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CONTINUOUS IMAGES OF N* WHICH ARE HOMEOMORPHIC TO N*

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Abstract: In this paper we characterize the closed subsets T of a Parovičenko space X such that the space we obtain from X collapsing T to one point is also a Parovičenko space.

1. Introduction

All spaces will be Tychonoff.

We recall that a space X is said to be an F-space if two disjoint cozero sets of X are completely separated. Other characterizations of F-spaces can be found in [3].

For the sake of brevity we will say a space X to be a G-int space if every nonempty G_δ of X has nonempty interior.

As usual, we denote by $\mathbb{N}$ a countably infinite discrete space. It is well known that the space $\mathbb{N}^* = \beta\mathbb{N} \setminus \mathbb{N}$

(a) is a compact space of weight $\mathfrak{c}$;

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(b) is 0-dimensional;
(c) is a G-int space;
(d) is an F-space;
(e) has no isolated points.

Parovičenko proved that, under CH, all spaces having the above properties are homeomorphic to $\mathbb{N}^*$ and every compact space whose weight is less than or equal to c is a quotient of $\mathbb{N}^*$, hence a remainder of a compactification of $\mathbb{N}$ ([6]).

A space with properties (a)–(e) is said to be a Parovičenko space. One could conjecture that a continuous image of a Parovičenko space under a very simple map, for instance, a map which collapses two points to one point, is still a Parovičenko space. But it was proved that this is not generally true. In fact the property of F-space is not preserved, in general, by this kind of map. The sets of two points of a compact F-space such that, collapsing them to one point, we obtain an F-space were characterized in [2] Th. 1.1.

In Sec. 2 of this paper we will extend that result to all (Tychonoff) spaces. Furthermore we obtain a characterization of the closed subsets T of a normal F-space X such the space X/T obtained by collapsing T to one point is still an F-space.

In Sec. 3 we give some results about continuous images of G-int spaces. Finally, in Sec. 4, we use the results of the previous sections to give a characterization of the closed subsets T of a Parovičenko space X such that X/T is a Parovičenko space. Under CH this is equivalent to characterize the closed subsets T of $\mathbb{N}^*$ such that $\mathbb{N}^*/T$ is homeomorphic to $\mathbb{N}^*$.

2. Images of F-spaces under quotient maps

We recall that a point x of the space X is said to be a P-point if every G_δ subset G of X, such that $x \in G$, is a neighborhood of x. It was proved in [5] that, under CH, $\mathbb{N}^*$ has 2^c P-points. Every point in $\overline{M} \setminus M$, where M is a countable subset of $\mathbb{N}^*$, is not a P-point.

It is clear that x is a non-P-point if and only if $x \in \overline{F} \setminus F$, where F is an F_σ. Moreover, one has

Lemma 2.1. *Let X be a space and $x \in X$. Then x is a non-P-point if and only if there is a cozero set $C \subseteq X$ such that $x \in \overline{C} \setminus C$.*

Proof. Let x be a non-P-point and let G a G_δ such that $x \in G \setminus \operatorname{Int}(G)$. One has $G = \bigcap_{n\in\mathbb{N}} V_n$, where V_n is open for every n. Put

$B_n = X \setminus V_n$. Then $x \in \overline{\bigcup_{n\in\mathbb{N}} B_n} \setminus \left(\bigcup_{n\in\mathbb{N}} B_n\right)$. Since X is Tychonoff, for every n there is a cozero set C_n such that $x \notin C_n$ and $B_n \subseteq C_n$. Then $C = \bigcup_{n\in\mathbb{N}} C_n$ is a cozero set such that $x \in \overline{C} \setminus C$. The converse is obvious. ◊

For a map $f : X \to Y$, a *fiber* is a set of the form $f^{-1}(y)$, where $y \in f(X)$. We say that a fiber is nontrivial if its cardinality is greater than 1.

Lemma 2.2. *Let $g : X \to \mathbb{R}$ be a continuous map, $C = \mathrm{Coz}(g)$ be a cozero set of X and let $f : X \to Y$ be a quotient map such that C is disjoint from every nontrivial fiber of f. Then $f(C)$ is a cozero set of Y.*

Proof. Since g is constant (in fact equal to 0) on the nontrivial fibers of f, we can define a real-valued function h on Y such that $h \circ f = g$. Then h is continuous, because f is a quotient map. Clearly $\mathrm{Coz}(h) = f(C)$. ◊

Proposition 2.3. *Let X be an F-space and let $f : X \to Y$ be a quotient map with only one nontrivial fiber $f^{-1}(y) = \{x_1, x_2\}$. Then Y is an F-space if and only if x_1 or x_2 is a P-point.*

Proof. Suppose that both x_1 and x_2 are non-P-points. Let A_1, A_2 be disjoint cozero sets in X such that $x_i \in A_i$ $i = 1, 2$. By Lemma 2.1, $x_i \in \overline{C_i} \setminus C_i$, where C_i is a cozero set, $i = 1, 2$. For each i, put $D_i = A_i \cap C_i$. Then $x_i \in \overline{D_i} \setminus D_i$. D_1 and D_2 are disjoint cozero sets which do not meet the only nontrivial fiber of f. Therefore, by Lemma 2.2, $f(D_1)$ and $f(D_2)$ are disjoint cozero sets of Y. But clearly y is in the intersection of their closures, and this implies that they are not completely separated.

Conversely, we can suppose that x_1 is a P-point. Let C_1, C_2 be disjoint cozero sets of Y. Then their inverse images $F_i = f^{-1}(C_i)$, $i = 1, 2$, are disjoint cozero sets in X and so they are completely separated. Let g be a continuous real-valued map such that $g(F_1) = 0$, $g(F_2) = 1$. First suppose $g(x_1) = g(x_2)$. This happens, in particular, if $y \in C_1$ or $y \in C_2$. Since g is constant on the only nontrivial fiber of f, there exists a continuous real-valued map h defined on Y such that $h \circ f = g$. One has $h(C_1) = 0$, $h(C_2) = 1$, then C_1 and C_2 are completely separated.

Now suppose $g(x_1) \neq g(x_2)$. Then $x_1 \notin F_i$ and so $x_1 \notin \overline{F_i}$, because x_1 is a P-point ($i = 1, 2$). Since x_1 is not in the closed set $\overline{F_1} \cup \overline{F_2} \cup \{x_2\}$, we can find a continuous real-valued map s such that $s(x_1) = g(x_2) - g(x_1)$ and $s(\overline{F_1} \cup \overline{F_2} \cup \{x_2\}) = 0$. Put $t = g + s$. Then one has:

$$t(x_1) = g(x_1) + g(x_2) - g(x_1) + = g(x_2) = g(x_2) + s(x_2) = t(x_2),$$
$$t(F_1) = 0, \qquad t(F_2) = 1.$$

Since t is constant on the only nontrivial fiber of f, we can prove, as before, that C_1 and C_2 are completely separated. ◊

The result of the above proposition was already known for compact F-spaces (see Th. 1.1 in [2]).

Proposition 2.4. *Let X be an F-space and let $f : X \to Y$ be a quotient map such that the union B of the nontrivial fibers is finite. Then Y is an F-space if and only if in every fiber of f there is at most one non-P-point.*

Proof. Suppose that $x_1, x_2 \in X$ are non-P-points and belong to the same fiber. We can find disjoint cozero sets A_1, A_2 which do not meet the closed set $B \setminus \{x_1, x_2\}$ and such that $x_i \in A_i$, $i = 1, 2$. As in Prop. 2.3, we can prove that Y is not an F-space.

Conversely, we can use the following easy fact: if a quotient map collapses only two points x_1, x_2 to one point y, then y is a P-point if and only if both x_1 and x_2 are P-points. Therefore the proof can be done inductively, using Prop. 2.3. ◊

A subset T of a space X is said to be a *P-set* if, for every G_δ subset G of X such that $T \subseteq G$, one has $T \subseteq \mathrm{Int}(G)$. Finite sets of P-points are closed P-sets.

If the space X is normal, Prop. 2.4 can be generalized.

Theorem 2.5. *Let X be a normal F-space and let $f : X \to Y$ be a quotient map with only one nontrivial fiber $T = f^{-1}(y)$. Then Y is an F-space if and only if either T is a P-set or there is $z \in T$ such that $T \setminus \{z\}$ is a P-set.*

Proof. Suppose Y is an F-space and T is a not a P-set. This means that there is a G_δ subset G of X such that $T \subseteq G$ and $T \not\subseteq \mathrm{Int}(G)$. Suppose $z \in T \setminus \mathrm{Int}(G)$ and put $F = X \setminus G$. Then $z \in \overline{F} \setminus F$. Let $F = \bigcup_{n \in \mathbb{N}} B_n$, where B_n is closed for every n. Since X is normal, for each n there is a cozero set C_n such that $T \cap C_n = \emptyset$ and $B_n \subseteq C_n$. Then $C = \bigcup_{n \in \mathbb{N}} C_n$ is a cozero set, disjoint from T, such that $z \in \overline{C}$.

Suppose $T \setminus \{z\}$ is not a P-set. Then there is a G_δ subset G_1 of X such that $T \setminus \{z\} \subseteq G_1$ and $(T \setminus \{z\}) \setminus \mathrm{Int}(G_1)$ contains at least one point x. Let U, W be disjoint open neighborhood of z and x respectively. Then $G_2 = G_1 \cup U$ is a G_δ which contains T. Furthermore $x \in G_2 \setminus \setminus \mathrm{Int}(G_2)$. In fact, if $x \in V \subseteq G_2$, where V is open, then $x \in V \cap \cap W \subseteq G_1$, since W is disjoint from U, and this is a contradiction. As before, we can find a cozero set C', disjoint from T, such that $x \in \overline{C'}$.

Let E, E' be disjoint cozero neighborhoods of z and x respectively. Then $D = C \cap E$ and $D' = C' \cap E'$ are disjoint cozero sets which do not meet T and such that $z \in \overline{D}$, $x \in \overline{D'}$. By Lemma 2.2, the images of D and D' are disjoint cozero sets in Y. But y is in the intersection of their closures, hence they are not completely separated. Contradiction.

We will prove the converse in the case $T \setminus \{z\}$ is a P-set. The other case is similar and easier. Suppose C_1, C_2 are disjoint cozero sets of Y. We can start defining F_i, $i = 1, 2$ and g as in Prop. 2.3. The cases $y \in C_1$ and $y \in C_2$ are proved exactly in the same way, since g is constant on T. If $y \notin C_i$, $i = 1, 2$, then $T \setminus \{z\}$ is disjoint from F_i, $i = 1, 2$ and so it is also disjoint from $\overline{F_i}$, because it is a P-set. Since $\overline{F_1}$ and $\overline{F_2}$ are disjoint, we can suppose, without loss of generality, that $z \notin \overline{F_1}$. Then T is disjoint from $\overline{F_1}$ and we can find a continuous real-valued function t, defined on X, such that $t(\overline{F}_1) = 0$ and $t(T \cup \overline{F_2}) = 1$. Since t is constant on T, we can prove, as in Prop. 2.3, that C_1 and C_2 are completely separated. ◊

Proposition 2.6. *Let X be a normal F-space and let $f : X \to Y$ be a quotient map. Suppose that $M = \{y \in Y \mid |f^{-1}(y)| > 1\}$ is a closed discrete subset of Y. Then Y is an F-space if and only if, for every $y \in M$, either $f^{-1}(y)$ is a P-set or there is z_y such that $f^{-1}(y) \setminus \{z_y\}$ is a P-set.*

Proof. For every $y \in M$, put $T_y = f^{-1}(y)$. If Y is an F-space, we can prove that every T_y satisfies the required properties in the same way as in the above theorem, taking E and E' disjoint from the union of the other nontrivial fibers.

Conversely, suppose C_1, C_2 are disjoint cozero sets of Y and let $F_i = f^{-1}(C_i)$, $i = 1, 2$. Then $\overline{F_1}$, $\overline{F_2}$ are disjoint. Let y be a point in M. As in the proof of the above theorem, T_y cannot meet both $\overline{F_1}$ and $\overline{F_2}$. Put $M_1 = \{y \in M \mid T_y \cap \overline{F_1} \neq \emptyset\}$, $M_2 = M \setminus M_1$. Let

$$H_i = \overline{F_i} \cup \left(\bigcup_{y \in M_i} T_y \right), \quad i = 1, 2.$$

H_1 and H_2 are disjoint saturated closed subsets of X. Then there exists a continuous real-valued map t on X, constant on the fibers of f, such that $t(F_1) = 0$, $t(F_2) = 1$. This implies that C_1 and C_2 are completely separated. ◊

Note that, for the "only if" part of the last four results, the hypothesis that X is an F-space is not necessary.

3. Images of G-int spaces under quotient maps

We give a result about continuous images of G-int spaces under closed irreducible maps.

Theorem 3.1. *If X is a G-int space and $f : X \to Y$ is is a closed irreducible (surjective) map, then Y is also a G-int space.*

Proof. Let $G \subseteq Y$ be a nonempty G_δ. Then $f^{-1}(G)$ is also a G_δ, hence it contains a nonempty open set U. Since f is onto, $f(U) \subseteq \subseteq f(f^{-1}(G)) = G$ and $Y \setminus f(U) \subseteq f(X \setminus U)$. By hypothesis, $f(X \setminus U)$ is a proper closed subset of Y. One has

$$Y \setminus G \subseteq Y \setminus f(U) \subseteq f(X \setminus U)$$

and so $Y \setminus f(X \setminus U)$ is a nonempty open set which is contained in G. $\Diamond$

Let X be a G-int space and T a closed subset of X. Obviously, if T is a non-open G_δ, X/T is not a G-int space. We prove that the converse is also true.

Theorem 3.2. *Let X be a G-int space and let $f : X \to Y$ be a quotient map with one nontrivial fiber $T = f^{-1}(y)$. Then Y is a G-int space if and only if either T is open or it is not a G_δ.*

Proof. We have only to prove that, if T is not a G_δ, then Y is a G-int space. Let G be a G_δ subset of Y. The only nontrivial case is when $y \in G$. In this case, $G_1 = f^{-1}(G)$ is a G_δ of X which properly contains T. Therefore $G_1 \cap (X \setminus T)$ is a nonempty G_δ, hence it contains a nonempty open subset U. Since U is saturated with respect to f, $f(U)$ is a nonempty open subset of G. $\Diamond$

Corollary 3.3. *Let X be a G-int space and let $f : X \to Y$ be a quotient map with one nontrivial fiber $T = f^{-1}(y)$. Suppose T satisfies one of the following conditions:*

(i) *T is a P-set;*

(ii) *T is nowhere dense;*

(iii) *$T = \overline{A}$, where A is an open F_σ (in particular, a cozero set).*

Then Y is a G-int space.

Proof. (i) and (ii) are obvious.

(iii) In a G-int space the closure of an open non-closed F_σ is never a G_δ (in particular, the closure of a non-closed cozero set is never a zero set). In fact, if A is an open non-closed F_σ and G is a closed G_δ containing A, then $G \setminus A = G \cap (X \setminus A)$ is a nonempty G_δ, so it contains a nonempty open set (disjoint from A). $\Diamond$

The statement (ii) was essentially known (see Lemma 1.4.2 in [7]). It could be also deduced by Th. 3.1. In fact, it is easy to see that a quotient map with one nontrivial fiber T is always closed and it is irreducible if and only if T is nowhere dense.

Corollary 3.4. *Let X be a G-int space with no isolated points and let $f : X \to Y$ be a quotient map with one nontrivial fiber $T = f^{-1}(y)$. Then Y is a G-int space with no isolated points if and only if T it is not a G_δ.*

Proposition 3.5. *Let X be G-int space with no isolated points and let $f : X \to Y$ be a quotient map. Suppose $M = \{y \in Y \mid |f^{-1}(y)| > 1\}$ is a closed discrete subset of Y. Then Y is a G-int space with no isolated points if and only if, for every $y \in M$, $f^{-1}(y)$ is not a G_δ.*

Proof. Let Y be a G-int space. Suppose that there is $y \in M$ such that $T = f^{-1}(y)$ is a G_δ. Then we can write $T = \bigcap_{n\in\mathbb{N}} V_n$, where each V_n is open and disjoint from $\bigcup_{z\in M\setminus\{y\}} f^{-1}(z)$, which is a closed subset of X. Since every V_n is saturated with respect to f, the singleton $\{y\}$ is a nonempty G_δ with empty interior, contradiction.

Conversely, by hypothesis, no point of M is a G_δ, and this implies that M is not a G_δ. Otherwise, for any $y \in M$, $\{y\} = M \cap W_y$, where W_y is an open neighborhood of y disjoint from $M \setminus \{y\}$, hence $\{y\}$ would be a G_δ. Therefore, if $G \subseteq Y$ is a nonempty G_δ, then $G \setminus M$ must be nonempty. Then $f^{-1}(G \setminus M) = f^{-1}(G) \cap (X \setminus f^{-1}(M))$ is a nonempty G_δ in X and so it contains a nonempty open set U. Since U is saturated with respect to f, $f(U)$ is a nonempty open subset of G. $\Diamond$

4. Continuous images of Parovičenko spaces

Let X be a Parovičenko space and let T be a closed subset of X. We want to find conditions ensuring that X/T is also a Parovičenko space.

The following two propositions are essentially known.

Proposition 4.1. *Let X be a Parovičenko space and let $f : X \to Y$ be a continuous surjective map with one nontrivial fiber T. If T is a not open P-set, then Y is a Parovičenko space.*

Proof. Y is an F-space by Lemma 1.4.1 in [7]. The other properties can be easily proved. $\Diamond$

Proposition 4.2. *Suppose X is a Parovičenko space and let $f : X \to$ $\to Y$ be a continuous surjective map with one nontrivial fiber T. If T is a finite set, then Y is a Parovičenko space if and only if at most one point of T is a non-P-point.*

Proof. It follows from from Th. 1.1 and Lemma 1.2 in [2]) and from Lemma 1.4.2 in [7] (or from Prop. 2.4).

Remark 4.3. It is compatible that $\mathbb{N}^*$ does not contain any P-points. Under this hypothesis, no space obtained by collapsing finitely many points of $\mathbb{N}^*$ to one point can be a Parovičenko space.

We want to generalize Prop. 4.2. We need two lemmas.

Lemma 4.4. *Let X be a compact 0-dimensional F-space and let A be an open non-closed F_σ in X. Then $|\overline{A} \setminus A| \geq 2^{\mathfrak{c}}$. Equivalently, if G is a closed non-open G_δ in X, then $|G \setminus \mathrm{Int}(G)| \geq 2^{\mathfrak{c}}$.*

Proof. It is easy to prove that one has $A = \bigcup_{n \in \mathbb{N}} C_n$, where the C_n's are nonempty, clopen and pairwise disjoint. If we choose $x_n \in C_n$, for every n, then $M = \{x_n \mid n \in \mathbb{N}\}$ is discrete. Since X is an F-space and M is countable, M is C^*-embedded in X, hence $|\overline{M} \setminus M| = |\mathbb{N}^*| = 2^{\mathfrak{c}}$. One has $\overline{M} \setminus M \subseteq \overline{A} \setminus A$, because every point of A belongs to a C_n, which contains only one element of M. ◊

Lemma 4.5. *Let X be a Parovičenko space and let T be a closed not open subset of X. Suppose $T = S \cup \{z\}$, where S is a P-set and $z \notin S$. Then T is not a G_δ.*

Proof. First suppose that S is closed. If T were a G_δ, then $\{z\} = T \cap \cap (X \setminus S)$ would be a G_δ and this is impossible, since X is a G-int space with no isolated points.

If S is not closed, then one has $T = \overline{S}$. Suppose T is a G_δ. Since S is a P-set, there is an open set U such that $S \subseteq U \subseteq T$. By hypothesis, T is not open, then it must be $S = U$. This implies $|T \setminus \mathrm{Int}(T)| = 1$, which is impossible by Lemma 4.4. Therefore T is not a G_δ. ◊

Finally we can prove:

Theorem 4.6. *Let X be a Parovičenko space and let $f : X \to Y$ be a continuous surjective map with one nontrivial fiber $T = f^{-1}(y)$, where T is not open. Then Y is a Parovičenko space if and only if T is a P-set or there is $z \in T$ such that $S = T \setminus \{z\}$ is a P-set.*

Proof. The "only if" part follows from Th. 2.5. Conversely, the case when T is a P-set has been already considered in Prop. 4.1. In the other case, Y is a G-int space with no isolated points by Lemma 4.5 and Cor. 3.4. Moreover Y is an F-space again by Th. 2.5. The other properties are easy to prove. ◊

The proof of the following corollary is easy. It can be also deduced from Props 2.6 and 3.5 and from Lemma 4.5.

Corollary 4.7. *Let X be a Parovičenko space and let $f : X \to Y$ be a continuous surjective map with finitely many nontrivial fibers $T_i =$*

$= f^{-1}(y_i)$, $i = 1, \ldots n$. *Then Y is a Parovičenko space if and only if, for every i, either T_i is a P-set or there is $z_i \in T_i$ such that $T_i \setminus \{z_i\}$ is a P-set.*

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