th. 4 are screw surfaces which carry translation surfaces at the same time. Taking two helices $l_1(u), l_2(v)$ with parallel axis

$$l_1(u) = (p_1\,u, r\cos u, r\sin u)^T,\ r > 0\ p_1 \neq 0,$$

$$l_2(v) = (p_2\,v, s\cos v, s\sin v)^T,\ s > 0\ p_2 \neq 0,$$

then

$$f(u,v) = l_1(u) + l_2(v),\ (u,\ v) \in \mathbb{R}^2.$$

is the local representation of a translation surface, which is a screw surface at the same time (see E. Müller [7]). Iff the constants are related by

$$r : s = -\varepsilon\, p_1 : p_2 \quad (\varepsilon = \pm 1)$$

the surface is part of a surface of Th. 4. In the special case $\varepsilon = -1$ and $p_1 = p_2$, $f(\mathbb{R}^2)$ is part of a right helicoid.

Remark 7. Considering the case $k > 0$, that means

$$g(u) = \alpha \sinh(\sqrt{k}\, u) + \beta \cosh(\sqrt{k}\, u) = \alpha_1 \exp(\sqrt{k} u) + \beta_1 \exp(-\sqrt{k} u)$$

where $(\alpha_1, \beta_1) \in \mathbb{R}^2 \backslash \{0, 0\}$, the selection of the screwing parameter p according to $p^2 = 4\, k\, \alpha_1\, \beta_1$ gives the euclidean minimal screw surfaces, that is the well known 1-parameter family of screw surfaces containing the catenoid and the right helicoid.

Remark 8. In case of $k > 0$ with $\alpha = \beta$ we have

$$g(u) = \alpha \exp(\sqrt{k}\, u),\ (\alpha \neq 0) \tag{37}$$

and the cross sections are *logarithmic spirals*. They are shadow lines of the surface with respect to light centers on the axis g, so these surfaces belong to the class of (euclidean) screw surfaces carrying a 1-parameter family of *plane shadow lines* (see O. Röschel [11]). The sections with planes containing g (*meridians*) are exponential lines; together with the cross sections they form the net of *affine lines of curvature*. Surfaces with cross sections determined by (37) are the only surfaces from Th. 4 with *plane affine lines of curvature*. The surfaces determined by (37) are also the only surfaces of Th. 4 with *flat affine metric*, that is the scalar curvature κ of the affine metric G is zero: Calculating κ in case of surfaces (29) gives $\kappa = 0 \Leftrightarrow g\ddot{g} - \dot{g}^2 = 0$. This gives (37).

4.3. Screw surfaces with spacelike axis

Theorem 5. *Let $\Phi \subset \mathbb{R}^3_1$ be a non ruled surface which is not an improper affine sphere. If Φ has planar affine normals and Φ is a screw surface with respect to transformations* (9) *(spacelike axis), then Φ admits a local representation*

$$f(u,v) = (X(u) \cosh v + Y(u) \sinh v, X(u) \sinh v + Y(u) \cosh v, p\, v)^T,$$

where $(X(u),\ Y(u), 0)$ represents the cross section in the plane $z = 0$ with

$$X(u) = g(u)\cosh u - \dot{g}(u)\sinh u$$
$$Y(u) = -\dot{g}(u)\cosh u + g(u)\sinh u,$$

or

$$X(u) = -g(u)\sinh u + \dot{g}(u)\cosh u$$
$$Y(u) = \dot{g}(u)\sinh u - g(u)\cosh u,$$

and $g(u)$ *is given by* (30) *or* (31), *where* $k \in \mathbb{R}\backslash\{0,1\}$. *The induced connection* ∇ *of these surfaces is again locally symmetric and projectively flat.*

The proof of Th. 5 is analogous to that of Th. 4.

4.4. Screw surfaces with isotropic axis

Theorem 6. *Let* $\Phi \subset \mathbb{R}^3_1$ *be surface which admits a cubic screw motion* (10). *If* Φ *has planar affine normals then* Φ *is an improper affine sphere and the following two cases are possible*

(a) Φ *is a Cayley ruled surface admitting the representation*

$$f(u,v) = p\begin{pmatrix} \sqrt{2}\,v \\ p\,v\,(v-\beta) \\ \dfrac{p\sqrt{2}}{3}\,v^3 \end{pmatrix} + u\begin{pmatrix} 0 \\ 1 \\ \sqrt{2}\,v \end{pmatrix} \quad (p \neq 0), \tag{38}$$

with the equation

$$3\,p^2\,z = 3\,p\,x\,y - x^3 - \frac{3\,p\,\beta}{\sqrt{2}}x^2. \tag{39}$$

(b) Φ *has the representation*

(40)

$$f(u,v) = \left(u^3+\beta u^2 v+p\left(\frac{v^3}{3}+v\right), u^3+\beta u^2 v+p\left(\frac{v^3}{3}-v\right), \beta u^2+p\,v^2\right)^T,$$

with $\beta \neq 0$, $p \neq 0$. *This means* Φ *is generated by the Neil parabola* $(u^3, \beta u^2)$ *in the isotropic plane* $x = y$. *In a suitable coordinate system* Φ *solves the equation*

$$\beta^3\,(6\,p^2\,z - 6\,p\,x\,y + 2\,x^3)^2 = 9\,p\,(2\,p\,y - x^2)^3, \tag{41}$$

so the surfaces (40) *are algebraic of order six.*

Proof. We take the generating curve in the plane $x = y$

$$(c_1(u), c_2(u) = c_1(u), c_3(u))^T,$$

where u is taken from some suitable interval I. Applying (10) gives the representation

$$\text{(42)}\quad \begin{aligned} f(u,v) = \Big(& c_1(u) + c_3(u)\,v + p\left(\frac{v^3}{3} + v\right), c_1(u) + \\ & + c_3(u)\,v + p\left(\frac{v^3}{3} - v\right), c_3(u) + p\,v^2\Big)^T. \end{aligned}$$

According to (4) we calculate

$$\text{(43)}\quad D_{11} = 2p\,(\ddot{c}_1\dot{c}_3 - \dot{c}_1\ddot{c}_3),\ D_{12} = 2p\,\dot{c}_3^2,\ D_{22} = -4p^2\,\dot{c}_1.$$

If $\dot{c}_3 = 0$ in I we have $D_{11} = D_{12} = 0$ so that f is degenerate. Hence locally we can take $c_3(u) = u$. Then (43) becomes

$$\text{(44)}\quad D_{11} = 2p\,\ddot{c}_1,\quad D_{12} = 2p,\quad D_{22} = -4p^2\,\dot{c}_1.$$

If the surface has planar affine normals so these normals are necessarily parallel to the invariant planes $x = y$, that means

$$\text{(45)}\quad \xi^1(u,v) = \xi^2(u,v).$$

A straightforward calculation of ξ shows the equivalence of (45) with

$$\text{(46)}\quad \dot{c}_1(u) = \sqrt{Au + B},\ (A,\ B) \in \mathbb{R}^2\backslash\{0,0\}.$$

Calculating the third component of the affine normal ξ using (46) we get

$$\xi = \mu(1,1,0)^T,$$

where $\mu \neq 0$ is a constant. So Φ is an improper affine sphere.

In case of $A = 0$ we have

$$c_1(u) = \beta\,u + \beta_1,$$

where we can take $\beta_1 = 0$ without restriction. Inserting this into (42) gives

$$\text{(47)}\quad f(u,v) = \left(p\left(\frac{v^3}{3}+v\right)+u\,(\beta+v), p\left(\frac{v^3}{3}+v\right)+u\,(\beta+v),\ v^2+u\right)^T.$$

Applying the coordinate transformation

$$\text{(48)}\quad x_1 = \frac{1}{\sqrt{2}}(x-y),\quad x_2 = z,\ x_3 = \frac{1}{\sqrt{2}}(x+y)$$

gives

$$f(u,v)=\left(\sqrt{2}\,p\,v,\,u+p\,v^2,\,\sqrt{2}\left(p\frac{v^3}{3}+u\,(\beta+v)\right)\right)^T \tag{49}$$

solving the equation (in the previous notation $x:=x_1{}'$, $y:=x_2{}'$, $z:=$ $=x_3{}'$)

$$3\,p^2\,z=3\,p\,x\,y-x^3+\frac{3\,p\,\beta}{\sqrt{2}}\,(2\,p\,y-x^2). \tag{50}$$

Changing to osculating lines as parametric lines taking

$$\varphi(u',v')=(u'+\beta\,p\,(v'-\beta),v'-\beta)=(u,v),$$

gives $f':=f\circ\varphi$ with

$$f'(u',v')=\begin{pmatrix} p\sqrt{2}\,(v'-\beta)\\ p\,v'\,(v'-\beta)\\ \frac{p\sqrt{2}}{3}(v'^3-\beta^3)\end{pmatrix}+u'\begin{pmatrix}0\\1\\ \sqrt{2}\,v'\end{pmatrix}, \tag{51}$$

that is except for a translation of the coordinate system representation (38).

In case of $A\neq 0$ we have from (46)

$$c_1(u)=\frac{2}{3\,A}(A\,u+B)^{3/2},$$

by omitting an additive constant. Reparametrizing the generating curve (using the previous notation) by

$$(c_1(u),c_2(u),c_3(u))=(u^3,u^3,\beta\,u^2),\ (\beta\neq 0),$$

gives (40). Applying (47) and then eliminating u and v finally gives (41). ◇

Remark 9. The Cayley surface is well known to be equiaffine homogeneous ([8, p. 93]). The surface admits a one parameter family of cubic screw transformations (see for instance [1, p. 100], [12, p. 80], [17, p. 124]). Iff $\beta=0$ the surfaces (39) are *right helicoids* with respect to cubic screw transformations; in this case in every point S of the surface the generator e_S intersects the orbit of S orthogonally (in the sense of Minkowski inner product (7)) (see [17, p. 120], [2, p. 311]). The surfaces (38) and (39) respectively are affinely equivalent to the special Cayley surface

$$z=x\,y-\frac{x^3}{3}.$$

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