Mathematica Pannonica
17/1 (2006), 49–59

ON THE SCHWAB–BORCHARDT MEAN II

Edward **Neuman**

Department of Mathematics, Southern Illinois University, Carbondale, IL 62901-4408, USA

József **Sándor**

Department of Pure Mathematics, Babeş–Bolyai University, 400084 Cluj-Napoca, Romania

Received: April 2005

MSC 2000: 26 D 15

Keywords: Schwab–Borchardt mean, inequalities, Seiffert means, R-hypergeometric functions.

Abstract: This paper deals with the inequalities involving the Schwab–Borchardt mean and other bivariate means including those introduced by H.-J. Seiffert in [12], [13]. The main results of this paper are obtained using a representation of the Schwab–Borchardt mean in terms of the R-hypergeometric functions of two variables. The Ky Fan type inequalities for the particular means contained in the family of the Schwab–Borchardt means are also included.

1. Introduction

The Schwab–Borchardt mean of two numbers $x \geq 0$ and $y > 0$, denoted by $SB(x, y) \equiv SB$, is defined as

E-mail addresses: edneuman@math.siu.edu, jjsandor@hotmail.com

$$
(1.1)\qquad SB(x,y)=\begin{cases}\dfrac{\sqrt{y^2-x^2}}{\arccos(x/y)}, & 0\le x<y\\[2mm] \dfrac{\sqrt{x^2-y^2}}{\operatorname{arccosh}(x/y)}, & y<x\\[2mm] x, & x=y.\end{cases}
$$

The notation SB used for the mean under discussion has been introduced in [7]. Other symbols employed to denote this mean are C_{14} (see [1, p. 257]) and L_{14} (see [4]). The Schwab–Borchardt mean is the iterative mean, i.e.,

$$
SB=\lim_{n\to\infty}x_n=\lim_{n\to\infty}y_n\,,
$$

where

$$
x_0=x,\quad y_0=y,\quad x_{n+1}=(x_n+y_n)/2,\quad y_{n+1}=\sqrt{x_{n+1}y_n},
$$

$n=0,1,\ldots$ (see [4, (2.3)], [1, p. 257], [2]). The invariance property

$$
(1.2)\qquad SB(x,y)=SB\left(\frac{x+y}{2},\sqrt{\frac{x+y}{2}y}\right)
$$

will be frequently used in this paper. Other elementary properties of the mean in question include monotonicity in its variables, i.e., $SB(x,y)$ increases with an increase in either x or y and the homogeneity of degree one

$$
(1.3)\qquad SB(\lambda x,\lambda y)=\lambda SB(x,y)\qquad(\lambda>0).
$$

In what follows the unweighted power mean of order $p\in\mathbb{R}$ of x and y will be denoted by A_p. Recall that

$$
A_p(x,y)\equiv A_p=\begin{cases}\left(\frac{x^p+y^p}{2}\right)^{1/p}, & p\neq 0\\ \sqrt{xy}, & p=0.\end{cases}
$$

For the sake of notation we will write G, A, and Q for the power means of order 0, 1, and 2, respectively. The last of these three means is often called the root-mean-square. Other means used in this paper include two means introduced by H.-J. Seiffert

$$
(1.4)\qquad P(x,y)\equiv P=\frac{x-y}{2\arcsin\left(\dfrac{x-y}{x+y}\right)}
$$

(see [12]),

$$(1.5) \qquad T(x,y) \equiv T = \frac{x-y}{2\arctan\left(\dfrac{x-y}{x+y}\right)}$$

(see [13]), the logarithmic mean

$$(1.6) \qquad L(x,y) \equiv L = \frac{x-y}{\ln x - \ln y} = \frac{x-y}{2\operatorname{arctanh}\left(\dfrac{x-y}{x+y}\right)}$$

(see, e.g., [5]) and a mean introduced in [7]

$$(1.7) \qquad M(x,y) \equiv M = \frac{x-y}{2\operatorname{arcsinh}\left(\dfrac{x-y}{x+y}\right)}.$$

Inequalities involving Seiffert means are derived in [14], [10]–[11], and in [8]. It has been pointed out in [7, (2.8)] that the means defined in (1.4)–(1.7) are particular cases of the Schwab–Borchardt mean, i.e.,

$$(1.8) \quad L = SB(A,G), \ P = SB(G,A), \ M = SB(Q,A), \ T = SB(A,Q).$$

Also, they satisfy the inequalities

$$(1.9) \qquad G \le L \le P \le A \le M \le T \le Q$$

(see [7, (2.10)]). Equalities hold in (1.9) if $x = y$. It is worth mentioning that these particular means satisfy the Ky Fan inequalities. Let $0 < < x, y \le \frac{1}{2}$ and let $x' = 1 - x$ and $y' = 1 - y$. Writing G' for $G(x', y')$, L' for $L(x', y')$, etc., we have

$$(1.10) \qquad \frac{G}{G'} \le \frac{L}{L'} \le \frac{P}{P'} \le \frac{A}{A'} \le \frac{M}{M'} \le \frac{T}{T'}$$

(see [7, Prop. 2.2]).

This paper is a continuation of our earlier work [7] and deals mostly with inequalities for the Schwab–Borchardt mean and particular means mentioned in this section. In Sec. 2 we give some results for the R-hypergeometric functions of two variables. The main results of this paper are contained in Sec. 3. Utilizing a relationship between the mean under discussion and the R-hypergeometric function we prove several new inequalities. Employing the so-called Ky Fan rules, which have been obtained in [9], we derive the Ky Fan type inequalities for means that satisfy the chain of inequalities (1.10).

2. The R-hypergeometric functions of two variables

In this section we give the definition of the bivariate R-hypergeometric functions. Some results for these functions are also included here.

In what follows the symbols $\mathbb{R}_+$ and $\mathbb{R}_>$ will stand for the non-negative semi-axis and the set of positive numbers, respectively. Let $b = (b_1, b_2) \in \mathbb{R}^2_+$. By μ_b, where

$$\mu_b(t) = \frac{\Gamma(b_1 + b_2)}{\Gamma(b_1)\Gamma(b_2)} t^{b_1 - 1}(1 - t)^{b_2 - 1}$$

we will denote the Dirichlet measure on the interval $[0, 1]$. It is well-known that μ_b is the probabillity measure on its domain. This in turn implies that

$$\int_0^1 \mu_b(t)\, dt = 1$$

(see, e.g., [5]). Also, let $X = (x, y) \in \mathbb{R}^2_>$. Following [5] the R-hypergeometric function $R_p(b; X)$ $(p \in \mathbb{R})$ is defined as

$$R_p(b; X) = \int_0^1 (u \cdot X)^p \mu_b(t)\, dt, \tag{2.1}$$

where $u = (t, 1 - t)$ and $u \cdot X = tx + (1 - t)y$ is the dot product of u and X. Many of the important special functions, including celebrated Gauss' hypergeometric function ${}_2F_1$ and the complete elliptic integrals R_K and R_E admit the integral representation (2.1). For more details, the interested reader is referred to Carlson's monograph [5]. A nice feature of the R-hypergeometric function is its permutation symmetry in both parameters and variables. It follows from (2.1) that

$$R_p(b_1, b_2; x, y) = R_p(b_2, b_1; y, x). \tag{2.2}$$

For later use, let us record Carlson's inequality [3, Th. 3]

$$\big[R_p(b; X)\big]^{1/p} \le \big[R_q(b; X)\big]^{1/q} \tag{2.3}$$

$(p, q \ne 0,\ p \le q)$.

We will need the following.

Proposition 2.1. *Let $p < 0$, $b \in \mathbb{R}^2_+$, and let $X, Y \in \mathbb{R}^2_>$. Then the following inequality*

$$R_p\big(b;\lambda X+(1-\lambda)Y\big)\le\big[R_p(b;X)\big]^{\lambda}\big[R_p(b;Y)\big]^{1-\lambda} \tag{2.4}$$

holds true for all $0\le\lambda\le 1$.

Proof. Inequality (2.4) states that the function R_p is logarithmically convex (log-convex) in its variables. For the proof of (2.4) let us note that the function $z\to z^p$ $(z>0)$ is log-convex provided $p<0$. Thus the following inequality

$$(\lambda r+(1-\lambda)s)^p\le(r^p)^{\lambda}(s^p)^{1-\lambda} \tag{2.5}$$

holds true for positive numbers r and s $(0\le\lambda\le 1)$.

Making use of (2.1), (2.5) and Hölder's inequality for integrals we obtain

$$\begin{aligned}R_p\big(b;\lambda X+(1-\lambda)Y\big)&=\int_0^1\big[u\cdot(\lambda X+(1-\lambda)Y)\big]^p\mu_b(t)\,dt=\\&=\int_0^1\big[\lambda(u\cdot X)+(1-\lambda)(u\cdot Y)\big]^p\mu_b(t)\,dt\le\\&\le\int_0^1\big[(u\cdot X)^p\mu_b(t)\big]^{\lambda}\big[(u\cdot Y)^p\mu_b(t)\big]^{1-\lambda}dt\le\\&\le\left[\int_0^1(u\cdot X)^p\mu_b(t)\,dt\right]^{\lambda}\left[\int_0^1(u\cdot Y)^p\mu_b(t)\,dt\right]^{1-\lambda}=\\&=\big[R_p(b;X)\big]^{\lambda}\big[R_p(b;Y)\big]^{1-\lambda}.\end{aligned}$$

The proof is complete. ◊

When $p=-\frac{1}{2}$ and $b=\left(\frac{1}{2},1\right)$, then the corresponding R-hypergeometric function is denoted by R_C (see [5]), i.e.,

$$R_C(x,y)=R_{-1/2}\left(\frac{1}{2},1;x,y\right). \tag{2.6}$$

A formula connecting R_C with the mean SB appears in [5] (also see [2, (2.31)]). We have

$$SB(x,y)=\big[R_C(x^2,y^2)\big]^{-1}. \tag{2.7}$$

3. Main results

To this end we will always assume that $x,y>0$. We are in a position to prove the following.

Theorem 3.1. *The following inequalities*

$$y\,SB(y,x) \le SB^2(x,y) \le \left[SB^2(y,x) + y^2\right]/2, \tag{3.1}$$

$$A\sqrt{y}\,SB(\sqrt{y},\sqrt{A}) \le SB^2(x,y), \tag{3.2}$$

and

$$SB(y,x) \le SB^2(\sqrt{x},\sqrt{A}) \tag{3.3}$$

hold true.

Proof. In order to prove the first inequality in (3.1) we shall show first that the following formula

$$R_{-1}\left(\frac{1}{2}, 1; x^2, y^2\right) = y^{-1}R_C(y^2, x^2) \tag{3.4}$$

is valid. To this aim we employ a known transformation for the R-hypergeometric functions [5, (5.9–19)]

$$R_{-a_1}(b_1, b_2; x, y) = y^{b_1-a_1}R_{-b_1}(a_1, a_2; x, y),$$

where $a_1 + a_2 = b_1 + b_2$. Letting $a_1 = 1$, $a_2 = \frac{1}{2}$, $b_1 = \frac{1}{2}$, $b_2 = 1$ and next replacing x by x^2 and y by y^2 we obtain, using the permutation symmetry (2.2),

$$R_{-1}\left(\frac{1}{2}, 1; x^2, y^2\right) = y^{-1}R_{-1/2}\left(1, \frac{1}{2}; x^2, y^2\right) =$$
$$= y^{-1}R_{-1/2}\left(\frac{1}{2}, 1; y^2, x^2\right) = y^{-1}R_C(y^2, x^2),$$

where in the last step we have used formula (2.6). The inequality in question is established with the aid of (2.7), (3.4), and the inequality

$$\left[R_{-1}\left(\frac{1}{2}, 1; x^2, y^2\right)\right]^{-1} \le \left[R_C(x^2, y^2)\right]^{-2}$$

which follows easily from Carlson's result (2.3). We have

$$y\,SB(y,x) = y\left[R_C(y^2, x^2)\right]^{-1} = \left[R_{-1}\left(\frac{1}{2}, 1; x^2, y^2\right)\right]^{-1} \le$$
$$\le \left[R_C(x^2, y^2)\right]^{-2} = SB^2(x,y).$$

For the proof of the second inequality in (3.1) we use Lemma 2.3 in [6] to obtain

$$R_C(x^2,y^2)R_{1/2}\left(\frac{1}{2},1;x^2,y^2\right)\geq 1.$$

Application of

$$R_{1/2}\left(\frac{1}{2},1;x^2,y^2\right)=\frac{1}{2}\left[x+y^2R_C(x^2,y^2)\right]$$

(see [5, Table 8.5-1]) gives

$$\frac{1}{2}R_C(x^2,y^2)\left[x+y^2R_C(x^2,y^2)\right]\geq 1.$$

Making use of (2.7) we obtain

$$SB^2(x,y)\leq\frac{1}{2}\left[xSB(x,y)+y^2\right].$$

Since $xSB(x,y)\leq SB^2(y,x)$ (see the first inequality in (3.1)) the desired result follows. Inequality (3.2) follows from the first inequality in (3.1). Replacing x by A and y by $\sqrt{Ay}$ and next using the invariance property (1.2) we obtain

$$\sqrt{Ay}\,SB(\sqrt{Ay},A)\leq SB^2(A,\sqrt{Ay})=SB^2(x,y). \tag{3.5}$$

Application of the homogeneity property (1.3) gives $SB(\sqrt{Ay},A)=$ $=\sqrt{A}\,SB(\sqrt{y},\sqrt{A})$. Combining this with (3.5) gives the inequality (3.2). In order to prove (3.3) we use the first inequality in (3.1) with y replaced by A and x replaced by $\sqrt{Ax}$. The result is

$$A\left[SB(A,\sqrt{Ax})\right]\leq SB^2(\sqrt{Ax},A).$$

Application of the invariance property (1.2) to the first member and the homogeneity property (1.3) to the second member of the last inequality compeletes the proof of (3.3). ◊

Inequalities for particular means defined in Sec. 1 are contained in Cor. 3.2. Therein we will also use means of an arbitrary order. For instance, the symbol P_r $(r\neq 0)$ will stand for the first Seiffert mean of order r which is defined as

$$P_r(x,y)\equiv P_r=\left[P(x^r,y^r)\right]^{1/r}.$$

Other means of an arbitrary order are derived from means of order one in the same way.

Corollary 3.2. *Let $r>0$. The following inequalities*

$$\begin{aligned}&GP\leq L^2\leq (G^2+P^2)/2, \qquad LA\leq P^2\leq (L^2+A^2)/2,\\ &AT\leq M^2\leq (A^2+T^2)/2, \quad MQ\leq T^2\leq (M^2+Q^2)/2,\end{aligned} \tag{3.6}$$

$$(3.7)\qquad (GA_r^2P_r)^{1/4} \le L_{2r} \le P_r,$$

and

$$(3.8)\qquad (A_{2r}A_r^2M_r)^{1/4} \le P_{2r} \le M_r$$

hold true. Inequalities (3.7) *and* (3.8) *are reversed if* $r < 0$.

Proof. Inequalities (3.6) follow immediately from the first inequality in (3.1) and from (1.8). For instance, substituting $x := A$ and $y := G$ in (3.1) and utilizing the first two formulas in (1.8) we obtain $GP \le \le L^2 \le (G^2 + P^2)/2$. The remaining three inequalities in (3.6) can be established in an analogous manner. We omit further details. For the proof of inequalities (3.7) we substitute $x := A$, $y := G$ and $A := A_{1/2}$ in (3.2) and also we let $x := G$, $y := A$ and $A := A_{1/2}$ in (3.3). The result is

$$(3.9)\qquad \sqrt{G}\,A_{1/2}\,SB(\sqrt{G}, \sqrt{A_{1/2}}\,) \le L^2(x,y) \le SB^4(\sqrt{G}, \sqrt{A_{1/2}}\,).$$

Use of $P(x,y) = S(G,A)$ (see (1.8)) yields $P(\sqrt{x},\sqrt{y}) = SB(\sqrt{G},\sqrt{A_{1/2}})$. This in conjunction with (3.9) gives

$$\sqrt{G}\,A_{1/2}\,P(\sqrt{x}, \sqrt{y}\,) \le L^2(x,y) \le P^4(\sqrt{x}, \sqrt{y}\,).$$

The inequalities (3.7) now follow by letting $x := x^{2r}$, $y := y^{2r}$ $(r \ne 0)$ and raising all members of the resulting inequalities to the power of $1/(4r)$. Inequalities (3.8) are derived from (3.2) and (3.3) in a similar fashion. We let $x := G$, $y := A$ and $A := A_{1/2}$ in (3.2) and $x := A$, $y := G$ and $A := A_{1/2}$ in (3.3). The resulting inequalities read as follows

$$(3.10)\qquad \sqrt{A}\,A_{1/2}\,SB(\sqrt{A}, \sqrt{A_{1/2}}\,) \le P^2(x,y) \le SB^4(\sqrt{A}, \sqrt{A_{1/2}}\,).$$

Using $M(x,y) = SB(Q,A)$ (see (1.8)) we obtain $M(\sqrt{x},\sqrt{y}) = = SB(\sqrt{A},\sqrt{A_{1/2}})$. Application of the last formula to (3.10) yields

$$\sqrt{A}\,A_{1/2}\,M(\sqrt{x}, \sqrt{y}\,) \le P^2(x,y) \le M^4(\sqrt{x}, \sqrt{y}\,).$$

The last inequalities are generalized to means of an arbitrary order in the same way as the one used earlier in the proof of (3.7). We omit further details. ◊

Theorem 3.3. *Let* $x_1, x_2, y_1, y_2 > 0$. *Then*

$$SB(x_1,y_1)SB(x_2,y_2) \le SB^2\left(\sqrt{\frac{x_1^2+x_2^2}{2}}, \sqrt{\frac{y_1^2+y_2^2}{2}}\right). \tag{3.11}$$

Proof. In order to establish (3.11) we let $p=-\frac{1}{2}$, $b=\left(\frac{1}{2},1\right)$, $X = (x_1^2,y_1^2)$, $Y=(x_2^2,y_2^2)$ and $\lambda=\frac{1}{2}$ in (2.4). The result is

$$R_C^2\left(\frac{x_1^2+x_2^2}{2}, \frac{y_1^2+y_2^2}{2}\right) \le R_C(x_1^2,y_1^2)R_C(x_2^2,y_2^2).$$

Raising both sides of the last inequality to the power of -1 and next using (2.7) we arrive at (3.11). ◊

Corollary 3.4. *Let $z>0$ and let Q and A stand for the root-mean-square and the arithmetic mean, respectively, of x and y. Then*

$$SB(x,z)SB(y,z) \le SB^2(Q,z), \tag{3.12}$$

$$SB(x,A)SB(y,A) \le M^2(x,y), \tag{3.13}$$

$$SB(z,x)SB(z,y) \le SB^2(z,Q), \tag{3.14}$$

$$SB(A,x)SB(A,y) \le T^2(x,y), \tag{3.15}$$

and

$$PM \le A^2, \qquad LT \le A^2. \tag{3.16}$$

Proof. Inequality (3.12) follows from (3.11) by letting $x_1=x$, $x_2=y$ and $y_1=y_2=z$ while (3.13) is a special case of (3.12) when $z=A$. For the proof of (3.14) we let $x_1=x_2=z$, $y_1=x$ and $y_2=y$ in (3.11) while (3.15) is a special case of (3.14) when $z=A$. Inequalities (3.16) follow from (3.12) and (3.14), respectively. To see this we let $x:=G$, $y:=Q$ and $z:=A$. Since the root-mean-square of G and Q is equal to A, the assertion follows. ◊

Before we will state the last result of this paper, let us recall the Ky Fan rules which have been derived in [9, Lemma 2.1]. They can be employed to obtain the Ky Fan type inequalities from the existing Ky Fan inequalities.

Let a, a', b, and b' be positive numbers.

(i) If $a \le b$ and $\frac{a}{a'} \le \frac{b}{b'} \le 1$ or if $b \le a$ and $1 \le \frac{a}{a'} \le \frac{b}{b'}$, then

$$\frac{1}{a'} - \frac{1}{a} \le \frac{1}{b'} - \frac{1}{b}. \tag{3.17}$$

(ii) If $a' \le b'$ and $\frac{a}{a'} \le \frac{b}{b'}$, then

$$aa' \le bb'. \tag{3.18}$$

(iii) If $\frac{a}{a'} \le \frac{b}{b'}$, then

$$\frac{a}{a'} \le \frac{a+b}{a'+b'} \le \frac{b}{b'} \tag{3.19}$$

and

$$\frac{a}{a'} \le \frac{\sqrt{ab}}{\sqrt{a'b'}} \le \frac{b}{b'}. \tag{3.20}$$

In what follows we will assume that x and y are positive numbers both not bigger than $1/2$ and we will write G' for $G(x', y')$, L' for $L(x', y')$, etc., where $x' = 1 - x$ and $y' = 1 - y$.

Theorem 3.5. *The following inequalities*

$$\frac{1}{G'} - \frac{1}{G} \le \frac{1}{L'} - \frac{1}{L} \le \frac{1}{P'} - \frac{1}{P} \le \frac{1}{A'} - \frac{1}{A} \le \frac{1}{M'} - \frac{1}{M} \le \frac{1}{T'} - \frac{1}{T} \tag{3.21}$$

and

$$GG' \le LL' \le PP' \le AA' \le MM' \le TT' \tag{3.22}$$

are valid.

Proof. In order to establish inequalities (3.21) it suffices to apply (3.17) to (1.10) and (1.9). Inequalities (3.22) are obtained in an analogous manner utilizing (3.18), (1.10) and (1.9). ◊

More Ky Fan type inequalities for the means under discussion can be obtained with the rule (iii) applied to (1.10) and (1.9). We omit further details.

References

[1] BORWEIN, J. M. and BORWEIN, P. B.: Pi and the AGM – A Study in Analytic Number Theory and Computational Complexity, Wiley, New York, 1987.

[2] BRENNER, J. L. and CARLSON, B. C.: Homogeneous mean values: weights and asymptotics, *J. Math. Anal. Appl.* **123** (1987), 265–280.

[3] CARLSON, B. C.: A hypergeometric mean value, *Proc. Amer. Math. Soc.* **16** (1965), 759–766.

[4] CARLSON, B. C.: Algorithms involving arithmetic and geometric means, *Amer. Math. Monthly* **78** (1971), 496–505.

[5] CARLSON, B. C.: Special Functions of Applied Mathematics, Academic Press, New York, 1977.

[6] NEUMAN, E.: Bounds for symmetric elliptic integrals, *J. Approx. Theory* **122** (2003), 249–259.

[7] NEUMAN, E. and SÁNDOR, J.: On the Schwab–Borchardt mean, *Math. Pannonica* **14/2** (2003), 253–266.

[8] NEUMAN, E. and SÁNDOR, J.: On certain means of two arguments and their extensions, *Int. J. Math. Math. Sci.*, **16** (2003), 981–993.

[9] NEUMAN, E. and SÁNDOR, J.: On the Ky Fan inequality and related inequalities, II, *Bull. Austral. Math. Soc.* **72** (2005), 87–107.

[10] SÁNDOR, J.: On certain inequalities for means, III, *Arch. Math (Basel)* **76** (2001), 34–40.

[11] SÁNDOR, J.: Über zwei Mittel von Seiffert, *Wurzel* **36** (2002), 104–107.

[12] SEIFFERT, H.-J.: Problem 887, *Nieuw Arch. Wisk.* **11** (1993), 176.

[13] SEIFFERT, H.-J.: Aufgabe 16, *Wurzel* **29** (1995), 87.

[14] SEIFFERT, H.-J.: Ungleichungen für einen bestimmten Mittelwert, *Nieuw Arch. Wisk.* **13** (1995), 195–198.