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# CONTINUOUS DEPENDENCE OF $\delta$ ON $\varepsilon$ AS A SELECTION PROPERTY

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**Abstract**: We give a negative answer to a question of Malešič and Repovš, about continuous selections of the modulus of continuity. We also provide a shorter proof for a theorem of the same authors, which generalizes a classical result about continuous dependence of $\delta$ on $\varepsilon$ in the definition of continuous function.

## 1. Introduction

Let $(X, d)$, $(Y, \rho)$ be metric spaces, and let $C(X, Y)$ be the space of continuous functions from $X$ to $Y$, equipped with the uniform convergence topology. Then for every $(x, \varepsilon, f) \in X \times ]0, +\infty[ \times C(X, Y)$ there exists a positive number $\delta(x, \varepsilon, f)$, such that whenever $d(x, x') < < \delta(x, \varepsilon, f)$ we have $\rho(f(x), f(x')) < \varepsilon$.

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One could wonder whether the map $\delta(x, \varepsilon, f)$ may be chosen to be continuous with respect to the product topology on $X \times ]0, +\infty[ \times \times C(X, Y)$. The answer is affirmative and different proofs have been provided (see [6], [1], [3], [2]). Among them, the shortest and most elegant one is due to G. De Marco [1, 2].

In [5, Th. 1.4] J. Malešič and D. Repovš developed the idea of G. De Marco in a more general setting, providing a criterion for the existence of a continuous map $\delta_P(x, \varepsilon, f, d, \rho)$, where $P$ is a continuity-like predicate (which includes, in particular, the case of continuity and of upper semi-continuity). In the first section of the present paper, we provide a shorter and more direct proof of that criterion.

In the second section we give a negative answer to a conjecture about $\alpha$-continuity [5, Conj. 3.6]. At the same time, we show that under some slight modifications of the hypotheses such a conjecture may become true — see Ths. 2 and 3.

## 2. Modulus of continuity

Before tackling the subject, we need a preliminary result which is of some independent interest.

**Proposition 1.** *Let $X$ be a normal, countably paracompact space, and $\varphi$ a map from $X$ to $]0, +\infty[$ (with no assumption about continuity). Then the following are equivalent:*

(1) $\forall x \in X : \exists V$ *neighbourhood of* $x : \inf \varphi(V) > 0$;

(2) *there exists a continuous* $h: X \to ]0, +\infty[$, *such that* $\forall x \in \in X : h(x) < \varphi(x)$.

**Proof.** Suppose first that (2) holds: given an arbitrary $\bar{x} \in X$, take $\hat{\varepsilon} > 0$ with $\hat{\varepsilon} < h(\bar{x})$. Then the continuity of $h$ at $\bar{x}$ implies that there exists a neighbourhood $V$ of $\bar{x}$ with $h(\bar{x}) - \hat{\varepsilon} \leq h(y) \leq h(\bar{x}) + \hat{\varepsilon}$ for every $y \in V$. Then, for every $y \in V$, we have in particular that $\varphi(y) > > h(y) \geq h(\bar{x}) - \hat{\varepsilon}$, so that $\inf \varphi(V) \geq h(\bar{x}) - \hat{\varepsilon} > 0$. Thus, (1) is fulfilled.

Suppose now that (1) holds. Then there exists an open covering $\mathcal{V}$ of $X$, such that $\forall V \in \mathcal{V} : c_V = \inf \varphi(V) > 0$. If $\chi_V$ denotes the characteristic function of each $V \in \mathcal{V}$, then $g_V = c_V \chi_V$ is lower semi-continuous. Therefore, $g = \sup_{V \in \mathcal{V}} g_V$ is lower semi-continuous, too, and is nonzero everywhere. Observe that $g(x) \leq \varphi(x)$ for every $x \in X$, because $\varphi(x) \geq c_V$ whenever $x \in V \in \mathcal{V}$. By a well-known result of Dowker–Katetov (see [4, Exc. 5.5.20]), there exists a continuous function $h: X \to R$ such that $0 < h(x) < g(x) \leq \varphi(x)$ for every $x \in X$. ◊

Let us recall some definitions from [5]. Let $X$, $Y$ be metric spaces, and $F$ the set of all (not necessarily continuous) maps from $X$ to $Y$. Suppose also to have fixed a compatible metric $\hat{\rho}$ on $Y$, and endow $F$ with the metric $\tilde{\rho}$ defined by:

$$\tilde{\rho}(f, g) = \sup_{x \in X} \min\left\{1, \rho\big(f(x), g(x)\big)\right\},$$

for $f, g \in F$. Finally, let $M_X$ and $M_Y$ be the sets of all compatible metrics on $X$ and $Y$, respectively, endowed with the metrics

$$\operatorname{dist}(d, d') = \sup_{x, x' \in X} \min\{1, |d(x, x') - d'(x, x')|\}$$

and

$$\text{dist'}(\rho, \rho') = \sup_{y, y' \in Y} \min\{1, |\rho(y, y') - \rho'(y, y')|\}.$$

Given a predicate $P$ on $X \times X \times ]0, +\infty[ \times F \times M_X \times M_Y$, we denote by $P^+$ the subset of $X \times X \times ]0, +\infty[ \times F \times M_X \times M_Y$ consisting of all 6-tuples $(x, x', \varepsilon, f, d, \rho)$ for which the proposition $P(x, x', \varepsilon, f, d, \rho)$ is valid. A map $f: X \to Y$ is said to be $P$-continuous if for each $x \in X$, $\varepsilon > 0$, $d \in M_X$ and $\rho \in M_Y$, there exists a neighbourhood $U$ of $x$ in $X$, such that $\{x\} \times U \times \{\varepsilon\} \times \{f\} \times \{d\} \times \{\rho\} \subseteq P^+$. We also denote by $F_P$ the set of all $P$-continuous maps from $X$ into $Y$; the predicate $P$ is said to be continuity-like if the set $F_P$ is nonempty.

Finally, the multivalued map $\Delta_P: X \times ]0, +\infty[ \times F_P \times M_X \times M_Y \to$ $\to ]0, +\infty[$, defined by the formula:

$$\Delta_P(x, \varepsilon, f, d, \rho) = \left\{\delta > 0 \,\middle|\, \forall x' \in S_d(x, \delta)\colon (x, x', \varepsilon, f, d, \rho) \in P^+\right\}$$

— where $S_d(x, \delta) = \left\{x' \in X \,\middle|\, d(x, x') < \delta\right\}$ — is said to be the modulus (of continuity) of the predicate $P$. Observe that, for every $(x, \varepsilon, f, d, \rho) \in$ $\in X \times ]0, +\infty[ \times F_P \times M_X \times M_Y$, the set $\Delta_P(x, \varepsilon, f, d, \rho)$ is either of the form $]0, \delta']$ (for some $\delta' > 0$), or $]0, +\infty[$.

We have the following immediate consequence of Prop. 1.

**Corollary 1.** *Let $P$ be a continuity-like predicate on $X \times X \times$ $\times ]0, +\infty[ \times F \times M_X \times M_Y$. Then the following are equivalent:*

(i) *for every $(x, \varepsilon, f, d, \rho) \in X \times ]0, +\infty[ \times F_P \times M_X \times M_Y$, there are a neighbourhood $W$ of $(x, \varepsilon, f, d, \rho)$ in $X \times ]0, +\infty[ \times F_P \times M_X \times M_Y$ and a $\delta > 0$, such that:*

$$\forall(x',\varepsilon',f',d',\rho') \in W{:}\, \delta \in \Delta_P(x',\varepsilon',f',d',\rho');$$

(ii) *there exists a continuous selection* $\hat{\delta}\colon X\times\,]0,+\infty[\,\times F_P\times M_X\times$ $\times\, M_Y \to ]0,+\infty[$ *of the modulus* $\Delta_P$.

**Proof.** If (ii) holds, then let $(x,\varepsilon,f,d,\rho) \in X\times\,]0,+\infty[\,\times F_P\times M_X\times$ $\times\, M_Y$, and take $\vartheta > 0$ with $\hat{\delta}(x,\varepsilon,f,d,\rho) > \vartheta$: by continuity of $\hat{\delta}$, there exists a neighbourhood $W$ of $(x,\varepsilon,f,d,\rho)$ in $X\times\,]0,+\infty[\,\times F_P\times M_X\times$ $\times\, M_Y$ such that

$$\hat{\delta}(x,\varepsilon,f,d,\rho) - \vartheta \le \hat{\delta}(x',\varepsilon',f',d',\rho') \le \hat{\delta}(x,\varepsilon,f,d,\rho) + \vartheta$$

for every $(x',\varepsilon',f',d',\rho') \in W$. Since every $\Delta_P(x',\varepsilon',f',d',\rho')$ is an initial segment of $]0,+\infty[$, we have that

$$\hat{\delta}(x,\varepsilon,f,d,\rho) - \vartheta \in \Delta_P(x',\varepsilon',f',d',\rho')$$

for every $(x',\varepsilon',f',d',\rho') \in W$.

Conversely, if (i) holds, then let $\varphi\colon X\times\,]0,+\infty[\,\times F_P\times M_X\times M_Y \to$ $\to ]0,+\infty[$ be defined by:

$$\varphi(x,\varepsilon,f,d,\rho) = (1/2)\min\{1, \sup\Delta_P(x,\varepsilon,f,d,\rho)\}.$$

Observe that condition (1) of Prop. 1 is fulfilled for $\varphi$. Indeed, given $(x,\varepsilon,f,d,\rho) \in X\times\,]0,+\infty[\times F_P\times M_X\times M_Y$, let $W$ a neighbourhood of $(x,\varepsilon,f,d,\rho)$ and $\delta > 0$, such that ($\sharp$) holds. Then it is easily seen that $\inf\varphi(W) \ge (1/2)\min\{1,\delta\} > 0$.

By Prop. 1, there exists a continuous $h\colon X\times\,]0,+\infty[\,\times F_P\times M_X\times$ $\times\, M_Y \to ]0,+\infty[$, with $h(\mathrm{w}) \le \varphi(\mathrm{w}) = (1/2)\min\{1,\sup\Delta_P(\mathrm{w})\}$ for every $\mathrm{w} \in X\times\,]0,+\infty[\,\times F_P\times M_X\times M_Y$. Since it is clear that $\varphi(\mathrm{w}) <$ $< \sup\Delta_P(\mathrm{w})$, and $\Delta_P(\mathrm{w})$ is an initial segment of $]0,+\infty[$, we have that $h$ is a selection of $\Delta_P$. $\Diamond$

**Corollary 2.** *A sufficient condition, for the modulus* $\Delta_P$ *of a continuity-like predicate* $P$ *on* $X\times X\times\,]0,+\infty[\,\times F\times M_X\times M_Y$ *to have a continuous selection, is that:*

$\forall(x,\varepsilon,f,d,\rho) \in X\times\,]0,+\infty[\,\times F_P\times M_X\times M_Y$:

$\exists\, W_1\times W_2\times W_3\times W_4\times W_5$ *neighbourhood of* $(x,\varepsilon,f,d,\rho)$:

$\forall x',x'' \in W_1{:}\,\forall(\varepsilon',f',d',\rho') \in W_2\times W_3\times W_4\times W_5$:

$$(x',x'',\varepsilon',f',d',\rho') \in P^+.$$

**Proof.** It will suffice to prove condition (i) of Cor. 1. Given an arbitrary $(x,\varepsilon,f,d,\rho) \in X\times\,]0,+\infty[\,\times F_P\times M_X\times M_Y$, let $W_1\times W_2\times W_3\times W_4\times$ $\times\, W_5$ be a neighbourhood of it, for which the property of the above statement is fulfilled. We may suppose $W_1$ to be of the form $S_d(x,\vartheta)$:

putting $\delta = \vartheta/3$ and $W = S_d(x, \vartheta/3) \times W_2 \times W_3 \times (W_4 \cap S_{\text{dist}}(d, \vartheta/3)) \times \times W_5$, we have that every $(x', \varepsilon', f', d', \rho') \in W$ is also in $W_1 \times W_2 \times \times W_3 \times W_4 \times W_5$, and moreover every $x'' \in S_{d'}(x', \delta) = S_{d'}(x', \vartheta/3)$ is in turn in $W_1$ (because $d(x, x'') \le d(x, x') + d(x', x'') < d(x, x') + + d'(x', x'') + (\vartheta/3) < (\vartheta/3) + (\vartheta/3) + (\vartheta/3) = \vartheta$). Therefore formula (♯) of condition (i), in the statement of Cor. 1, is fulfilled. ◊

Let us recall here another definition from [5]. For a predicate $P$ on $X \times ]0, +\infty[ \times F \times M_X \times M_Y$, we put $P_0^+ = P^+ \cap (X \times X \times ]0, +\infty[ \times \times F_P \times M_X \times M_Y)$. The definition of $P$-continuous map implies that $\text{diag} X \times ]0, +\infty[ \times F_P \times M_X \times M_Y \subseteq P_0^+$, where diag$X$ is the diagonal of $X$. Now we are in condition to prove [5, Th. 1.4]

**Corollary 3.** *Let $P$ be a continuous-like predicate on $X \times X \times \times ]0, +\infty[ \times F \times M_X \times M_Y$. Then the following two properties are equivalent:*

(a) *the set* $\text{diag} X \times ]0, +\infty[ \times F_P \times M_X \times M_Y$ *lies in the interior of the set $P_0^+$;*

(b) *there exists a continuous selection $\hat{\delta}$ of $\Delta_P$.*

**Proof.** By Cor. 1, it will suffice to prove that condition (i) of Cor. 1 is equivalent to condition (a) above.

Suppose first that condition (a) holds. Given $(x, \varepsilon, f, d, \rho) \in X \times \times ]0, +\infty[ \times F_P \times M_X \times M_Y$, we know that there must exist a $\delta > 0$ and open neighbourhoods $V_1, V_2, V_3, V_4$ of $\varepsilon$ in $]0, +\infty[$, of $f$ in $F_P$, of $d$ in $M_X$ and of $\rho$ in $M_Y$, respectively, such that $S_d(x, \delta) \times S_d(x, \delta) \times V_1 \times \times V_2 \times V_3 \times V_4 \subseteq P_0^+$. Therefore, putting $W = S_d(x, \delta) \times V_1 \times V_2 \times V_3 \times V_4$, we have that:

$$\forall (x', \varepsilon', f', d', \rho') \in W : \forall x'' \in S_d(x, \delta) : (x', x'', \varepsilon', f', d', \rho') \in P^+,$$

i.e.: $\forall \mathrm{w} \in W : \delta \in \Delta_P(\mathrm{w})$.

The proof that condition (i) of Cor. 1 implies condition (a) above is completely similar (we consider an arbitrary element of $\text{diag} X \times \times ]0, +\infty[ \times F_P \times M_X \times M_Y$, and we find a neighbourhood of it in $X \times \times X \times ]0, +\infty[ \times F_P \times M_X \times M_Y$, which is included in $P_0^+$). ◊

## 3. Degree of discontinuity

Now we use the results obtained so far to produce a counterexample to [5, Conj. 3.6].

We recall that if $X$ is a metrizable space and $(Y, \rho)$ a metric space, a function $\alpha : X \to [0, +\infty]$ is said to be a degree of discontinuity (with respect to $\rho$) for a function $f: X \to Y$, if:

$$\forall \varepsilon > 0 : \forall x \in X : \exists V \text{ neighbourhood of } x : \forall x' \in V : \rho\big(f(x), f(x')\big) < \alpha(x) + \varepsilon.$$

Of course, every $\alpha: X \to [0, +\infty]$ is a degree of discontinuity (with respect to any $\rho \in M_Y$) for every continuous function $h$ from $X$ to $Y$. This implies, in particular, that the predicate $P_\alpha$ on $X \times X \times ]0, +\infty[ \times \times F \times M_X \times M_Y$, defined by:

$$P_\alpha(x, x', \varepsilon, f, d, \rho) \Longleftrightarrow \rho\big(f(x), f(x')\big) < \alpha(x) + \varepsilon,$$

is always continuity-like. Of course, a generic function $f: X \to Y$ belongs to $F_{P_\alpha}$ if and only if there is a $\rho \in M_Y$ such that $\alpha$ is a degree of discontinuity for $\alpha$ with respect to $\rho$.

**Example 1.** *Let $X$ and $Y$ be the real line (endowed with the Euclidean topology). Then there is a lower semi-continuous (actually, continuous) function $\alpha^\sharp: R \to [0, +\infty[$, such that $\Delta_{P_{\alpha^\sharp}}$ has no continuous selection.*

**Proof.** Let $\alpha^\sharp$ to be the map on $R$ with constant value 1. Observe that $\alpha^\sharp$ is a degree of discontinuity (with respect to the Euclidean metric $\rho^\sharp$ on $R$) for the function $f^\sharp: R \to R$, defined by:

$$f^\sharp(x) = \begin{cases} -1 & \text{if } x < 0; \\ 0 & \text{if } x = 0; \\ 1 & \text{if } x > 0. \end{cases}$$

By contradiction, suppose condition (i) of Cor. 1 is satisfied. Then there would be a neighbourhood $W$ of $(0, (1/2), f^\sharp, \rho^\sharp, \rho^\sharp)$ in $R \times \times ]0, +\infty[ \times F_{P_{\alpha^\sharp}} \times M_R \times M_R$, and a $\hat{\delta} > 0$, such that:

$$\forall (x', \varepsilon', f', d', \rho') \in W : \forall x'' \in S_{d'}\left(x', \hat{\delta}\right) : \rho'\left(f'(x'), f'(x'')\right) < 1 + \varepsilon'.$$

In particular, there would be a $\vartheta > 0$ such that:

$$\forall x' \in S_{\rho^\sharp}(0, \vartheta) : \forall x'' \in S_{\rho^\sharp}\left(x', \hat{\delta}\right) : \rho^\sharp\left(f^\sharp(x'), f^\sharp(x'')\right) < 1 + (1/2).$$

Take $x' \in R$ with $0 < x' < \min\{\hat{\delta}, \vartheta\}$, and $x'' \in R$ with $x'' < 0$ and $x' + |x''| = |x' - x''| = \rho^\sharp(x', x'') < \min\{\hat{\delta}, \vartheta\}$. Then $x' \in S_{\rho^\sharp}(0, \vartheta)$, $x'' \in S_{\rho^\sharp}\left(x', \hat{\delta}\right)$, but

$$\rho^\sharp\left(f^\sharp(x'), f^\sharp(x'')\right) = |f^\sharp(x') - f^\sharp(x'')| = |1 - (-1)| = 2 > 1 + (1/2).$$

A contradiction. $\Diamond$

Consider now the case of a 4-variable predicate $P$ on $X \times X \times$ $\times\, ]0, +\infty[\, \times M_X$, and define (as in the 6-variable case) $P^+$ to be the set of all quadruplets $(x, x', \varepsilon, d)$ such that the proposition $P(x, x', \varepsilon, d)$ holds. We will say that the predicate $P$ is admissible if

$$\forall\, (x, \varepsilon, d) \in X \times\, ]0, +\infty[\, \times M_X \colon \exists\, U \text{ neighbourhood of } x\colon$$
$$\{x\} \times U \times \{\varepsilon\} \times \{d\} \subseteq P^+$$

(such a notion replaces that of continuity-like). As is natural to guess, the multi-valued map $\Delta_P \colon X \times\, ]0, +\infty[\, \times M_X \to\, ]0, +\infty[$ will be defined by:

$$\Delta_P(x, \varepsilon, d) = \{\delta > 0 \mid \forall x' \in S_d\,(x, \delta)\colon (x, x', \varepsilon, d) \in P^+\}.$$

It is very easy to check that Cors. 2, 3 and 4 hold as well if we suppose $P$ to be a 4-variable admissible predicate, instead of a 6-variable continuity-like predicate.

Using the above-introduced definitions, we obtain two positive results in the same vein of the Malešič–Repovš conjecture.

**Theorem 2.** *Let $f$ be a function from $X$ to $Y$, $\rho \in M_Y$ and $\alpha$ a lower semi-continuous function from $X$ to $[0, +\infty]$, which is a degree of discontinuity for $f$ with respect to $\rho$. Then the predicate $P_{\alpha,f,\rho}$ on $X \times X \times\, ]0, +\infty[\, \times M_X$, defined by:*

$$P_{\alpha,f,\rho}(x, x', \varepsilon, d) \iff \rho\big(f(x), f(x')\big) < 2\alpha(x) + \varepsilon,$$

*is admissible, and there exists a continuous selection for the multi-valued mapping $\Delta_{P_{\alpha,f,\rho}}$.*

**Proof.** Since $\alpha$ is a degree of discontinuity for $f$ with respect to $\rho$, we clearly have that the predicate $P$ is admissible.

By Cor. 2, to prove the existence of a continuous selection for $\Delta_{P_{\alpha,f,\rho}}$ it will suffice to show that for every $(x, \varepsilon, d) \in X \times\, ]0, +\infty[\, \times$ $\times M_X$, there is a neighbourhood $W_1 \times W_2 \times W_3$ of it, such that:

$$\forall x', x'' \in W_1 \colon \forall (\varepsilon', d') \in W_2 \times W_3 \colon \rho\big(f(x'), f(x'')\big) < 2\alpha(x') + \varepsilon'$$

(observe that, in this case, the metric $d'$ is not involved in the final inequality).

Let $(x, \varepsilon, d) \in X \times\, ]0, +\infty[\, M_X$: the lower semi-continuity of $\alpha$ implies that there exists a neighbourhood $W'$ of $x$ such that:

$$\forall x' \in W' \colon \alpha(x) - \alpha(x') < \varepsilon/5.$$

On the other hand, the fact that $\alpha$ is a degree of discontinuity for $f$ implies that there is a neighbourhood $W''$ of $x$ such that:

$$\forall x' \in W'' : \rho\big(f(x), f(x')\big) < \alpha(x) + (\varepsilon/5).$$

Put $W_1 = W' \cap W''$, $W_2 = ](4/5)\varepsilon, (6/5)\varepsilon[$ and $W_3 = M_X$: then $W_1 \times \times W_2 \times W_3$ is a neighbourhood of $(x, \varepsilon, d)$ in $X \times ]0, +\infty[ \times M_X$, and we claim that

$$\forall x', x'' \in W_1 : \forall \varepsilon' \in W_2 : \forall d' \in W_3 : \rho\big(f(x'), f(x'')\big) < 2\alpha(x') + \varepsilon'.$$

Indeed, let $x', x'' \in W_1$ and $\varepsilon' \in W_2$. We have:

$$\begin{aligned}\rho\big(f(x'), f(x'')\big) &\le \rho\big(f(x'), f(x)\big) + \rho\big(f(x), f(x'')\big) \le \\ &\le \alpha(x) + (\varepsilon/5) + \alpha(x) + (\varepsilon/5) = \\ &= 2\alpha(x) + (2/5)\varepsilon \le 2\big(\alpha(x') + (\varepsilon/5)\big) + (2/5)\varepsilon = \\ &= 2\alpha(x') + (4/5)\varepsilon < 2\alpha(x') + \varepsilon'. \ \Diamond\end{aligned}$$

**Theorem 3.** *Let $f$ be a function from $X$ to $Y$, and $\rho \in M_Y$. Suppose that the* oscillation $\omega_{f,\rho} : X \to [0, +\infty[$ *of $f$ with respect to $\rho$, defined by:*

$$\omega_{f,\rho}(x) = \inf\big\{\operatorname{diam}_\rho\big(f(V)\big) \,\big|\, V \text{ neighbourhood of } x\big\},$$

*is a lower semi-continuous (actually, continuous) function. Then the predicate $P_{f,\rho}$ on $X \times X \times ]0, +\infty[ \times M_X$, defined by:*

$$P_{f,\rho}(x, x', \varepsilon, d) \iff \big(f(x), f(x')\big) < \omega_{f,\rho}(x) + \varepsilon,$$

*is admissible, and there exists a continuous selection for the multi-valued mapping $\Delta_{P_{f,\rho}}$.*

**Proof.** The admissibility of $P_{f,\rho}$ is a direct consequence of the definition of $\omega_{f,\rho}$.

To prove the existence of a continuous selection for $\Delta_{P_{f,\rho}}$, we will use again Cor. 2. Let $(x, \varepsilon, d) \in X \times ]0, +\infty[ \times M_X$: by the definition of $\omega_{f,\rho}$, there will exist an open neighbourhood $W'$ of $x$ such that $\operatorname{diam}_\rho f(W') < \omega_{f,\rho}(x) + (\varepsilon/3)$, whence:

$$\forall x', x'' \in W' : \rho\big(f(x'), f(x'')\big) < \omega_{f,\rho}(x) + (\varepsilon/3).$$

Moreover, the lower semi-continuity of $\omega_{f,\rho}$ at $X$ gives us a neighbourhood $W''$ of $x$ such that

$$\forall x' \in W'' : \omega_{f,\rho}(x') > \omega_{f,\rho}(x) - (\varepsilon/3).$$

Put $W_1 = W' \cap W''$, $W_2 = ](2/3)\varepsilon, (4/3)\varepsilon[$ and $W_3 = M_X$: we claim that

$$\forall x', x'' \in W_1 : \forall \varepsilon' \in W_2 : \forall d' \in W_3 : \rho\big(f(x'), f(x'')\big) < \omega_{f,\rho}(x') + \varepsilon'.$$

Indeed, for $x', x'' \in W' \cap W''$ and $\varepsilon' \in ](2/3)\varepsilon, (4/3)\varepsilon[$ we have:

$$\rho(f(x'), f(x'')) \leq \omega_{f,\rho}(x) + (\varepsilon/3) < \omega_{f,\rho}(x') + (\varepsilon/3) + (\varepsilon/3) = \\ = \omega_{f,\rho}(x') + (2/3)\varepsilon < \omega_{f,\rho}(x') + \varepsilon'. \ \Diamond$$

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