

CENTRALIZERS AND THE ISOMORPHISM PROBLEM FOR NEARRINGS

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Abstract: In this note we study various types of centralizers in nearing modules of a nearing R . We then go on to apply this to the isomorphism problem for nearrings which deals with the problem of finding canonical R -ideals so that faithful R -modules are isomorphic modulo these canonical R -ideals obtaining two results. One of these generalizes a recent result of Stuart Scott and the other generalizes a result of the author.

1. Introduction

In [6], this author began referring to the following problem as the isomorphism problem for nearrings:

Given two isomorphic nearrings R and S and two faithful modules G and H of R and S , respectively, are there canonical normal subgroups N of G and M of H so that G/N and H/M are isomorphic?

Actually, a formulation of this problem more appropriate for the nearing setting would be to replace normal subgroups by R - and S -ideals so that it reads as follows:

Given an isomorphism φ from a nearing R to a nearing S and two faithful modules G and H of R and S , respectively, are there respective canonical R - and S -ideals of N of G and M of H so that there is a group isomorphism β from G/N to H/M such that

$$((g + N)r)\beta = (g + N)\beta(r\varphi)$$

for all $g \in G$ and all $r \in R$?

Calling an endomorphism nearring of a group generated by automorphisms an automorphism nearring, an affirmative answer was obtained to the original problem in [6] when G and H are finite perfect groups and R and S are compatible automorphism nearings of G and H , respectively, by using the centers of G and H as the canonical normal subgroups. A generalization of this automorphism nearring result will appear in [3]. In a series of papers, Stuart Scott has obtained a succession of results culminating in [11] with an affirmative answer to the second formulation of the problem for compatible nearings using a type of ideal similar to the Fitting subgroup of a group when the module modulo this Fitting type ideal has no factors that are central in a certain sense.

A precise statement of Scott's result will be given in Sec. 3 of this paper where we will prove a more general result for compatible endomorphism nearings replacing the assumption of having none of these central type factors by a weaker assumption requiring the factor of the module by its Fitting type ideal to be perfect. In Sec. 4 of this paper we will generalize the result of [6], the most significant part of which is that the perfect assumption on the groups may be replaced by the weaker assumption of having the groups modulo their centers perfect. Further, we also will see that our isomorphism is not only a group isomorphism, but is actually an isomorphism of the form in the second statement of the isomorphism problem.

When considering the second statement of the isomorphism problem, we can simplify both its statement and our notation by identifying the nearings R and S in which case it takes on the form:

Given two faithful modules G and H of a nearring R , are there canonical R -ideals N of G and M of H so that G/N and H/M are isomorphic R -modules?

In fact, this last formulation of the isomorphism problem is the one studied in Scott's work and is the one we will consider in this paper.

Throughout this paper R will always denote a left, 0-symmetric nearring with identity. We also will use [4] as our basic reference on nearings and will follow notational conventions and terminology used in it with one exception: tame conditions will refer to those used in Scott's work. Thus, when we say an R -module G is tame, we mean that each R -subgroup of G is an R -ideal of G , which is equivalent to:

for each $g \in G$ and $r \in R$, given an element $x_1 \in G$ there exists an element $\beta \in R$ such that

$$(g + x_1)r - gr = x_1\beta.$$

When we say an R -module G is 2-tame, we mean: for each $g \in G$ and $r \in R$, given two elements $x_1, x_2 \in G$ there exists an element $\beta \in R$ such that

$$(g + x_1)r - gr = x_1\beta \text{ and } (g + x_2)r - gr = x_2\beta.$$

When we say an R -module G is 3-tame, we mean: for each $g \in G$ and $r \in R$, given three elements $x_1, x_2, x_3 \in G$ there exists an element $\beta \in R$ such that

$$(g + x_1)r - gr = x_1\beta, (g + x_2)r - gr = x_2\beta, \text{ and } (g + x_3)r - gr = x_3\beta.$$

The final tame condition we will encounter is for an R -module G to be compatible, by which we mean: for each $g \in G$ and $r \in R$ there exists an element $\beta \in R$ such that

$$(g + x)r - gr = x\beta$$

for all $x \in G$. A nearring R is called tame (2-tame, 3-tame, compatible) if it possesses a faithful tame (2-tame, 3-tame, compatible) module G . Note that if G is a faithful compatible R -module, then the group of inner automorphisms of G , $\text{Inn}(G)$, is contained in R . (Use $r = 1$.) Moreover, if R is an endomorphism nearring of G , then R is compatible if and only if $\text{Inn}(G) \subseteq R$ [5, Prop. 1.2]. In the terminology of [4], a tame endomorphism nearring is the same as our compatible endomorphism nearring.

Centralizers of subsets of R -modules of a nearring R play an important role in the study of the isomorphism problem. Unfortunately, several types of centralizers, often indicated by the same name and denoted by the same notation, have been used in the literature over the years causing readers to have to pay careful attention to which one is being used. In an effort to rectify this situation, we are going to begin with a careful development of centralizers using different terms and notation for the various types of centralizers.

2. Centralizers

When we apply a group theoretic term in this paper to a module G of a nearring R , its meaning and notation for it will be the usual group theoretic ones. As a first illustration of this, suppose that X is

a subset of G . When we speak of the centralizer of X in G , denoted $C_G(X)$, we will mean the usual group centralizer of X in G :

$$C_G(X) = \{g \in G \mid g + x = x + g \text{ for all } x \in X\},$$

But there are other types of centralizers that play more important roles when studying nearring modules. One, which first appeared in [9], is what this author has taken to calling the module centralizer of X in G denoting it by $MC_G(X)$ in [7]:

$$MC_G(X) = \{g \in G \mid gr + x = x + gr \text{ for all } x \in X \text{ and for all } r \in R\}.$$

In the next proposition we record some elementary properties of the module centralizer.

Proposition 2.1. (i) $MC_G(X)$ is an R -subset of G contained in $C_G(X)$.

(ii) If R is distributively generated by a multiplicative semigroup S and G is an (R, S) -module, then $MC_G(X)$ is an R -subgroup of G and is an R -ideal of G if X is an R -ideal of G .

(iii) If G is a tame R -module, then $MC_G(X)$ is an R -ideal of G .

Proof. (i) easily follows from the definition of $MC_G(X)$.

(ii) Observe that

$$MC_G(X) = \{g \in G \mid g\alpha + x = x + g\alpha \text{ for all } x \in X \text{ and for all } \alpha \in S\}$$

when R is distributively generated by S . Now if $g, h \in MC_G(X)$, $\alpha \in S$ and $x \in X$, it is easily checked that $(g - h)\alpha + x = x + (g - h)\alpha$ giving us that $MC_G(X)$ is an R -subgroup of G . If in addition X is an R -ideal of G , then for any $g \in MC_G(X)$, $h \in G$, $\alpha \in S$, and $x \in X$,

$$\begin{aligned} (g^h)\alpha + x &= -h\alpha + g\alpha + h\alpha + x = -h\alpha + g\alpha + x^{-h\alpha} + h\alpha = \\ &= -h\alpha + x^{-h\alpha} + g\alpha + h\alpha = x + (g^h)\alpha \end{aligned}$$

giving us that $MC_G(X)$ is an R -ideal of G .

(iii) This is a generalization of [9, Prop. 9.1] where X is taken to be an R -ideal of G with the same proof: If $g \in MC_G(X)$, we have $gR \subseteq C_G(X)$. Thus the sum of the R -ideals gR , $g \in MC_G(X)$, is an R -ideal contained in $C_G(G)$. It now follows that this sum is the same as $MC_G(X)$ giving us $MC_G(X)$ is an R -ideal of G . \diamond

Another type of centralizer introduced by Stuart Scott that has played an important role in his work actually involves a distributivity condition from which a commutativity condition follows. To define it, let us first introduce the following terminology: we will say that an R -subgroup H of G is R -distributive with a subset X of G if

$$(h+x)r = hr + xr \text{ for all } h \in H, x \in X, \text{ and } r \in R.$$

We then define the distributor of X , denoted $D_G(X)$, to be the union of all R -subgroups H that are R -distributive with X . Scott denotes $D_G(X)$ by $C_G(X)$ in [11] and calls it the centralizer of X in G , but we will not do so to avoid confusion with the usual group centralizer $C_G(X)$. In the next proposition, we list some basic properties of $D_G(X)$.

Proposition 2.2. (i) $D_G(X)$ is an R -subset of G contained in $MC_G(X)$.

(ii) If R is distributively generated by a multiplicative semigroup S , G is an (R, S) -module, and X is an R -subgroup of G , then $D_G(X) = MC_G(X)$.

(iii) If G is a 3-tame R -module, then $D_G(X)$ is an R -ideal of G .

Proof. (i) It is immediate from its definition that $D_G(X)$ is an R -subset of G . For its containment in $MC_G(x)$, first note that by using $r = -1$ in the distributivity condition $(h+x)r = hr + xr$ gives us $D_G(X) \subseteq C_G(X)$. Now since $D_G(X)$ is an R -subset of G , it follows that $hr \in C_G(X)$ for all $h \in H$ and $r \in R$, which in turn gives us that $D_G(X) \subseteq MC_G(X)$.

(ii) To show the required opposite containment in this setting, let $x \in X$, $g \in MC_G(X)$, and $r \in R$. Express r in the form

$$r = \varepsilon_1\alpha_1 + \cdots + \varepsilon_n\alpha_n$$

where $\varepsilon_i = \pm 1$ and $\alpha_i \in S$. We then obtain

$$\begin{aligned} (g+x)r &= (g+x)(\varepsilon_1\alpha_1 + \cdots + \varepsilon_n\alpha_n) \\ &= \varepsilon_1(g\alpha_1 + x\alpha_1) + \cdots + \varepsilon_n(g\alpha_n + x\alpha_n). \end{aligned}$$

Now using the fact that X is an R -subgroup of G , we can rewrite this as

$$(g+x)r = \varepsilon_1g\alpha_1 + \cdots + \varepsilon_ng\alpha_n + \varepsilon_1x\alpha_1 + \cdots + \varepsilon_nx\alpha_n = gr + xr.$$

Thus $MC_G(X) \subseteq D_G(X)$.

(iii) This is [11, Th. 31.1]. \diamond

Whether $MC_G(X)$ or $D_G(X)$ is used typically depends on the context in which one is working. If one is not working in a distributively generated setting, then $D_G(X)$ may be the more suitable centralizer to use (although notice by Prop. 2.2(iii) that one must pay the price of having G being 3-tame before we are assured that $D_G(X)$ is even an R -subgroup of G). In the case where R is distributively generated by a multiplicative semigroup S and G is an (R, S) -module—let us henceforth call this the d. g. setting— $MC_G(X)$ is often the preferred centralizer since it is always an R -subgroup by Prop. 2.1(ii). Moreover,

$D_G(X)$ agrees with $MC_G(X)$ if X is an R -subgroup of G (a case where centralizers are frequently used) in the d. g. setting by Prop. 2.2(ii).

As is done in group theory for $C_G(X)$, the definitions of $MC_G(X)$ and $D_G(X)$ can be modified to case where G is replaced by an R -subgroup H of G or X is replaced by a subset X/K of G/K where K is an R -ideal of G . For example,

$$MC_H(X) = \{h \in H | hr + x = x + hr \forall x \in X \text{ and } \forall r \in R\}$$

and

$$MC_G(X/K) = \{g \in G | gr + x + K = x + gr + K \forall x \in X \text{ and } \forall r \in R\}.$$

The reader can easily supply the definitions of $D_H(X)$ and $D_G(X/K)$.

Two important facts concerning centralizers of isomorphic factors are the following:

Proposition 2.3. *Suppose that $K \leq L$ and $M \leq N$ are R -ideals of an R -module G such that L/K and N/M are isomorphic R -modules.*

(i) [2, Prop. 1] *If G is a compatible R -module, then $MC_G(L/K) = MC_G(N/M)$.*

(ii) [11, Th. 31.2] *If G is a 3-tame R -module, then $D_G(L/K) = D_G(N/M)$.*

Following the convention stated at the beginning of this section, when we say an R -module G is nilpotent (solvable) in this paper, we will mean that G is a nilpotent (solvable) group. In [11], Scott uses stronger versions of nilpotency and solvability, called \mathbf{R} -nilpotency and \mathbf{R} -solvability, which can be defined as follows: An R -module G is R -nilpotent (R -solvable) if G has a series of R -ideals

$$(1) \quad 0 = G_0 < G_1 < G_2 < \dots < G_n = G$$

such that $D_G(G_{i+1}/G_i) = G$ ($G_{i+1} \leq D_G(G_{i+1}/G_i)$) for each i . In fact, Scott defines G to be R -solvable if for each i , $R/\text{Ann}_R(G_{i+1}/G_i)$ is a ring and G_{i+1}/G_i is a ring module of $R/\text{Ann}_R(G_{i+1}/G_i)$ (indicated by saying G_{i+1}/G_i is a ring module in [11]), but this is equivalent to our definition by the following elementary result whose proof involves a routine argument that will be left to the reader.

Proposition 2.4. *Suppose that G is a faithful R -module. Then R is a ring and G is a ring module of R if and only if $D_G(G) = G$.*

It is easy to see an R -nilpotent (R -solvable) module is a nilpotent (solvable) group. Conversely, in the d. g. setting, if an R -module G is a nilpotent (solvable) group, it is easy to see that G is R -nilpotent (R -solvable) by using the lower central series (derived series) of G for the series in (1).

Continuing to follow usual group theory conventions, the Fitting subgroup of (an additive) group G , denoted

$$F(G),$$

is the sum of the normal nilpotent subgroups of G . If G is an R -module, by the Fitting ideal of G , denoted

$$MF(G),$$

we shall mean the sum of the R -ideals of G that are nilpotent groups. We shall call the sum of the R -nilpotent R -ideals of G the distributive Fitting ideal of G and denote it by

$$DF(G).$$

If G is 3-tame, $DF(G)$ the same as $L(V)$ in [11]. Observe that

$$DF(G) \leq MF(G) \leq F(G)$$

with $DF(G) = MF(G)$ in the d. g. setting and $MF(G) = F(G)$ if R is the endomorphism nearring of G generated by $\text{Inn}(G)$, $I(G)$. If G is a finite group, it is well-known (see [8], for example) that $F(G)$ is the maximal nilpotent normal subgroup of G and equals the intersection of the group centralizers of the principal factors of G . In [10] and [11], Scott takes his Fitting submodule of a 3-tame R -module G to be the intersection of the distributors $D_G(H/K)$ over all type 2 factors H/K of G . The next theorem, which is a restatement of Th. 6.4 of [10] using our notation, shows that his Fitting submodule is the same as $DF(G)$ when G is 3-tame and R satisfies the descending chain condition on right ideals (dcr) thereby giving us a result similar to the one in the finite group case.

Theorem 2.5. *If G is a 3-tame R -module and R satisfies dcr, then:*

(i) $DF(G) = \bigcap D_G(H/K)$ where the intersection runs over all type 2 factors H/K of G .

(ii) $DF(G)$ is the maximal R -nilpotent R -ideal of G .

As an immediate consequence of Prop. 2.2(ii) and Th. 2.5, we have the following important special case.

Corollary 2.6. *If R is a compatible endomorphism nearring of a group G and R satisfies dcr, then:*

(i) $MF(G) = \bigcap MC_G(H/K)$ where the intersection runs over all type 2 factors H/K of G .

(ii) $MF(G)$ is the maximal nilpotent R -ideal of G .

Recall that a group G is perfect if G equals its commutator subgroup, G' , or equivalently if G contains no normal subgroup N such

that G/N is abelian. In the same spirit as we defined R -nilpotency and R -solvability, if G is an R -module that contains no R -ideal I such that $D_G(G/I) = G$, we will say that G is an \mathbf{R} -perfect module. It is easy to see in the d. g. setting that G is perfect if and only if G is R -perfect. In [11], Scott calls an R -module G perfect if G contains no R -ideal I such that G/I is a ring module. By Prop. 2.4, Scott's definition of perfect is equivalent to our definition of R -perfect.

The next result tells us that perfect modules are monogenic in many important instances.

Proposition 2.7. *Suppose G is an R -module and R satisfies docr.*

(i) [6, Lemma 4] *If R is a compatible endomorphism nearring of G , then G is a monogenic R -module if and only if G/G' is a monogenic. In particular, if G is perfect, then G is monogenic.*

(ii) [11, Prop. 10.5] *If G is a tame R -module and G is R -perfect, then G is monogenic.*

Sec. 3 of [3] contains versions of the first part of Prop. 2.7 with alternative assumptions to our docr assumption.

A large part of [11] is devoted to proving the following remarkable result appearing as Th. 32.3.

Theorem 2.8. *Suppose that G and H are faithful 3-tame R modules and R satisfies docr. Further, suppose that G and H are both R -perfect. If $K < L$ are R -ideals of G and $M < N$ are R -ideals of H such that L/K and N/M are isomorphic type 2 R -modules, then*

$$G/D_G(L/K) \simeq H/D_H(N/M)$$

as R -modules.

Two problems arising from Th. 2.8 deserving further investigation are whether shorter proofs of this theorem can be found and whether the assumptions of assuming G and H are R -perfect modules can be weakened.

We conclude this section with a consequence of Ths. 2.5(i) and 2.8 we shall need in the next section.

Corollary 2.9. *Suppose that G and H are faithful 3-tame R -modules and R satisfies docr. Further, suppose that G and H are R -perfect. Then*

$$\text{Ann}_R(G/DF(G)) = \text{Ann}_R(H/DF(H)).$$

Proof. By Th. 2.5(i), it follows that

$$\text{Ann}_R(G/DF(G)) = \cap \text{Ann}_R(G/D_G(L/K))$$

and

$$\text{Ann}_R(H/DF(H)) = \cap \text{Ann}_R(H/D_H(N/M))$$

where the intersections run over the type 2 factors L/K of G and N/M of H . By Th. 3.13 of [1] we know that the type 2 factors of G and H represent the all the isomorphism classes of type 2 R -modules (otherwise the annihilators of the socle series of G and H would properly contain $J_2(R)$) and hence our corollary now follows by Th. 2.8. \diamond

3. Isomorphism modulo the Fitting ideal

Sec. 39 of [11] gives various types of similar isomorphism results. Of these, Cor. 39.4 is the one stated in the third form of the isomorphism problem given in the introduction of this paper and we shall state it as our next theorem. In this statement, an R -module G is called Z -constrained if G contains no R -ideals $K < L$ such that $D_G(L/K) = G$.

Theorem 3.1. *Suppose that G and H are faithful compatible R -modules and R satisfies docr. If $G/DF(G)$ is Z -constrained, then $G/DF(G)$ and $H/DF(H)$ are isomorphic R -modules.*

A Z -constrained R -module is easily seen to be R -perfect. The purpose of this section is to extend Th. 3.1 to the case where R is an endomorphism nearring of both G and H and $G/MF(G)$ and $H/MF(H)$ (which are the same as $G/DF(G)$ and $H/DF(H)$) are perfect (which is the same as R -perfect). We first do the special case when G and H are perfect.

Lemma 3.2. *Suppose R is a compatible endomorphism nearring of two groups G and H and R satisfies docr. If G and H are both perfect groups, then $G/MF(G)$ and $H/MF(H)$ are isomorphic R -modules.*

Proof. As noted in the proof of Cor. 2.9, by Th. 3.13 of [1] there exist factors of R -ideals $L_1/K_1, \dots, L_n/K_n$ of G representing the isomorphism classes of type 2 R -modules. Likewise, there exist factors of R -ideals $N_1/M_1, \dots, N_n/M_n$ of H representing the isomorphism classes of type 2 R -modules, where we may assume our labeling is such that $L_i/K_i \simeq N_i/M_i$ for each i . By Prop. 2.3 and Cor. 2.6 it follows that

$$MF(G) = \cap_i MC_G(L_i/K_i) \text{ and } MF(H) = \cap_i MC_H(N_i/M_i)$$

and hence $G/MF(G)$ and $H/MF(H)$ are subdirect sums of

$$\oplus \sum_i G/MC_G(L_i/K_i) \text{ and } \oplus \sum_i H/MC_H(N_i/M_i),$$

respectively. By Th. 2.8, we may identify $G/MC_G(L_i/K_i)$ and

$H/MC_H(N_i/M_i)$. Let us do so and set

$$A_i = G/MC_G(L_i/K_i) = H/MC_H(N_i/M_i)$$

to simplify our notation. Also, let us denote $\bigoplus \sum A_i$ by A and denote the images of $G/MF(G)$ and $H/MF(H)$ in A by B and C , respectively. We are going to complete this proof by showing that $B = C$.

To begin to accomplish this, let τ_b denote the inner automorphism an element $b \in B$ induces on B . Since B is a compatible R -module, there exists an element $r_b \in R$ such that

$$x\tau_b = xr_b$$

for all $x \in B$. In fact, we claim the action of r_b on A is the same as conjugation by b . To verify our claim, let π_i denote the projection from $A = \bigoplus \sum A_i$ onto A_i and let $b = \sum b_i$ where $b_i \in A_i$. For any $a = \sum a_i$ in A where $a_i \in A_i$, there exist $x_i \in B$ such that $x_i\pi_i = a_i$ since B is a subdirect sum of the A_i . Using the fact that each π_i is an R -homomorphism,

$$\begin{aligned} ar_b &= \sum a_i r_b = \sum x_i \pi_i r_b = \sum (x_i r_b) \pi_i = \sum (-b + x_i + b) \pi_i = \\ &= \sum (-b_i + a_i + b_i) = a^b \end{aligned}$$

and our claim is verified. Likewise we also have that for each $c \in C$ there exists $r_c \in R$ such that the action of r_c on A is conjugation by c . Because of this, for any commutator $[b, c]$, $b \in B, c \in C$,

$$[b, c] = -b + b^c = -b + br_c \in B$$

and

$$[b, c] = (-c)^b + c = (-c)r_b + c \in C.$$

Hence the subgroup $[B, C]$ of A generated by the commutators $[b, c]$, $b \in B, c \in C$, which is an R -subgroup of A , will then be an R -ideal of each of the compatible modules B and C .

We next show that $[B, C] = B$. This will in effect complete our proof since by symmetry we will also have $[B, C] = C$. To see this, suppose $[B, C] < B$. Let $\bar{R} = R/\text{Ann}_R(B)$, in which case B is a faithful \bar{R} -module. By Cor. 2.9, C is also a faithful \bar{R} -module. Since the socle series of $B/[B, C]$ has finite length and since socle series factors are direct sums of type 2 R -modules, there is a maximal R -ideal M of B with $[B, C] \leq M$. Since B/M is a type 2 \bar{R} -module, C contains a factor P/Q of \bar{R} -ideals P and Q isomorphic to B/M . As $r_c = \tau_c$ acts as the identity on B/M for each $c \in C$, the same result holds for P/Q . But

then P/Q is abelian forcing B/M to be abelian which is impossible since B is perfect. Thus we do have $[B, C] = B$ and our proof is complete. \diamond

To prove the isomorphism result in the more general case where $G/MF(G)$ and $H/MF(H)$ are perfect, we shall reduce to the perfect case using minimal covers as was done in the proof of Th. 39.2 in [11]. For the reader not familiar with covers, let us sketch the details from [11]. Given a set S of isomorphism classes of type 2 R -modules and a faithful 2-tame R -module G , let $G(S)$ denote the set of all factors L/K of G whose isomorphism class lies in S . An R -ideal N of G covers $G(S)$ if $N + K/K \geq L/K$ for all $L/K \in G(S)$. If R satisfies dcr and the elements of S are the isomorphism classes on nonring type 2 R -modules, then $G(S)$ has a unique minimal cover W in G by [11, Th. 14.6] and W is R -perfect by [11, Cor. 15.2]. Moreover, if H is another faithful 2-tame R -module and U is the minimal cover of $H(S)$, then by [11, Th. 17.3],

$$(2) \quad \text{Ann}_R(W) = \text{Ann}_R(U).$$

Theorem 3.3. *Suppose R is a compatible endomorphism nearring of two groups G and H and R satisfies dcr. If $G/MF(G)$ and $H/MF(H)$ are both perfect groups, then $G/MF(G)$ and $H/MF(H)$ are isomorphic R -modules.*

Proof. We use the notation in the paragraph preceding this theorem. Observe that we must have

$$W + MF(G) = G,$$

for if $W + MF(G) < G$ then $G/(W + MF(G))$ contains a nonring type 2 factor since $G/MF(G)$ is perfect. But then W would not cover $G(S)$. Thus

$$G/MF(G) = (W + MF(G))/MF(G) \simeq W/W \cap MF(G).$$

Since $W \cap MF(G)$ is easily seen to be the maximal nilpotent R -ideal of W , we must have $MF(W) = W \cap MF(G)$ by Cor. 2.6(ii). Thus

$$G/MF(G) \simeq W/MF(W)$$

as R -modules. Likewise,

$$H/MF(H) \simeq U/MF(U)$$

as R -modules. Let

$$\bar{R} = R/\text{Ann}_R(W).$$

By equation (2), we also have

$$\bar{R} = R/\text{Ann}_R(U).$$

As W and U are both perfect, Lemma 3.2 now gives us that

$$W/MF(W) \simeq U/MF(U)$$

as \overline{R} -modules. Since this is an R -isomorphism as well, we then have the desired R -isomorphism $G/MF(G) \simeq H/MF(H)$. \diamond

4. Automorphism nearrings

Recall from the introduction that an automorphism nearring of a group G is a nearring generated by a group of automorphisms of G . In [6] the following theorem was proved where $Z(G)$ denotes the center of G .

Theorem 4.1. *Suppose that G and H are perfect finite groups and that R and S are compatible automorphism nearrings of G and H , respectively. If R and S are isomorphic, then $G/Z(G)$ and $H/Z(H)$ are isomorphic groups.*

In this section we shall strengthen this to the next result following the identification of R and S convention mentioned in the introduction.

Theorem 4.2. *Suppose R is a compatible automorphism nearring of two groups G and H and R satisfies docr. If $G/Z(G)$ and $H/Z(H)$ are both perfect groups, then $G/Z(G)$ and $H/Z(H)$ are isomorphic R -modules.*

The change of the finiteness condition from finite groups to docr and the change from simply isomorphic as groups to R -isomorphic are actually minor changes. The most significant change is from perfect to center by perfect and, of course, is similar to the change from Lemma 3.2 to Th. 3.3. Here, however, we will not use the first result to prove the second as we did in the last section. Instead we will modify the proof of Th. 4.1 to obtain the proof of Th. 4.2. We first prove a lemma generalizing Lemma 3 of [6]. In the statement of this lemma, $MZ(G)$, called the module center of G , is

$$MZ(G) = MC_G(G).$$

Observe that $MZ(G) = Z(G)$ if R is an automorphism nearring of G .

Lemma 4.3. *Suppose that R is an endomorphism nearring of a group G . If $G/MZ(G)$ is a monogenic R -module, then every distributive element of R is an endomorphism of G .*

Proof. Suppose that

$$G/MZ(G) = (g + MZ(G))R$$

where $g \in G$. Let $x, y \in G$ and α be a distributive element of R . Writing

$x = gr + z_1$ and $y = gs + z_2$ where $r, s \in R$ and $z_1, z_2 \in MZ(G)$, we have

$$(x + y)\alpha = (gr + z_1 + gs + z_2)\alpha = (g(r + s) + z_1 + z_2)\alpha.$$

Now since $MZ(G) = D_G(G)$ by Prop. 2.2(ii) and $(r + s)\alpha = r\alpha + s\alpha$, we obtain

$$\begin{aligned} (x + y)\alpha &= g(r + s)\alpha + z_1\alpha + z_2\alpha = gr\alpha + z_1\alpha + gs\alpha + z_2\alpha = \\ &= (gr + z_1)\alpha + (gs + z_2)\alpha = x\alpha + y\alpha \end{aligned}$$

and hence α is an endomorphism of G . \diamond

We now prove Th. 4.2. Let $A = \text{Aut}(G) \cap R$ and $B = \text{Aut}(H) \cap R$ where $\text{Aut}(G)$ and $\text{Aut}(H)$ denote the automorphism groups of G and H , respectively. Since $G/MZ(G)$ is monogenic by Prop. 2.8 and since the elements of B are distributive invertible elements of R , Lemma 4.3 gives us that $B \leq A$. Likewise we have $A \leq B$, and hence $A = B$.

Keeping in mind that both $\text{Inn}(G)$ and $\text{Inn}(H)$ are contained in R , let

$$K_1 = \text{Inn}(G), \quad K_2 = \text{Inn}(H), \quad \text{and} \quad N = K_1 \cap K_2.$$

As we have done previously in this paper, let us denote the inner automorphism $g \in G$ induces on G by τ_g . Let L be the inverse image of N under the homomorphism $g \rightarrow \tau_g$, which is a normal subgroup of G . In fact, L is an R -ideal of G . To verify this it suffices to show that for any $l \in L$, $l\sigma \in L$ for all $\sigma \in A$ since A additively generates R . This follows since

$$\tau_{l\sigma} = \tau_l^\sigma \in N$$

as N is a normal subgroup of A .

We next claim that the automorphisms in K_2 act trivially on G/L . For any $g \in G$ and $\sigma \in K_2$, we have

$$[\tau_g, \sigma] = \tau_g^{-1}\sigma^{-1}\tau_g\sigma \in K_1 \cap K_2 = N.$$

Thus $[\tau_g, \sigma] = \tau_l$ for some $l \in L$ and

$$\tau_{g\sigma} = \tau_g^\sigma = \tau_g[\tau_g, \sigma] = \tau_g\tau_l = \tau_{g+l}.$$

This gives us $g\sigma = g + l + z$ for some $z \in Z(G)$. As $Z(G) \leq L$, $g\sigma + L = g + L$ and hence K_2 acts trivially on G/L as claimed.

We now proceed in a manner similar to the last paragraph of the proof of Lemma 3.2. The trivial action of K_2 on G/L forces $L = G$, for suppose we had $L < G$. Let M a maximal R -ideal of G with $L \leq M$. Since R is a compatible endomorphism nearring of H , there is a factor P/Q of R -ideals of H that is R -isomorphic to G/M . As $K_2 = \text{Inn}(H)$

acts trivially G/M it acts trivially on P/Q . Hence the factor P/Q is abelian. But then G/M is abelian violating our assumption that $G/Z(G)$ is perfect.

Now that we have $L = G$, we have $\text{Inn}(G) = K_1 = N \leq K_2$. Interchanging the roles of G and H , we have $K_2 = \text{Inn}(H) \leq K_1$ and hence $\text{Inn}(G) = \text{Inn}(H)$. Thus for each $g \in G$, there exists an $h \in H$ such that $\tau_g = \tau_h$. The composite

$$g + Z(G) \rightarrow \tau_g = \tau_h \rightarrow h + Z(H)$$

is easily checked to be a group isomorphism from $G/Z(G)$ onto $H/Z(H)$. Further, it is an R -isomorphism since for any $\sigma \in A$,

$$\begin{aligned} (g + Z(G))\sigma &= g\sigma + Z(G) \rightarrow \tau_{g\sigma} = \tau_g^\sigma \\ &= \tau_h^\sigma = \tau_{h\sigma} \rightarrow h\sigma + Z(H) = (h + Z(G))\sigma \end{aligned}$$

and our proof is complete. \diamond

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