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SOME GRADED RADICALS OF GRADED RINGS

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Abstract: In this paper two new graded radicals α^* and $\tilde{\alpha}$ of graded rings which can be associated with a given radical α of ordinary associative rings, are introduced and some results relating to these are proved.

1. Introduction

In this paper we introduce two graded radicals α^* and $\tilde{\alpha}$ of graded rings, which can be associated with a given radical α of ordinary associative rings, and prove some results relating to them. Throughout the paper we consider G-graded (associative) rings R, where G is a multiplicative group with identity element e. For general notation and terminology of graded rings we refer to [4], for radical theory of ordinary associative rings to [2], [6], [7], [8], and for radical theory of graded rings to [3], [9]. In particular, if R is a graded ring we denote by $h(R)$ the set of all homogeneous elements of R. The symbols $\trianglelefteq_h$, $\trianglelefteq_{hp}$, $\trianglelefteq_{hsp}$, $\trianglelefteq_{he}$, $\trianglelefteq_{ph}$, $\trianglelefteq_{sph}$, $\trianglelefteq_{eh}$ denote respectively a homogeneous, graded prime, graded semiprime, graded essential, prime homogeneous, semiprime homogeneous, and an essential homogeneous ideal of R. If $I \trianglelefteq_h R$, the annihilator of I in R, that is, the set $\{r \in R : rx = 0, xr = 0, \forall x \in I\}$ is denoted by $\text{ann}^R I$. By I_G we mean the largest homogeneous ideal of R in I, where I is an ideal of R.

We now mention some definitions and results which we may use in the sequel.

Definition 1.1. A non-empty class α of graded rings is called a *graded radical class* (or a *graded radical*) if α satisfies the following conditions:

(i) α is graded homomorphically closed.

(ii) Each graded ring R contains a largest homogeneous ideal in α, denoted by $\alpha(R)$, that is, $\alpha(R)$ is the sum of all the homogeneous ideals of R in α.

(iii) $\alpha(R/\alpha(R)) = 0$.

We say that R is an *α-radical ring* if $\alpha(R) = R$ and *α-semisimple* if $\alpha(R) = 0$.

If α, γ are two graded radicals, we write $\alpha \subseteq \gamma$ if the α-radical class is contained in the γ-radical class or equivalently $\alpha(R) \subseteq \gamma(R)$ for all R. Clearly $\alpha \subseteq \gamma$ if and only if $S_\gamma \subseteq S_\alpha$, where S_α denotes the class of α-semisimple rings.

Theorem 1.2. (i) *If $I \trianglelefteq_h R$, then* $\text{ann}^R I \trianglelefteq_h R$.

(ii) *If $P \trianglelefteq_{hp} (\trianglelefteq_{ph}) R$ and $I \trianglelefteq_h R$, then $P \cap I \trianglelefteq_{hp} (\trianglelefteq_{ph}) I$.*

(iii) *If $J \trianglelefteq_{hsp} (\trianglelefteq_{sph}) I \trianglelefteq_h R$, then $J \trianglelefteq_h R$.*

(iv) *If $I \trianglelefteq_{he} R$ such that I is a graded semiprime ring, then* $\text{ann}^R I = 0$ *and R is itself a graded semiprime ring.*

We have the following result for ordinary associative rings R (see [8, Th. 29]).

Theorem 1.3. *The upper radical of a semisimple class S is hereditary if and only if S is closed under essential extensions, that is, if $I \trianglelefteq_e R$ and $I \in S$, then $R \in S$.*

We state here its graded version.

Theorem 1.4. *The upper graded radical of a graded semisimple class S is graded hereditary if and only if S is closed under graded essential extensions.*

2. Graded radical α^*

We shall show that we can associate with a given radical α of ordinary associative rings a graded radical α^* of graded rings. First, we prove two lemmas which have their own interest.

Lemma 2.1. *Let R be a graded ring and $J \trianglelefteq_h I \trianglelefteq_h R$. If $r \in h(R)$, then the mapping $\theta : J \longrightarrow I/J$, defined by $\theta(x) = rx + J$, $\forall x \in J$, is a graded ring homomorphism and its kernel $K \trianglelefteq_h I$.*

Proof. We have, $\forall\ x,\ y \in J$, $\theta(x+y) = r(x+y) + J = (rx+J)+$ $+(ry+J) = \theta(x) + \theta(y)$, and $\theta(xy) = rxy + J = J = rxry + J =$ $= (rx+J)(ry+J) = \theta(x)\theta(y)$. Hence θ is a graded ring homomorphism of degree (k, e), where $k = \deg(r)$. Its kernel $K = \{x \in J : rx \in J\}$ is a homogeneous ideal of J. We shall show that $K \trianglelefteq_h I$. Let $x \in K$ and $a \in I$, then $rax \in Ix \subseteq J$, so $ax \in K$. Also $r(xa) = (rx)a \in JI \subseteq J$, whence $xa \in K$. Thus $K \trianglelefteq_h I$. $\Diamond$

Lemma 2.2. *Let S be a graded hereditary class of graded rings, which is closed under graded extensions and graded isomorphisms. Let I be a homogeneous ideal of a graded ring R, minimal with respect to the property that $R/I \in S$. Let J be such a homogeneous ideal of I, the $J = I$.*

Proof. We define θ as in Lemma 2.1, then its kernel $K \trianglelefteq_h I$. Also $\mathrm{Im}(\theta) = (rJ + J)/J \trianglelefteq_h I/J \in S$. Hence $J/K \cong \mathrm{Im}(\theta) \in S$. Now $I/K/$ $/J/K \cong I/J \in S$, so $I/K \in S$. Hence, by the minimality of J, $K = J$, so $rx \in J$, $\forall x \in J$. Similarly $xr \in J$, $\forall\ x \in J$. Since r is arbitrary, $J \trianglelefteq_h R$. Since $R/J/I/J \cong R/I \in S$, and $I/J \in S$, so $R/J \in S$. By the minimality of I, we have $J = I$. $\Diamond$

Let R be a graded ring and let $\{C_\lambda\}_{\lambda\in\Lambda}$ be the family of all homogeneous ideals of R such that $R/C_\lambda \in S_\alpha\ \forall\ \lambda \in \Lambda$, where S_α is the semisimple class of the given radical α. We define $\alpha^*(R) =$ $= \cap_{\lambda\in\Lambda} C_\lambda = R^*$ say. Clearly, $\alpha(R) \subseteq \alpha^*(R)$, and $R/R^* \in S_\alpha$. By taking S to be the class of all graded rings in S_α in Lemma 2.2, we get the following:

Theorem 2.3. $\alpha^*(R^*) = R^*$, *where* $R^* = \alpha^*(R)$.

Also we note that $\alpha(R) = 0$ if and only if $\alpha^*(R) = 0$.

Theorem 2.4. α^* *is a graded radical of graded rings.*

Proof. We shall show that conditions (i), (ii) and (iii) of Def. 1.1 are satisfied.

Let R^*/I be a graded homomorphic image of R^* and let $\alpha^*(R^*/I) =$ $= K/I$, where $K \trianglelefteq_h R^*$. Hence $R^*/K \cong (R^*/I)/(K/I) \in S_\alpha$, so $K \supseteq$ $\supseteq \alpha^*(R^*) = R^*$. Thus $K = R^*$ and $\alpha^*(R^*/I) = R^*/I$, satisfying (i).

Now let $S \trianglelefteq_h R$, and let $S^* = \alpha^*(S)$. Then $S^*/(S^* \cap R^*) \cong$ $\cong (S^*+R^*)/R^* \in S_\alpha$, so $S^* = \alpha^*(S^*) \subseteq S^* \cap R^* \subseteq R^*$. This proves (ii).

Since $R/R^* \in S_\alpha$, $\alpha^*(R/R^*) = 0$, and (iii) is also satisfied. $\Diamond$

It is now easy to prove the following

Theorem 2.5. *The class S_G of all graded rings which belong to S_α as ordinary associative rings is a graded semisimple class and α^* is the upper graded radical of S_G.*

In [3] the authors introduce a graded radical α^G. For a graded ring R they define $\alpha^G(R)$ to be the largest homogeneous α-ideal of R. In general $\alpha^G(R) \neq \alpha^*(R) \neq (\alpha(R))_G$ as shown by the following example.

Example 2.6. We take $\alpha = \beta$, the prime radical and R to be the group ring $F_p[C_p]$, where C_p is a cyclic group of order p and F_p is a field of p elements. Then $\beta(R)$ is the augmentation ideal of R, $\beta^*(R) = R$, and $\beta^G(R) = 0$, $\beta_G(R) = 0$, $(\beta(R))_G = 0$, where β_G denotes the graded prime radical.

We recall that a graded radical α is graded supernilpotent if $\beta_G \subseteq \alpha$ and α is graded hereditary see [3].

We shall say that a graded radical α is supernilpotent graded if $\beta^* \subseteq \alpha$ and α is graded hereditary.

Theorem 2.7. *If α is a supernilpotent radical of ordinary associative rings, then α^* is supernilpotent graded.*

Proof. Since α is supernilpotent, $\beta \subseteq \alpha$, so $S_\alpha \subseteq S_\beta$, whence $S_{\alpha^*} \subseteq \subseteq S_{\beta^*}$, and so $\beta^* \subseteq \alpha^*$. Now we show that α^* is graded hereditary. Let R be a graded ring and $I \trianglelefteq_{he} R$ such that $I \in S_{\alpha^*}$. We show that I, in fact, is an essential ideal of R. Let $0 \neq K \trianglelefteq R$ and suppose that $K \cap I = 0$. Then $IK = 0 = KI$, so $K \subseteq \operatorname{ann}^R I \trianglelefteq_h R$. Thus $\operatorname{ann}^R I \neq 0$. Since $I \trianglelefteq_{he} R$, $L = I \cap \operatorname{ann}^R I \neq 0$. But then $L^2 = 0$, and $L \trianglelefteq_h I \in \in S_{\alpha^*} \subseteq S_{\beta^*}$, so $L = 0$, a contradiction. Hence $K \cap I \neq 0$, and I is an essential ideal. Since α is hereditary, $R \in S_\alpha$ by Th. 1.3, so $R \in S_{\alpha^*}$. Hence α^* is graded hereditary by Th. 1.4. ◊

Corollary 2.8. *β^* is the least supernilpotent graded radical.*

Remark 2.9. We note that S_{β^*} consists of all semiprime graded rings, that is, graded rings which have no non-zero nilpotent ideals and S_{β_G} consists of all graded semiprime rings, that is, graded rings which have no nonzero homogeneous nilpotent ideals. Hence $S_{\beta^*} \subseteq S_{\beta_G}$, so $\beta_G \subseteq \subseteq \beta^*$. It follows, therefore, that a supernilpotent graded radical is also graded supernilpotent but the converse need not be true. For example, β_G is graded supernilpotent but not supernilpotent graded, for $\beta^* \not\subseteq \beta_G$ as shown by Ex. 2.6.

We have a theorem corresponding to Th. 2 in [3]. First, we define a weakly special graded class.

Definition 2.10. We call a non-empty class $\mathcal{K}$ of graded rings *weakly special graded* if

(i) $\mathcal{K}$ consists of semiprime graded rings,

(ii) $I \trianglelefteq_h R$ and $R \in \mathcal{K}$, then $I \in \mathcal{K}$ and

(iii) $I \trianglelefteq_{eh} R$ and $I \in \mathcal{K}$, then $R \in \mathcal{K}$.

Again, by some modifications in the proof of Ryabukhin's theorem (see [6, Th. 11.5]), we can prove the following

Theorem 2.11. *A graded radical α is supernilpotent graded if and only if it coincides with the graded upper radical determined by a weakly special graded class $\mathcal{K}$. Then for any graded ring R, $\alpha(R) = \cap_{\lambda\in\Lambda} I_\lambda$, where $\{I_\lambda : \lambda \in \Lambda\}$ is the family of all those homogeneous ideals I_λ of R for which $R/I_\lambda \in \mathcal{K}$, and thus an α-semisimple graded ring is a graded subdirect sum of rings from $\mathcal{K}$.*

3. Graded radical $\tilde{\alpha}$

Corresponding to a special class and special radical of ordinary associative rings are defined a graded special class and graded special radical of graded rings (see [3]). We now define a special graded class.

Definition 3.1. We shall say that a non-empty class $\mathcal{K}$ of graded rings R is *special graded* if

(i) $\mathcal{K}$ consists of prime graded rings,

(ii) $I \trianglelefteq_h R$ and $R \in \mathcal{K}$, then $I \in \mathcal{K}$ and

(iii) $I \trianglelefteq_{eh} R$ and $I \in \mathcal{K}$, then $R \in \mathcal{K}$.

Thus a special graded class is also weakly special graded, and so by Th. 2.11, determines a supernilpotent graded radical α which we call *special graded.* Then for any graded ring R, $\alpha(R) = \cap_{\lambda\in\Lambda} P_\lambda$, where $\{P_\lambda : \lambda \in \Lambda\}$ is the family of all those prime homogeneous ideals P_λ of R such that $R/P_\lambda \in \mathcal{K}$.

Since a prime graded ring is also graded prime, a special graded radical is also graded special, but the converse may not be true. For example, β_G is graded special but not special graded, for it is not supernilpotent graded by Remark 2.9.

Theorem 3.2. *Let α be a supernilpotent radical of ordinary associative rings such that the class $\mathcal{K}$ of all prime graded rings in S_α is non-empty. Then $\mathcal{K}$ is a special graded class.*

Proof. We need to verify only 3.1 (ii) and (iii). Since any nonzero ideal of a prime ring is a prime ring (ii) is satisfied. Now let R be a graded ring with $I \trianglelefteq_{eh} R$, $I \in \mathcal{K}$. Since α is hereditary $R \in S_\alpha$ by Th. 1.3. Also if A, B are nonzero ideals of R, then $I \cap A$ and $I \cap B$ are nonzero ideals of I. If $AB = 0$, then $(I \cap A)(I \cap B) = 0$, a contradiction, so $AB \neq 0$ and R is prime. Hence $R \in \mathcal{K}$. ◊

Thus $\mathcal{K}$ determines a special graded radical $\tilde{\alpha}$. Clearly $\alpha^* \subseteq \tilde{\alpha}$.

Theorem 3.3. *$\tilde{\alpha}$ is the smallest graded radical among special graded radicals containing α^*.*

Proof. Let γ be a special graded radical such that $\gamma \supseteq \alpha_*$. Then for a graded ring R, $\gamma(R) = \cap P_i$, where $P_i \trianglelefteq_{ph} R$ such that $R/P_i \in S_\gamma$. But $S_\gamma \subseteq S_{\alpha^*}$, so $R/P_i \in S_{\alpha^*}$ and $\tilde{\alpha}(R) \subseteq \gamma(R)$. ◊

Corollary 3.4. *$\tilde{\beta}$ is the least special graded radical.*

If α is a special radical, then α^* need not be special graded as shown by the following example by taking $\alpha = \beta$.

Example 3.5. Let R be the group ring $F_p[C_{p-1}(g)]$, where p is an odd prime and $C_{p-1}(g)$ is a cyclic group of order $p-1$, generated by g. Let I and K be the ideals of R generated by the idempotents $-(e+g+g^2+\cdots+g^{p-2})$ and $-(e-g+g^2-g^3+\cdots-g^{p-2})$, where e is the identity of $C_{p-1}(g)$. Then $I \neq 0$ and $K \neq 0$ but $IK \subseteq I \cap K = 0$. Hence R is not a prime ring. Moreover, I and K are prime ideals of R for $R/I \simeq F_p \simeq R/K$, so I and K are maximal and hence prime ideals of R. Thus (0) is a semiprime homogeneous ideal of R. Hence $R \in S_{\beta^*}$. But R has no prime homogeneous ideals of which (0) is the intersection. Hence β^* is not special graded. Also $\beta_G(R) = \beta(R) = \beta^*(R) = 0$, but $\tilde{\beta}(R) = R$.

Ryabukhin (see [6]) gave an example of a supernilpotent radical which is not special. We shall now show that a graded supernilpotent (respectively supernilpotent graded) radical may not be graded special (respectively special graded). Let $G = C_{p-1}(g)$ where p is an odd prime, and let $\mathcal{K}$ be the class of all G-graded rings R satisfying the conditions $x^p = x$, $px = 0$, $\forall x \in h(R)$. Then $\mathcal{K}$ is a graded radical graded semisimple class and each ring in $\mathcal{K}$ is a graded subdirect sum of graded fields in $\mathcal{F}$, where $\mathcal{F}$ consists of $F_p[C_{p-1}]$ and its graded subfields (see [5, Sect. 4]).

We now prove, by an example, the existence of a nonzero graded ring in $\mathcal{K}$ which does not contain a homogeneous ideal which is a finite graded field in $\mathcal{F}$.

Example 3.6. Let S be the set of symbols x_i, where $i \in \mathbb{Q}$, the set of rationals. We multiply these symbols by the rule $x_i x_j = x_k$ where $k = \max(i, j)$. Then S is a multiplicative commutative semigroup. Let R be the group ring $A[G]$, where A is the semigroup ring $F_p[S]$. We shall show that $y^p = y$, $\forall y \in h(R)$, by induction on the number of nonzero components of y. Suppose that $y \in R_k$ has only one nonzero component and let $y = ax_ig^k$. Then $y^p = a^px_i^pg^{kp} = ax_ig^k$. Now suppose $y^p = y$ if y has less than m components. Let $y = \sum_{j=1}^{m} a_{i_j}x_{i_j}g^k = b+c$ where $b = \sum_{j=1}^{m-1} a_{i_j}x_{i_j}g^k$ and $c = a_{i_m}x_{i_m}g^k$. Then $y^p = (b+c)^p = b^p +$ $+ \binom{p}{1}b^{p-1}c + \cdots + \binom{p}{p-1}bc^{p-1} + c^p$. Since p divides $\binom{p}{r}$, $1 \le r \le p-1$, $y^p = b^p + c^p$. By the inductive hypothesis $b^p = b$ and $c^p = c$. Also $py = 0$, $\forall y \in h(R)$, so $R \in \mathcal{K}$. Now let $0 \ne I \trianglelefteq_h R$ and let $0 \ne y \in$ $\in I_{g^k}$. Then we can write $y = \sum_{l=1}^{r} a_ly_{i_l}g^k$, $i_1 < i_2 < \cdots < i_r$, and $1 \le a_l \le p-1$. If $p \nmid (\sum_{l=1}^{r} a_l)$, then for every $j \ge i_r$, we have $0 \ne yx_jg^k = (\sum_{l=1}^{r} a_l)x_jg^{2k} \in I$ and so there are infinitely many elements in I. If $p|(\sum_{l=1}^{r} a_l)$, then for j with $i_{r-1} \le j < i_r$, we have $yx_jg^k = (\sum_{l=1}^{r-1} a_l)x_jg^{2k} + a_rx_{i_r}g^{2k} \in I$. Hence, in either case, I is infinite, so $I \notin \mathcal{F}$.

Theorem 3.7. *Let $\mathcal{K}_1$ be the subclass of graded rings in $\mathcal{K}$ that do not have any homogeneous ideal which is a graded field in $\mathcal{K}$. Then $\mathcal{K}_1$ is a graded weakly special class. The upper graded radical α determined by $\mathcal{K}_1$ is graded supernilpotent but not graded special.*

Proof. The class $\mathcal{K}_1 \ne \emptyset$ because of the above example. The rings in $\mathcal{K}_1$ are graded semiprime for they have no nonzero nilpotent homogeneous elements and clearly $\mathcal{K}_1$ is graded hereditary. Let $I \trianglelefteq_{he} R$ where R is a graded ring and $I \in \mathcal{K}_1$. Then, by Th. 1.2 (iv), $\mathrm{ann}^R I = 0$ and R is graded semiprime. Now let $r \in h(R)$ and $x \in h(I)$. Then $rx \in I$ and $0 = (rx)^p - rx = r^px - rx = (r^p - r)x$. Since $\mathrm{ann}^R I = 0$, $r^p - r = 0$. Also $0 = p(rx) = prx$ implies that $pr = 0$. Hence $R \in \mathcal{K}$. Suppose now that $0 \ne K \trianglelefteq_h R$, where K is a graded field in $\mathcal{F}$. Since $I \trianglelefteq_{he} R$, $K \cap$ $\cap I \ne 0$ but $K \cap I \trianglelefteq_h K$, so $K \cap I = K$. Hence $K \subseteq I$, a contradiction. Thus $R \in \mathcal{K}_1$ and so $\mathcal{K}_1$ is a graded weakly special class. Now any graded prime ring in $\mathcal{S}_\alpha$ is a graded field in $\mathcal{K}$, so in $\mathcal{F}$. But this is an α-radical ring. Hence no graded prime ring exists in $\mathcal{S}_\alpha$. Therefore α is graded supernilpotent but not graded special. $\Diamond$

Remark 3.8. We note that the graded rings R in $\mathcal{K}_1$ also satisfy $r^p =$ $= r$, $\forall r \in R$. Hence the rings in $\mathcal{K}_1$ are semiprime graded. Therefore α is a supernilpotent graded radical.

We can similarly prove the following theorem.

Theorem 3.9. *Let $\mathcal{K}_2$ be the subclass of graded rings in $\mathcal{K}$ that do not have any homogeneous ideal which is the field F_p. Then $\mathcal{K}_2$ is weakly special graded but not special graded. The upper graded radical determined by $\mathcal{K}_2$ is supernilpotent graded but not special graded.*

Remark 3.10. We note that $\mathcal{K}_1 \subseteq \mathcal{K}_2$, but $\mathcal{K}_1 \neq \mathcal{K}_2$, for $R = F_p[C_{p-1}(g)] \in \mathcal{K}_2$ but $R \notin \mathcal{K}_1$.

We have the following:

Theorem 3.11. *Every supernilpotent graded (graded supernilpotent) radical α whose graded semisimple class contains prime graded (graded prime) rings can be extended to a special graded (graded special) radical α_1, and α_1 is the least special graded (graded special) radical containing α.*

Proof. Similar to that of Thms. 3.2 and 3.3. ◊

Corollary 3.12. *Let α be a supernilpotent radical of ordinary associative rings such that $\mathcal{S}_\alpha$ contains prime graded rings, then $\alpha_1^* = \tilde{\alpha}$.*

We shall now consider graded rings with chain conditions on homogeneous ideals. First, a lemma.

Lemma 3.13. *Let R be a $\tilde{\beta}$-semisimple graded ring and $I \trianglelefteq_h R$ which is not a prime ring, then I contains two nonzero homogeneous ideals I_1, K_1 of R such that $I_1 K_1 = 0$.*

Proof. There exist two nonzero ideals A, B of I such that $AB = 0$. Since I is also $\tilde{\beta}$-semisimple we have $\cap_{\lambda \in \Lambda} P_\lambda = 0$, where $\{P_\lambda : \lambda \in \Lambda\}$ is the family of all prime homogeneous ideals of I. Now $AB \subseteq P_\lambda, \forall \lambda \in \Lambda$, so $A \subseteq P_\lambda$ or $B \subseteq P_\lambda$, but not all P_λ, $\lambda \in \Lambda$, contain A or B. Let $\Lambda_1 = \{\lambda \in \Lambda : A \subseteq P_\lambda\}$ and let $\Lambda = \Lambda_1 \dot{\cup} \Lambda_2$, where $\dot{\cup}$ denotes disjoint union. Then $A \subseteq I_1 = \cap_{\lambda \in \Lambda_1} P_\lambda$, $B \subseteq K_1 = \cap_{\lambda \in \Lambda_2} P_\lambda$. Hence I_1, K_1 are nonzero semiprime homogeneous ideals of I, so also homogeneous ideals of R by Th. 1.2 (iii), such that $I_1 K_1 \subseteq I_1 \cap K_1 = 0$. ◊

Theorem 3.14. *Let R be a $\tilde{\beta}$-semisimple graded ring satisfying the ascending chain condition (ACC) or the descending chain condition (DCC) on homogeneous ideals, then every nonzero homogeneous ideal of R contains a homogeneous ideal of R, which is a prime graded ring.*

Proof. Let R satisfy ACC and let $0 \neq I \trianglelefteq_h R$. If I is a prime ring, we are finished. If not, by Lemma 3.13, I contains two nonzero homogeneous ideals I_1 and K_1 of R such that $I_1 K_1 = 0$. If I_1 is a prime ring, we are done. Otherwise, there exist nonzero homogeneous ideals I_2, K_2 of R in I_1 such that $I_2 K_2 = 0$. Then the argument proceeds as in the ungraded case (see [1, Lemma 1.6])

Now let R satisfy DCC, and $0 \neq I \trianglelefteq_h R$. Let J be a (non-zero) minimal homogeneous ideal of R contained in I. Then J is a prime graded ring by Lemma 3.13, for $J^2 \neq 0$. ◊

Theorem 3.15. *Let α be a supernilpotent graded (graded supernilpotent) radical whose graded semisimple class $\mathcal{S}_\alpha$ contains prime graded*

(graded prime) rings, and let α_1 be the least special graded (graded special) radical containing α. Let R be a graded ring such that every nonzero homogeneous ideal of a graded homomorphic image of R in S_α contains a prime graded (graded prime) ring as its homogeneous ideal, then $\alpha(R) = \alpha_1(R)$.

Proof. Let $\alpha(R) \neq \alpha_1(R)$. Then $0 \neq \alpha_1(R)/\alpha(R) \trianglelefteq_h R/\alpha(R) \in S_\alpha$, so it contains a nonzero homogeneous ideal, say K, which is a prime (graded prime) ring. Since α and α_1 are both graded hereditary K is both an α_1-radical ring and an α_1-semisimple ring, but $K \neq 0$, a contradiction. $\Diamond$

Corollary 3.16. *Let α be a supernilpotent graded radical such that $\alpha \supseteq \tilde{\beta}$ and S_α contains prime graded rings. If α_1 is the least special graded radical containing α, then α coincides with α_1 on every graded ring R satisfying ACC or DCC on homogeneous ideals.*

Proof. By Th. 3.14 and Th. 3.15. $\Diamond$

We can similarly prove a corresponding result for a graded supernilpotent radical. Also, we have the following:

Theorem 3.17. *Let α be a graded supernilpotent radical with S_α containing graded prime rings, and let α_1 be the least graded special radical containing α. Let R be a graded ring such that in every graded semiprime homomorphic image of R the zero ideal (0) is a finite product of graded prime ideals, then $\alpha(R) = \alpha_1(R)$.*

Proof. Let R_1 be a graded homomorphic image of R in S_α, so R_1 is graded semiprime. Let $I \trianglelefteq_h R_1$, then we shall show that I has a homogeneous ideal which is a graded prime ring. If I is itself a graded prime ring we are done. Otherwise, there exists a finite set $\{P_1, P_2, \ldots, P_k\}$, $k \geq 2$, of graded prime ideals of R such that $\Pi_{i=1}^k P_i = 0$. Then $0 \neq \neq Q_i = I \cap P_i$ is a graded prime ideal of I such that $\Pi_{i=1}^k Q_i = 0$. There is no loss of generality in supposing that $\Pi_{i=1}^{k-1} Q_i \neq 0$. Then $K = \operatorname{ann}^I Q_k \neq 0$, for it contains $\Pi_{i=1}^{k-1} Q_i$. But $K \cap Q_k = 0$, for R_1 is graded semiprime. Hence (0) is a graded prime ideal of K, so K is a graded prime ring. The result then follows from Th. 3.15. $\Diamond$

Corollary 3.18. *α coincides with α_1 on every graded ring satisfying ACC or DCC on homogeneous ideals.*

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