

IMPROVEMENTS OF THE DIS- CREPANCY OF THE VAN DER CORPUT SEQUENCE

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Abstract: In a recent paper, Kritzer, Larcher, and Pillichshammer showed that the star discrepancy of the van der Corput sequence can be decreased if one applies a digital shift to the points of this sequence. In this paper we study the L_2 discrepancy of the shifted van der Corput sequence. We show that it is not possible to reduce the order of magnitude of the L_2 discrepancy in N by digitally shifting the van der Corput sequence. However, it is possible to reduce the constant in the “leading term”. Our proof is based on a thorough analysis of a sum of distances-to-the-nearest-integer.

1. Introduction

In this paper, we examine shifted versions of the van der Corput sequence. The van der Corput sequence is very well known in the

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theory of uniform distribution modulo one and is studied extensively in the literature, see for example [1, 2, 3, 5, 6, 7, 8, 10, 15, 16]. The van der Corput sequence is an infinite sequence $\gamma = (y_n)_{n \geq 0}$ in the unit interval $[0, 1)$, where for $n = a_{m-1}2^{m-1} + \dots + a_12 + a_0$ we have

$$y_n = \frac{a_0}{2} + \frac{a_1}{2^2} + \dots + \frac{a_{m-1}}{2^m}.$$

Let $\sigma = (\sigma_1, \sigma_2, \dots)^T \in \mathbb{Z}_2^\infty$. By a σ -shifted van der Corput sequence we mean the sequence $\gamma_\sigma = (y_n)_{n \geq 0}$ which is obtained by setting

$$y_n = \frac{a_0 \oplus \sigma_1}{2} + \frac{a_1 \oplus \sigma_2}{2^2} + \dots + \frac{a_{m-1} \oplus \sigma_m}{2^m},$$

where \oplus denotes addition in \mathbb{Z}_2 . The reason for considering shifted van der Corput sequences is that by this operation the distribution properties of the point set can be improved considerably. Let us, for example, consider the star discrepancy $D_N^*(\omega)$ of a sequence $\omega = (x_n)_{n \geq 0}$ in the unit interval which is defined by

$$D_N^*(\omega) := \sup_{0 \leq \alpha \leq 1} \left| \frac{A_N([0, \alpha))}{N} - \alpha \right|.$$

Here, $A_N([0, \alpha))$ denotes the number of indices n satisfying $0 \leq n < N$ and $x_n \in [0, \alpha)$. For the unshifted form of the van der Corput sequence, it is known that

$$ND_N^*(\gamma) \leq \frac{\log N}{3 \log 2} + 1$$

for all N (see, for example, [1]). By a result of Schmidt [18] (see also [4, 12]), this bound is best possible in the order of magnitude in N since there exists a constant $c > 0$ such that for any sequence in the unit interval its star discrepancy is larger than $c(\log N)/N$ for infinitely many values of $N \in \mathbb{N}$. Further for the unshifted form of the van der Corput sequence it is also known that the constant $1/(3 \log 2)$ is best possible, see [8]. In the recent paper [10] it was shown that there exists a shift vector $\tilde{\sigma}$ (which can be given explicitly) such that the $\tilde{\sigma}$ -shifted van der Corput sequence satisfies

$$ND_N^*(\gamma_{\tilde{\sigma}}) \leq \frac{\log N}{6 \log 2} + c\sqrt{\log N},$$

for all $N \in \mathbb{N}$, where $c > 0$ is a constant, and where the constant $1/(6 \log 2)$ is best possible for any shift $\sigma \in \mathbb{Z}_2^\infty$.

In this paper we consider the L_2 discrepancy of the σ -shifted van der Corput sequence which is, for a sequence ω in $[0, 1)$, defined by

$$L_{2,N}(\omega) := \left(\int_0^1 \left| \frac{A_N([0, \alpha))}{N} - \alpha \right|^2 d\alpha \right)^{1/2}.$$

It follows from the result of Roth in [17] (see also [4, 12]) that there exists a constant $c > 0$ such that for the L_2 discrepancy of any sequence ω in $[0, 1)$ we have

$$NL_{2,N}(\omega) \geq c\sqrt{\log N}$$

for infinitely many values of $N \in \mathbb{N}$. For the unshifted form of the van der Corput sequence, it is known that

$$(1) \quad (NL_{2,N}(\gamma))^2 \leq \left(\frac{\log N}{6 \log 2} \right)^2 + \left(\frac{11}{3} + \frac{2 \log 3}{\log 2} \right) \frac{\log N}{36 \log 2} + \frac{1}{3}$$

for all N (see, for example, [7]) and that the constant $1/(6 \log 2)$ is best possible, see, for example, [2, 15, 16]. Hence the L_2 discrepancy of the van der Corput sequence is not best possible in the order of magnitude in N . This, together with the fact that the star discrepancy can be reduced by digitally shifting the van der Corput sequence, raises the question whether there exists a shift $\sigma \in \mathbb{Z}_2^\infty$ such that the L_2 discrepancy of the σ -shifted version of the van der Corput sequence is best possible in the sense of Roth's result. The answer to this question will be given in the subsequent Sect. 2. In Sect. 3 we shall present some auxiliary results and Sect. 4 contains the proofs for the results from Sect. 2. In the proof of our main result will appear a sum of the form

$$\sum_{u=0}^{m-1} \|2^u \beta\| \varepsilon_u$$

where $\|\cdot\|$ denotes the distance-to-the-nearest-integer function, i.e., $\|x\| = \min(x - [x], 1 - (x - [x]))$, and where $\varepsilon_u \in \{-1, 1\}$. Our result depends only on the maximum of absolute value of this sum where the maximum is extended over all β with at most m non-zero digits in base 2 representation. Since we think that such sums are, apart from their application here, interesting on their own, we defer the analysis of these sums to a separate section (Sect. 5). Similar sums have already been analyzed in the recent papers [10] and [13].

2. The results

Our first result shows that there exist shifts $\sigma \in \mathbb{Z}_2^\infty$ such that the L_2 discrepancy of the σ -shifted van der Corput sequence is "small" on

the average. Before we state the result it is convenient throughout the paper to define for $m \in \mathbb{N}$ the quantity $l_m := \#\{1 \leq u \leq m : \sigma_u = 0\}$, that is the number of zero components among the first m components of the shift vector $\sigma = (\sigma_1, \sigma_2, \dots)^T$. Further define $\psi(m) := l_m - \frac{m}{2}$.

Theorem 1. *For all $\sigma \in \mathbb{Z}_2^\infty$ for which $\lim_{m \rightarrow \infty} \psi(m)^2/m$ exists, we have*

$$\limsup_{m \rightarrow \infty} \frac{1}{2^m} \sum_{N=1}^{2^m} \frac{(NL_{2,N}(\gamma_\sigma))^2}{m} < c_\sigma.$$

Here $0 < c_\sigma < \infty$ only depends on the shift σ .

Remark 1. It will follow from the proof that $c_\sigma \leq \frac{1}{16} \lim_{m \rightarrow \infty} \frac{\psi(m)^2}{m} + \frac{4}{3}$.

From Th. 1 we obtain

Corollary 1. *For any $\varepsilon > 0$ and for all $\sigma \in \mathbb{Z}_2^\infty$ for which $\lim_{m \rightarrow \infty} \psi(m)^2/m$ exists, we have*

$$\lim_{m \rightarrow \infty} \frac{1}{2^m} \#\{1 \leq N \leq 2^m : NL_{2,N}(\gamma_\sigma) < (\log N)^{\frac{1}{2} + \varepsilon}\} = 1.$$

We would hope from the result of Cor. 1 that there exists a shift $\sigma \in \mathbb{Z}_2^\infty$ such that the L_2 discrepancy of the σ -shifted version of the van der Corput sequence is of order $(\log N)^\beta/N$ with $\frac{1}{2} \leq \beta < 1$. This, however, is not the case.

Theorem 2. *There exists a constant $c > 0$ with the following property: for all $\sigma \in \mathbb{Z}_2^\infty$ we have*

$$NL_{2,N}(\gamma_\sigma) \geq c \log N,$$

for infinitely many values of $N \in \mathbb{N}$.

Th. 2 shows that digitally shifting the van der Corput sequence cannot decrease the order of magnitude of the L_2 discrepancy in N , but what about the constant in the “leading term”, i.e.,

$$(2) \quad q(\sigma) := \limsup_{N \rightarrow \infty} \frac{NL_{2,N}(\gamma_\sigma)}{\log N}$$

for shifts $\sigma \in \mathbb{Z}_2^\infty$? Although we could not calculate $q(\sigma)$ exactly for arbitrary $\sigma \in \mathbb{Z}_2^\infty$ we were able to prove that shifting does not increase this quantity (compared to the unshifted van der Corput sequence).

Theorem 3. *For the quantity $q(\sigma)$ defined in (2) we have*

$$\sup_{\sigma \in \mathbb{Z}_2^\infty} q(\sigma) = \frac{1}{6 \log 2}$$

and this value is attained for the shift $\sigma = (0, 0, \dots)^T$, i.e., for the

unshifted van der Corput sequence.

Of course it would now be of great interest to know how small $q(\sigma)$ can be.

Theorem 4. For the quantity $q(\sigma)$ defined in (2) we have

$$\inf_{\sigma \in \mathbb{Z}_2^\infty} q(\sigma) \leq \frac{1}{20 \log 2}.$$

The value $1/(20 \log 2)$ is attained for the shift $\sigma^* = (1, 0, 1, 0, \dots)^T$. We conjecture that this shift yields the smallest value for the L_2 discrepancy of all digital shifts, but a proof of this conjecture has to remain open for the moment.

3. Prerequisites

For $\sigma^{(m)} = (\sigma_1, \dots, \sigma_m)^T \in \mathbb{Z}_2^m$ the $\sigma^{(m)}$ -shifted Hammersley point set is a set $P_{\sigma^{(m)}}$ of 2^m points $\mathbf{x}_0, \dots, \mathbf{x}_{2^m-1}$ in the unit square $[0, 1]^2$, with

$$\mathbf{x}_n = (x_n, y_n), \quad 0 \leq n \leq 2^m - 1,$$

where for $n = a_{m-1}2^{m-1} + \dots + a_12 + a_0$ we have

$$x_n = \frac{n}{2^m} \quad \text{and} \quad y_n := \frac{a_0 \oplus \sigma_1}{2} + \frac{a_1 \oplus \sigma_2}{2^2} + \dots + \frac{a_{m-1} \oplus \sigma_m}{2^m}.$$

This is a generalization of the classical two-dimensional Hammersley point set which can be obtained by choosing $\sigma_1 = \dots = \sigma_m = 0$.

For any set $P = \{\mathbf{x}_0, \dots, \mathbf{x}_{N-1}\}$ of points in the unit square $[0, 1]^2$ the discrepancy function $\Delta_{2^m}(P, \cdot, \cdot)$ is defined as

$$\Delta_{2^m}(P, \alpha, \beta) = A_N([0, \alpha) \times [0, \beta)) - N\alpha\beta$$

for $0 \leq \alpha, \beta \leq 1$, where $A_N([0, \alpha) \times [0, \beta))$ denotes the number of indices n satisfying $0 \leq n < N$ and $\mathbf{x}_n \in [0, \alpha) \times [0, \beta)$.

Further for a sequence ω in the unit interval $[0, 1)$ we write

$$\Delta_N(\omega, \alpha) = A_N([0, \alpha)) - N\alpha.$$

We need some further notation: for $\alpha = a_1/2 + \dots + a_m/2^m$ with $a_i \in \{0, 1\}$ we say in the following that α is m -bit. For arbitrary $0 \leq \alpha \leq 1$ and $m \in \mathbb{N}$ we denote by $\alpha(m)$ the smallest m -bit number which is larger than or equal to α . If α is greater than $1 - 2^{-m}$, then we set $\alpha(m) = 1$.

Let $\sigma = (\sigma_1, \sigma_2, \dots)^T \in \mathbb{Z}_2^\infty$ and denote by $\sigma^{(m)}$ the vector consisting of the first m components of σ , i.e., $\sigma^{(m)} = (\sigma_1, \dots, \sigma_m)^T \in \mathbb{Z}_2^m$.

Let γ_σ denote the σ -shifted van der Corput sequence and $P_{\sigma^{(m)}}$ the $\sigma^{(m)}$ -shifted Hammersley set.

Let $m \in \mathbb{N}$ and $N \leq 2^m$. From [10, Sect. 5] we know that, for any $\alpha \in [0, 1]$,

$$\Delta_N(\gamma_\sigma, \alpha) = \Delta_{2^m}(P_{\sigma^{(m)}}, N/2^m, \alpha^{(m)}) + c_\alpha$$

with $-1 \leq c_\alpha \leq 2$. Therefore we obtain

$$\begin{aligned} (NL_{2,N}(\gamma_\sigma))^2 &= \int_0^1 \Delta_N(\gamma_\sigma, \alpha)^2 d\alpha = \\ &= \int_0^1 \Delta_{2^m}(P_{\sigma^{(m)}}, N/2^m, \alpha^{(m)})^2 d\alpha + \\ (3) \quad &+ 2 \int_0^1 c_\alpha \Delta_{2^m}(P_{\sigma^{(m)}}, N/2^m, \alpha^{(m)}) d\alpha + \int_0^1 c_\alpha^2 d\alpha = \\ &= \frac{1}{2^m} \sum_{l=1}^{2^m} \Delta_{2^m}(P_{\sigma^{(m)}}, N/2^m, l/2^m)^2 + \\ &+ \sum_{l=1}^{2^m} \Delta_{2^m}(P_{\sigma^{(m)}}, N/2^m, l/2^m) \int_{(l-1)/2^m}^{l/2^m} 2c_\alpha d\alpha + O(1). \end{aligned}$$

From [9, 10] we find that $|\Delta_{2^m}(P_{\sigma^{(m)}}, N/2^m, l/2^m)| \leq \frac{m}{3} + \frac{13}{9} - (-1)^m \frac{4}{9 \cdot 2^m}$ for any $0 \leq N, l \leq 2^m$. Therefore,

$$\begin{aligned} \frac{1}{2^m} \sum_{N=1}^{2^m} (NL_{2,N}(\gamma_\sigma))^2 &= \frac{1}{2^{2m}} \sum_{N=1}^{2^m} \sum_{l=1}^{2^m} \Delta_{2^m}(P_{\sigma^{(m)}}, N/2^m, l/2^m)^2 + O(m) = \\ (4) \quad &= \frac{m^2 - 4l_m m + 4l_m^2}{64} + O(m). \end{aligned}$$

The last equality follows from [11, Lemma 6].

4. The Proofs

First we give the

Proof of Theorem 1. Since $m^2 - 4l_m m + 4l_m^2 = 4\psi(m)^2$ the result follows from (4). \diamond

The bound on c_σ from Remark 1 can be obtained easily by following the considerations above.

Proof of Corollary 1. Let $\varepsilon > 0$, $y > 0$ and choose σ as in the statement of the corollary. From Th. 1 we know that there exists a constant $0 < c_\sigma < \infty$ such that

$$\begin{aligned} c_\sigma m &\geq \frac{1}{2^m} \sum_{N=1}^{2^m} (NL_{2,N}(\gamma_\sigma))^2 \geq \\ &\geq \frac{1}{2^m} \#\{1 \leq N \leq 2^m : NL_{2,N}(\gamma_\sigma) \geq y \cdot m^{\frac{1}{2}+\varepsilon}\} \cdot y^2 \cdot m^{1+2\varepsilon} = \\ &= \frac{1}{2^m} (2^m - \#\{1 \leq N \leq 2^m : NL_{2,N}(\gamma_\sigma) < y \cdot m^{\frac{1}{2}+\varepsilon}\}) \cdot y^2 \cdot m^{1+2\varepsilon} \end{aligned}$$

Therefore we obtain

$$\lim_{m \rightarrow \infty} \frac{1}{2^m} \#\{1 \leq N \leq 2^m : NL_{2,N}(\gamma_\sigma) < y \cdot m^{\frac{1}{2}+\varepsilon}\} = 1.$$

Since

$$\lim_{m \rightarrow \infty} \frac{1}{2^m} \#\left\{1 \leq N \leq 2^{\frac{m}{2}} : NL_{2,N}(\gamma_\sigma) < y \cdot m^{\frac{1}{2}+\varepsilon}\right\} = 0$$

we obtain

$$\begin{aligned} 1 &= \lim_{m \rightarrow \infty} \frac{1}{2^m} \#\left\{2^{\frac{m}{2}} < N \leq 2^m : NL_{2,N}(\gamma_\sigma) < y \cdot m^{\frac{1}{2}+\varepsilon}\right\} \leq \\ &\leq \lim_{m \rightarrow \infty} \frac{1}{2^m} \#\left\{2^{\frac{m}{2}} < N \leq 2^m : NL_{2,N}(\gamma_\sigma) < y \cdot \left(2 \frac{\log N}{\log 2}\right)^{\frac{1}{2}+\varepsilon}\right\} \leq \\ &\leq \lim_{m \rightarrow \infty} \frac{1}{2^m} \#\left\{1 \leq N \leq 2^m : NL_{2,N}(\gamma_\sigma) < y \cdot \left(2 \frac{\log N}{\log 2}\right)^{\frac{1}{2}+\varepsilon}\right\} \leq 1. \end{aligned}$$

Choose $y = \left(\frac{\log 2}{2}\right)^{\frac{1}{2}+\varepsilon}$ and the result follows. \diamond

We now give the

Proof of Theorem 2. For $\sigma^{(m)} = (\sigma_1, \dots, \sigma_m)^T \in \mathbb{Z}_2^m$ let $\tilde{\sigma}^{(m)} := (\sigma_m, \dots, \sigma_1)^T$. Then it is easily verified that

$$P_{\tilde{\sigma}^{(m)}} = \{(y, x) : (x, y) \in P_{\sigma^{(m)}}\}.$$

Hence for $0 \leq \alpha, \beta \leq 1$ we have

$$\Delta_{2^m}(P_{\sigma^{(m)}}, \alpha, \beta) = \Delta_{2^m}(P_{\tilde{\sigma}^{(m)}}, \beta, \alpha).$$

Let $2^{m-1} < N \leq 2^m$, then

$$\begin{aligned} \int_0^1 \Delta_{2^m}(P_{\sigma^{(m)}}, N/2^m, \alpha(m))^2 d\alpha &= \int_0^1 \Delta_{2^m}(P_{\tilde{\sigma}^{(m)}}, \alpha(m), N/2^m)^2 d\alpha = \\ &= \frac{1}{2^m} \sum_{l=1}^{2^m} \Delta_{2^m}(P_{\tilde{\sigma}^{(m)}}, l/2^m, N/2^m)^2 = \frac{1}{2^m} \sum_{2^m \alpha=1}^{2^m} \Delta_{2^m}(P_{\tilde{\sigma}^{(m)}}, \alpha, N/2^m)^2 = \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{2^m} \sum_{u_1, u_2=0}^{m-1} \|2^{u_1} N/2^m\| \cdot \|2^{u_2} N/2^m\| (-1)^{\tilde{\sigma}_{u_1+1} + \tilde{\sigma}_{u_2+1}} \sum_{2^m \alpha=1}^{2^m} \prod_{i=1}^2 (a_{m-u_i} \oplus a_{m+1-j(u_i)}).
 \end{aligned}$$

see [11, Lemma 1]. Here $\sum_{2^m \alpha=1}^{2^m}$ means summation over all $\alpha > 0$ m -bit and a_1, \dots, a_m denote the digits of these m -bit numbers. Further $\|\cdot\|$ denotes the distance-to-the-nearest-integer function. The quantities $j(u)$ depend on $\sigma^{(m)}, \alpha, N/2^m$ and u . Since they are not important for these considerations we omit their exact definition (see [11, Lemma 1]). For us it is enough to know [11, Lemma 2] which states that

$$\sum_{2^m \alpha=1}^{2^m} \prod_{i=1}^2 (a_{m-u_i} \oplus a_{m+1-j(u_i)}) = \begin{cases} 2^{m-2} & \text{if } u_1 \neq u_2, \\ 2^{m-1} & \text{if } u_1 = u_2. \end{cases}$$

Therefore we obtain

$$\begin{aligned}
 &\int_0^1 \Delta_{2^m}(P_{\sigma^{(m)}}, N/2^m, \alpha(m))^2 d\alpha = \\
 &= \frac{1}{4} \sum_{\substack{u_1, u_2=0 \\ u_1 \neq u_2}}^{m-1} \|2^{u_1} N/2^m\| \cdot \|2^{u_2} N/2^m\| (-1)^{\tilde{\sigma}_{u_1+1} + \tilde{\sigma}_{u_2+1}} + \\
 (5) \quad &\quad \quad \quad + \frac{1}{2} \sum_{u=0}^{m-1} \|2^u N/2^m\|^2 = \\
 &= \left(\frac{1}{2} \sum_{u=0}^{m-1} \|2^u N/2^m\| (-1)^{\tilde{\sigma}_{u+1}} \right)^2 + \frac{1}{4} \sum_{u=0}^{m-1} \|2^u N/2^m\|^2.
 \end{aligned}$$

From [13] it follows that $\sum_{u=0}^{m-1} \|2^u N/2^m\|^2 = O(m)$ and therefore we only have to analyze the sum

$$\sum_{u=0}^{m-1} \|2^u N/2^m\| (-1)^{\tilde{\sigma}_{u+1}}.$$

Since we think that this analysis is—apart from its application in this proof—of interest on its own we consider these problems in a separate section.

We shall show in Sect. 5 (Th. 6) that there exists a constant $c > 0$ such that for any $\sigma_1, \dots, \sigma_m \in \{0, 1\}$ we have

$$(6) \quad \max_{\beta \text{ } m\text{-bit}} \left| \sum_{u=0}^{m-1} \|2^u \beta\| (-1)^{\sigma_{u+1}} \right| > cm$$

for infinitely many values of $m \in \mathbb{N}$. Since for $N \leq 2^m$ the number $N/2^m$ is a m -bit number we obtain the result from Th. 2 from (6) together with (3) and (5). \diamond

Proof of Theorem 3. The result follows from (3), (5) and (1). \diamond

The proof of Th. 4 will be given at the end of Sect. 5.

5. A sum of distances-to-the-nearest-integer

In this section we are interested in sums of the form

$$(7) \quad \sum_{u=0}^{m-1} \|2^u \beta\| (-1)^{\sigma_{u+1}}$$

where β is an m -bit number and where $\sigma_1, \dots, \sigma_m$ are arbitrary given numbers in $\{0, 1\}$. To be more precise, we are interested in the maximum and the minimum of such sums over all m -bit numbers β . This problem was considered in [13] for the case $\sigma_i = 0$ for all $1 \leq i \leq m$. Similar sums have also been considered quite recently in [10].

First we determine the m -bit numbers for which the maximum and the minimum of the sum (7) is attained. W.l.o.g. we may assume that $\sigma_1 = 1$. We divide $\sigma^{(m)} = (\sigma_1, \dots, \sigma_m)$ into blocks

$$\sigma^{(m)} = C_0 C_1 \dots C_t$$

with

$$C_0 = \underbrace{11 \dots 1}_{y_0 \text{ bits}}, \quad \text{and} \quad C_k = \underbrace{00 \dots 0}_{x_k \text{ bits}} \underbrace{11 \dots 1}_{y_k \text{ bits}}, \quad 1 \leq k \leq t,$$

where $x_k, y_k \geq 1$, except for C_t which also may consist of zeros only. Note that $t = t(m)$. Formally, let $x_0 := 0, f_0 := 0$ and for $1 \leq r \leq t+1$,

$$f_r := \sum_{i=0}^{r-1} (x_i + y_i).$$

Of course we have $f_{t+1} = m$. We define

$$\zeta^{(0)} = \zeta^{(0)}(\sigma^{(m)}) = 0.z_1^{(0)} \dots z_m^{(0)},$$

$$\zeta^{(1)} = \zeta^{(1)}(\sigma^{(m)}) = 0.z_1^{(1)} \dots z_m^{(1)}$$

as follows in several steps. In each step we show for $0 \leq r \leq t$ how the digits $z_n^{(h)}, 1 \leq h \leq 2$, of $\zeta^{(h)}$ with $f_r + 1 \leq n \leq f_{r+1}$ are to be chosen.

STEP 1. Concerning C_0 , let

$$\zeta^{(0)} := \underbrace{0.11\dots 1\dots}_{y_0 \text{ bits}} \quad \text{and} \quad \zeta^{(1)} := 0.\underbrace{00\dots 0\dots}_{y_0 \text{ bits}}\dots$$

STEP 2. Concerning C_1 , let

$$\zeta^{(0)} := \begin{cases} \dots \underbrace{1010\dots 10}_{x_1 \text{ bits}} \underbrace{11\dots 1\dots}_{y_1 \text{ bits}} \dots & \text{if } x_1 \text{ even,} \\ \dots \underbrace{1010\dots 10100\dots 0\dots}_{x_1 \text{ bits}} \underbrace{}_{y_1 \text{ bits}} \dots & \text{if } x_1 \text{ odd,} \end{cases}$$

$$\zeta^{(1)} := \begin{cases} \dots \underbrace{0101\dots 0100\dots 0\dots}_{x_1 \text{ bits}} \underbrace{}_{y_1 \text{ bits}} \dots & \text{if } x_1 \text{ even,} \\ \dots \underbrace{0101\dots 01011\dots 1\dots}_{x_1 \text{ bits}} \underbrace{}_{y_1 \text{ bits}} \dots & \text{if } x_1 \text{ odd.} \end{cases}$$

STEP 3. Concerning C_r , $2 \leq r \leq t-1$, let for $1 \leq h \leq 2$

$$\zeta^{(h)} := \begin{cases} \dots \underbrace{1010\dots 10}_{x_r \text{ bits}} \underbrace{11\dots 1\dots}_{y_r \text{ bits}} \dots & \text{if } x_r \text{ even and } z_{f_{r-1}+x_{r-1}}^{(h)} = 0, \\ \dots \underbrace{1010\dots 10100\dots 0\dots}_{x_r \text{ bits}} \underbrace{}_{y_r \text{ bits}} \dots & \text{if } x_r \text{ odd and } z_{f_{r-1}+x_{r-1}}^{(h)} = 0, \\ \dots \underbrace{0101\dots 0100\dots 0\dots}_{x_r \text{ bits}} \underbrace{}_{y_r \text{ bits}} \dots & \text{if } x_r \text{ even and } z_{f_{r-1}+x_{r-1}}^{(h)} = 1, \\ \dots \underbrace{0101\dots 01011\dots 1\dots}_{x_r \text{ bits}} \underbrace{}_{y_r \text{ bits}} \dots & \text{if } x_r \text{ odd and } z_{f_{r-1}+x_{r-1}}^{(h)} = 1. \end{cases}$$

STEP 4. Concerning C_t , let for $1 \leq h \leq 2$

$$\zeta^{(h)} := \begin{cases} \dots \underbrace{1010\dots 101100\dots 0}_{x_t \text{ bits}} \underbrace{}_{y_t \text{ bits}} & \text{if } x_t \text{ even and } z_{f_{t-1}+x_{t-1}}^{(h)} = 0, \\ \dots \underbrace{1010\dots 10100\dots 0}_{x_t \text{ bits}} \underbrace{}_{y_t \text{ bits}} & \text{if } x_t \text{ odd and } z_{f_{t-1}+x_{t-1}}^{(h)} = 0, \\ \dots \underbrace{0101\dots 0100\dots 0}_{x_t \text{ bits}} \underbrace{}_{y_t \text{ bits}} & \text{if } x_t \text{ even and } z_{f_{t-1}+x_{t-1}}^{(h)} = 1, \\ \dots \underbrace{0101\dots 01100\dots 0}_{x_t \text{ bits}} \underbrace{}_{y_t \text{ bits}} & \text{if } x_t \text{ odd and } z_{f_{t-1}+x_{t-1}}^{(h)} = 1. \end{cases}$$

STEP 4a. If $t = 1$, let

$$\zeta^{(0)} := \begin{cases} 0.\underbrace{1010\dots 1011}_{x_1 \text{ bits}}\underbrace{00\dots 0}_{y_1 \text{ bits}} & \text{if } x_1 \text{ even,} \\ 0.\underbrace{1010\dots 101}_{x_1 \text{ bits}}\underbrace{00\dots 0}_{y_1 \text{ bits}} & \text{if } x_1 \text{ odd,} \end{cases}$$

$$\zeta^{(1)} := \begin{cases} 0.\underbrace{0101\dots 01}_{x_1 \text{ bits}}\underbrace{00\dots 0}_{y_1 \text{ bits}} & \text{if } x_1 \text{ even,} \\ 0.\underbrace{0101\dots 011}_{x_1 \text{ bits}}\underbrace{00\dots 0}_{y_1 \text{ bits}} & \text{if } x_1 \text{ odd.} \end{cases}$$

STEP 4b. If $\sigma^{(m)} = C_0$, let

$$\zeta^{(0)} = \zeta^{(1)} = 0.00\dots 0.$$

Note that, except for the case $\sigma^{(m)} = C_0$, $\zeta^{(0)} + \zeta^{(1)} = 1$.

Further, we divide $\sigma^{(m)} = (\sigma_1, \dots, \sigma_m)$ into blocks

$$\sigma^{(m)} = B_1 B_2 \dots B_w$$

with

$$B_k = \underbrace{11\dots 1}_{u_k \text{ bits}}\underbrace{00\dots 0}_{v_k \text{ bits}}, \quad 1 \leq k \leq w,$$

with $u_k, v_k \geq 1$, except for B_w which also may consist of ones only. Let $e_1 := 0$ and for $2 \leq r \leq w + 1$,

$$e_r := \sum_{i=1}^{r-1} (u_i + v_i).$$

Of course we have $e_{w+1} = m$. We define

$$\delta^{(0)} = \delta^{(0)}(\sigma^{(m)}) = 0.d_1^{(0)} \dots d_m^{(0)},$$

$$\delta^{(1)} = \delta^{(1)}(\sigma^{(m)}) = 0.d_1^{(1)} \dots d_m^{(1)}$$

as follows in several steps. In each step we show for $1 \leq r \leq w$ how the digits $d_n^{(h)}$, $1 \leq h \leq 2$, of $\delta^{(h)}$ with $e_r + 1 \leq n \leq e_{r+1}$ are to be chosen.

STEP 1. Concerning B_1 , let

$$\delta^{(0)} := \begin{cases} 0.\underbrace{1010\dots 10}_{u_1 \text{ bits}}\underbrace{11\dots 1}_{v_1 \text{ bits}}\dots & \text{if } u_1 \text{ even,} \\ 0.\underbrace{1010\dots 101}_{u_1 \text{ bits}}\underbrace{00\dots 0}_{v_1 \text{ bits}}\dots & \text{if } u_1 \text{ odd,} \end{cases}$$

$$\delta^{(1)} := \begin{cases} 0.\underbrace{0101\dots 01}_{u_1 \text{ bits}}\underbrace{00\dots 0}_{v_1 \text{ bits}}\dots & \text{if } u_1 \text{ even,} \\ 0.\underbrace{0101\dots 010}_{u_1 \text{ bits}}\underbrace{11\dots 1}_{v_1 \text{ bits}}\dots & \text{if } u_1 \text{ odd.} \end{cases}$$

STEP 2. Concerning B_r , $2 \leq r \leq w - 1$, let for $1 \leq h \leq 2$

$$\delta^{(h)} := \begin{cases} \dots \underbrace{1010\dots 10}_{u_r \text{ bits}}\underbrace{11\dots 1}_{v_r \text{ bits}}\dots & \text{if } u_r \text{ even and } d_{e_{r-1}+u_{r-1}}^{(h)} = 0, \\ \dots \underbrace{1010\dots 101}_{u_r \text{ bits}}\underbrace{00\dots 0}_{v_r \text{ bits}}\dots & \text{if } u_r \text{ odd and } d_{e_{r-1}+u_{r-1}}^{(h)} = 0, \\ \dots \underbrace{0101\dots 01}_{u_r \text{ bits}}\underbrace{00\dots 0}_{v_r \text{ bits}}\dots & \text{if } u_r \text{ even and } d_{e_{r-1}+u_{r-1}}^{(h)} = 1, \\ \dots \underbrace{0101\dots 010}_{u_r \text{ bits}}\underbrace{11\dots 1}_{v_r \text{ bits}}\dots & \text{if } u_r \text{ odd and } d_{e_{r-1}+u_{r-1}}^{(h)} = 1. \end{cases}$$

STEP 3. Concerning B_w , let for $1 \leq h \leq 2$

$$\delta^{(h)} := \begin{cases} \dots \underbrace{1010\dots 1011}_{u_w \text{ bits}}\underbrace{00\dots 0}_{v_w \text{ bits}} & \text{if } u_w \text{ even and } d_{e_{w-1}+u_{w-1}}^{(h)} = 0, \\ \dots \underbrace{1010\dots 101}_{u_w \text{ bits}}\underbrace{00\dots 0}_{v_w \text{ bits}} & \text{if } u_w \text{ odd and } d_{e_{w-1}+u_{w-1}}^{(h)} = 0, \\ \dots \underbrace{0101\dots 01}_{u_w \text{ bits}}\underbrace{00\dots 0}_{v_w \text{ bits}} & \text{if } u_w \text{ even and } d_{e_{w-1}+u_{w-1}}^{(h)} = 1, \\ \dots \underbrace{0101\dots 011}_{u_w \text{ bits}}\underbrace{00\dots 0}_{v_w \text{ bits}} & \text{if } u_w \text{ odd and } d_{e_{w-1}+u_{w-1}}^{(h)} = 1. \end{cases}$$

STEP 3a. If $w = 1$, let

$$\delta^{(0)} := \begin{cases} 0.\underbrace{1010\dots 1011}_{u_1 \text{ bits}}\underbrace{00\dots 0}_{v_1 \text{ bits}} & \text{if } u_1 \text{ even,} \\ 0.\underbrace{1010\dots 101}_{u_1 \text{ bits}}\underbrace{00\dots 0}_{v_1 \text{ bits}} & \text{if } u_1 \text{ odd,} \end{cases}$$

$$\delta^{(1)} := \begin{cases} 0.\underbrace{0101\dots 01}_{u_1 \text{ bits}}\underbrace{00\dots 0}_{v_1 \text{ bits}}, & u_1 \text{ even,} \\ 0.\underbrace{0101\dots 011}_{u_1 \text{ bits}}\underbrace{00\dots 0}_{v_1 \text{ bits}}, & u_1 \text{ odd.} \end{cases}$$

Observe that $\delta^{(0)} + \delta^{(1)} = 1$.

We now show the following

Lemma 1. Let $\sigma^{(m)} = (\sigma_1, \sigma_2, \dots, \sigma_m)^T \in \mathbb{Z}_2^m$ with $\sigma_1 = 1$, and let $0 \leq \kappa < 1$.

(a) *The minimum*

$$\min_{\beta \text{ m-bit}} \left(\kappa\beta + \sum_{u=0}^{m-1} \|2^u \beta\| (-1)^{\sigma_{u+1}} \right) =: \min_{\beta \text{ m-bit}} \Sigma(\kappa, \sigma^{(m)}, m, \beta)$$

is attained for $\beta = \delta^{(1)}(\sigma^{(m)})$ defined as above.

(b) *The minimum*

$$\min_{\beta \text{ m-bit}} \left(-\kappa\beta + \sum_{u=0}^{m-1} \|2^u \beta\| (-1)^{\sigma_{u+1}} \right) =: \min_{\beta \text{ m-bit}} \Sigma(-\kappa, \sigma^{(m)}, m, \beta)$$

is attained for $\beta = \delta^{(0)}(\sigma^{(m)})$ defined as above.

(c) *The maximum*

$$\max_{\beta \text{ m-bit}} \left(\kappa\beta + \sum_{u=0}^{m-1} \|2^u \beta\| (-1)^{\sigma_{u+1}} \right) =: \max_{\beta \text{ m-bit}} \Sigma(\kappa, \sigma^{(m)}, m, \beta)$$

is attained for $\beta = \zeta^{(0)}(\sigma^{(m)})$ defined as above.

(d) *The maximum*

$$\max_{\beta \text{ m-bit}} \left(-\kappa\beta + \sum_{u=0}^{m-1} \|2^u \beta\| (-1)^{\sigma_{u+1}} \right) =: \max_{\beta \text{ m-bit}} \Sigma(-\kappa, \sigma^{(m)}, m, \beta)$$

is attained for $\beta = \zeta^{(1)}(\sigma^{(m)})$ defined as above.

Proof. We show the results in (a), (b), (c), and (d) simultaneously by induction on m .

For $m = 1, 2$ the result is easily verified numerically.

Assume now the results in (a), (b), (c), and (d) have already been shown for $m - 1$. We start by showing (a).

In order to minimize $\Sigma(\kappa, \sigma^{(m)}, m, \beta)$ it is necessary that the first digit of β equals zero (otherwise $1 - \beta$ would yield a lower value). Since the first component of $\sigma^{(m)}$ is one, we obtain for such a β

$$\begin{aligned}
\kappa\beta + \sum_{u=0}^{m-1} \|2^u\beta\| (-1)^{\sigma_{u+1}} &= \kappa\beta - \|\beta\| + \sum_{u=1}^{m-1} \|2^u\beta\| (-1)^{\sigma_{u+1}} = \\
&= (\kappa - 1)\beta + \sum_{u=1}^{m-1} \|2^u\beta\| (-1)^{\sigma_{u+1}} = \\
&= \beta' \left(\frac{\kappa - 1}{2} \right) + \sum_{u=0}^{m-2} \|2^u\beta'\| (-1)^{\sigma'_{u+1}},
\end{aligned}$$

where $\beta' = 0.\beta_2 \dots \beta_m$ and $\sigma' = (\sigma_2, \dots, \sigma_m)^T$. However, $-1 < \frac{\kappa-1}{2} \leq 0$. If $\sigma_2 = 1$, we are done by the induction assumption for (b).

If, on the other hand, $\sigma_2 = 0$, minimizing

$$\beta' \left(\frac{\kappa - 1}{2} \right) + \sum_{u=0}^{m-2} \|2^u\beta'\| (-1)^{\sigma'_{u+1}}$$

is the same task as maximizing

$$-\beta' \left(\frac{\kappa - 1}{2} \right) + \sum_{u=0}^{m-2} \|2^u\beta'\| (-1)^{\tilde{\sigma}'_{u+1}},$$

where $\tilde{\sigma}' = (1 \oplus \sigma_2, \dots, 1 \oplus \sigma_m)^T$, which of course means that $\tilde{\sigma}'_1 = 1$. In this case, the result follows by the induction assumption for (c).

The proof of (b), (c), and (d) is similar. \diamond

We now get the following theorem.

Theorem 5. Let $\sigma^{(m)} = (\sigma_1, \sigma_2, \dots, \sigma_m)^T \in \mathbb{Z}_2^m$ with $\sigma_1 = 1$.

(a) The minimum

$$\min_{\beta \text{ } m\text{-bits}} \sum_{u=0}^{m-1} \|2^u\beta\| (-1)^{\sigma_{u+1}}$$

is attained if β equals either $\delta^{(0)}(\sigma^{(m)})$ or $\delta^{(1)}(\sigma^{(m)})$ given above.

(b) The maximum

$$\max_{\beta \text{ } m\text{-bits}} \sum_{u=0}^{m-1} \|2^u\beta\| (-1)^{\sigma_{u+1}}$$

is attained if β equals either $\zeta^{(0)}(\sigma^{(m)})$ or $\zeta^{(1)}(\sigma^{(m)})$ given above.

Proof. The result follows immediately from Lemma 1 by choosing $\kappa = 0$. \diamond

It is easy to compute the average of the sum (7) over all m -bit numbers.

Lemma 2. Let again $l_m := \#\{1 \leq u \leq m : \sigma_u = 0\}$ and $\psi(m) := l_m - \frac{m}{2}$. Then

$$\frac{1}{2^m} \sum_{\beta \text{ } m\text{-bit}} \sum_{u=0}^{m-1} \|2^u \beta\| (-1)^{\sigma_{u+1}} = \frac{\psi(m)}{2}.$$

Proof. For $0 \leq u \leq m - 1$ it is true that (see [14])

$$\sum_{\beta \text{ } m\text{-bit}} \|2^u \beta\| = 2^u \sum_{\beta \text{ } (m-u)\text{-bit}} \|\beta\| = 2^{m-2}.$$

Hence,

$$\begin{aligned} \frac{1}{2^m} \sum_{\beta \text{ } m\text{-bit}} \sum_{u=0}^{m-1} \|2^u \beta\| (-1)^{\sigma_{u+1}} &= \sum_{u=0}^{m-1} (-1)^{\sigma_{u+1}} \frac{1}{2^m} \sum_{\beta \text{ } m\text{-bit}} \|2^u \beta\| = \\ &= \sum_{\substack{u=0 \\ \sigma_{u+1}=0}}^{m-1} \frac{1}{4} - \sum_{\substack{u=0 \\ \sigma_{u+1}=1}}^{m-1} \frac{1}{4} = \\ &= \frac{1}{4} (\#\{1 \leq u \leq m : \sigma_u = 0\} - \#\{1 \leq u \leq m : \sigma_u = 1\}) = \\ &= \frac{1}{4} \left(\frac{m}{2} + \psi(m) - \left(\frac{m}{2} - \psi(m) \right) \right) = \frac{\psi(m)}{2}. \diamond \end{aligned}$$

We now want to find out more about the order of magnitude in m of the term

$$\max_{\beta \text{ } m\text{-bit}} \left| \sum_{u=0}^{m-1} \|2^u \beta\| (-1)^{\sigma_{u+1}} \right|$$

for given $\sigma^{(m)} \in \mathbb{Z}_2^m$. For this purpose, we discuss several different cases. W. l. o. g. we always assume $\sigma^{(m)} = (1, \sigma_2, \dots, \sigma_m)$, i.e., the first digit of σ equals 1.

As usual, for a real valued function f defined on \mathbb{N} we may often write $f(m) = O(m^\alpha)$ if there exists a constant $c > 0$ such that $|f(m)| \leq cm^\alpha$. Further we write $f(m) = \Theta(m^\alpha)$ if $f(m) = O(m^\alpha)$ and $f(m) \neq O(m^\beta)$ with $\beta < \alpha$.

CASE 1. $\sigma^{(m)} \in \mathbb{Z}_2^m$ is such that $\psi(m) = \Theta(m)$. Then it follows by Lemma 2 that

$$\max_{\beta \text{ } m\text{-bit}} \left| \sum_{u=0}^{m-1} \|2^u \beta\| (-1)^{\sigma_{u+1}} \right| = \Theta(m).$$

CASE 2. $\sigma^{(m)} \in \mathbb{Z}_2^m$ is such that $\psi(m) = O(m^\alpha)$, $\alpha < 1$. By Th. 5 (b)

we know that

$$\begin{aligned} \max_{\beta \text{ } m\text{-bit}} \sum_{u=0}^{m-1} \|2^u \beta\| (-1)^{\sigma_{u+1}} &= \sum_{u=0}^{m-1} \left\| 2^u \zeta^{(0)}(\sigma) \right\| (-1)^{\sigma_{u+1}} \\ &= \sum_{u=0}^{m-1} \left\| 2^u \zeta^{(1)}(\sigma) \right\| (-1)^{\sigma_{u+1}}. \end{aligned}$$

As above, we write $\sigma^{(m)} = C_0 C_1 \dots C_t$ with

$$C_0 = \underbrace{11 \dots 1}_{y_0 \text{ bits}} \quad \text{and} \quad C_k = \underbrace{00 \dots 0}_{x_k \text{ bits}} \underbrace{11 \dots 1}_{y_k \text{ bits}}, \quad 1 \leq k \leq t,$$

where $x_k, y_k \geq 1$, except for C_t which also may consist of zeros only. Formally, let $x_0 := 0, f_0 = 0$ and for $1 \leq r \leq t + 1$,

$$f_r := \sum_{i=0}^{r-1} (x_i + y_i).$$

Note that $f_{t+1} = m$. We denote the digits of $\zeta^{(0)} = \zeta^{(0)}(\sigma^{(m)})$ constructed above by

$$\zeta^{(0)} = 0.z_1^{(0)} \dots z_m^{(0)}.$$

Suppose $u \in \{f_r + x_r, \dots, f_{r+1} - 1\}$, $0 \leq r \leq t$ (this corresponds to $\sigma_{u+1} = 1$), then it follows by the construction of $\zeta^{(0)}$ that

$$\left\| 2^u \zeta^{(0)} \right\| = \left\| 2^u 0.z_1^{(0)} \dots z_m^{(0)} \right\| = \left\| 0.z_{u+1}^{(0)} \dots z_m^{(0)} \right\| \leq \frac{1}{4}.$$

What if $u \in \{f_r, \dots, f_r + x_r - 1\}$, $1 \leq r \leq t$ (this corresponds to $\sigma_{u+1} = 0$)?

Suppose, in the first place, $u \in \{f_r, \dots, f_r + x_r - 3\}$ for an $r \in \{1, \dots, t\}$ with $x_r \geq 3$.

Then we have

$$0.z_{u+1}^{(0)} z_{u+2}^{(0)} z_{u+3}^{(0)} z_{u+4}^{(0)} z_{u+5}^{(0)} \dots = \begin{cases} 0.1010z_{u+5}^{(0)} \dots & \text{or} \\ 0.0101z_{u+5}^{(0)} \dots & \text{or} \\ 0.011, & \end{cases}$$

which results in

$$(8) \quad \left\| 2^u \zeta^{(0)} \right\| \geq \frac{1}{4} + \frac{1}{16}.$$

On the other hand, whenever $x_r \geq 2$, and $u = f_r + x_r - 2$, we have

$$\begin{aligned} & \left\| 2^u \zeta^{(0)} \right\| + \left\| 2^{u+1} \zeta^{(0)} \right\| = \\ & = \left\| 0.z_{u+1}^{(0)} z_{u+2}^{(0)} z_{u+3}^{(0)} z_{u+4}^{(0)} \dots \right\| + \left\| 0.z_{u+2}^{(0)} z_{u+3}^{(0)} z_{u+4}^{(0)} \dots \right\| = \\ & = \begin{cases} \left\| 0.101z_{u+4}^{(0)} \dots \right\| + \left\| 0.01z_{u+4}^{(0)} \dots \right\| & \text{(Case (a)), or} \\ \left\| 0.010z_{u+4}^{(0)} \dots \right\| + \left\| 0.10z_{u+4}^{(0)} \dots \right\| & \text{(Case (b)), or} \\ \left\| 0.11 \right\| + \left\| 0.1 \right\| & \text{(Case (c)), or} \\ \left\| 0.01 \right\| + \left\| 0.1 \right\| & \text{(Case (d)).} \end{cases} \end{aligned}$$

In Case (a),

$$\begin{aligned} & \left\| 0.101z_{u+4}^{(0)} \dots \right\| + \left\| 0.01z_{u+4}^{(0)} \dots \right\| = 1 - 0.101z_{u+4}^{(0)} \dots + 0.01z_{u+4}^{(0)} \dots = \\ & = \frac{1}{2} - \frac{1}{2} 0.01z_{u+4}^{(0)} \dots + 0.01z_{u+4}^{(0)} \dots = \frac{1}{2} + \frac{1}{2} 0.01z_{u+4}^{(0)} \dots \geq \\ & \geq \frac{1}{2} + \frac{1}{8} = 2 \left(\frac{1}{4} + \frac{1}{16} \right). \end{aligned}$$

In Case (b),

$$\begin{aligned} & \left\| 0.010z_{u+4}^{(0)} \dots \right\| + \left\| 0.10z_{u+4}^{(0)} \dots \right\| = 0.010z_{u+4}^{(0)} \dots + 1 - 0.10z_{u+4}^{(0)} \dots = \\ & = \frac{1}{2} 0.10z_{u+4}^{(0)} \dots + 1 - 0.10z_{u+4}^{(0)} \dots = 1 - \frac{1}{2} 0.10z_{u+4}^{(0)} \dots \geq \\ & \geq 1 - \frac{1}{2} 0.11 = \frac{5}{8} = 2 \left(\frac{1}{4} + \frac{1}{16} \right). \end{aligned}$$

In Case (c) and Case (d), we obviously obtain

$$\left\| 2^u \zeta^{(0)} \right\| + \left\| 2^{u+1} \zeta^{(0)} \right\| = \frac{3}{4} \geq 2 \left(\frac{1}{4} + \frac{1}{16} \right).$$

So, in any of the Cases (a), (b), (c), and (d),

$$(9) \quad \left\| 2^u \zeta^{(0)} \right\| + \left\| 2^{u+1} \zeta^{(0)} \right\| \geq 2 \left(\frac{1}{4} + \frac{1}{16} \right).$$

From (8) and (9) we conclude

$$\sum_{u=f_r}^{f_r+x_r-1} \left\| 2^u \zeta^{(0)} \right\| \geq x_r \left(\frac{1}{4} + \frac{1}{16} \right)$$

for $1 \leq r \leq t$ whenever $x_r \geq 2$.

If, however, $x_r = 1$ and $u = f_r + x_r - 1 = f_r$, then

$$\|2^u \zeta^{(0)}\| = \left\| 0.z_{u+1}^{(0)} z_{u+2}^{(0)} z_{u+3}^{(0)} z_{u+4}^{(0)} \dots \right\| = \begin{cases} 0.011z_{u+4}^{(0)} \dots & \text{or} \\ 0.100z_{u+4}^{(0)} \dots \end{cases}$$

This implies again

$$\sum_{u=f_r}^{f_r+x_r-1} \|2^u \zeta^{(0)}\| \geq x_r \left(\frac{1}{4} + \frac{1}{16} \right).$$

We therefore obtain

$$\begin{aligned} \sum_{u=0}^{m-1} \|2^u \zeta^{(0)}\| (-1)^{\sigma_{u+1}} &= \sum_{r=1}^t \sum_{u=f_r}^{f_r+x_r-1} \|2^u \zeta^{(0)}\| - \sum_{r=0}^t \sum_{u=f_r+x_r}^{f_{r+1}-1} \|2^u \zeta^{(0)}\| \geq \\ &\geq \sum_{r=1}^t x_r \left(\frac{1}{4} + \frac{1}{16} \right) - \sum_{r=0}^t \sum_{u=f_r+x_r}^{f_{r+1}-1} \frac{1}{4} = \\ &= \left(\frac{1}{4} + \frac{1}{16} \right) \left(\frac{m}{2} + \psi(m) \right) - \frac{1}{4} \left(\frac{m}{2} - \psi(m) \right) = \\ &= \frac{m}{32} + \frac{\psi(m)}{2} = \frac{m}{32} + O(m^\alpha). \end{aligned}$$

We summarize:

Theorem 6. *There exists a constant $c > 0$ such that the inequality*

$$\beta \max_{m\text{-bit}} \left| \sum_{u=0}^{m-1} \|2^u \beta\| (-1)^{\sigma_{u+1}} \right| > cm$$

holds for infinitely many values of $m \in \mathbb{N}$.

Finally we use the results from this section to give the

Proof of Theorem 4. Let $\sigma^* = (1, 0, 1, 0, \dots)^T$. We show that $q(\sigma^*) = 1/(20 \log 2)$. The result then follows.

From (3) and the proof of Th. 2 we find that for $N \leq 2^m$ and for any shift $\sigma \in \mathbb{Z}_2^\infty$ we have

$$(NL_{2,N}(\gamma_\sigma))^2 = \left(\frac{1}{2} \sum_{u=0}^{m-1} \|2^u N/2^m\| (-1)^{\tilde{\sigma}_{u+1}} \right)^2 + O(m),$$

where $\tilde{\sigma}^{(m)} = (\tilde{\sigma}_1, \dots, \tilde{\sigma}_m)^T$ is the vector $\sigma^{(m)}$ with the components in reversed order.

We have to compute

$$(10) \quad \max_{\beta \text{ } m\text{-bit}} \left| \sum_{u=0}^{m-1} \|2^u \beta\| (-1)^{\tilde{\sigma}_{u+1}} \right|,$$

for $\tilde{\sigma}^{(m)} = (\tilde{\sigma}_1, \dots, \tilde{\sigma}_m) = (1, 0, 1, 0, \dots) \in \mathbb{Z}_2^m$ and $\tilde{\sigma}^{(m)} = (\tilde{\sigma}_1, \dots, \tilde{\sigma}_m) = (0, 1, 0, 1, \dots) \in \mathbb{Z}_2^m$. Note that it is enough to consider only the first one of these two cases.

From Th. 5 we know that the maximum (10) is attained for $\beta = \delta^{(0)}(\tilde{\sigma}^{(m)})$ or for $\beta = \zeta^{(0)}(\tilde{\sigma}^{(m)})$. For $\tilde{\sigma}^{(m)} = (1, 0, 1, 0, \dots)^T \in \mathbb{Z}_2^m$ we have

$$\delta^{(0)}(\tilde{\sigma}^{(m)}) = \begin{cases} 0.10011001 \dots 10011010, & \text{if } m = 4k, \\ 0.10011001 \dots 100110011, & \text{if } m = 4k + 1, \\ 0.10011001 \dots 1001100110, & \text{if } m = 4k + 2, \\ 0.10011001 \dots 10011001101, & \text{if } m = 4k + 3, \end{cases}$$

and

$$\zeta^{(0)}(\tilde{\sigma}^{(m)}) = \begin{cases} 0.110011001 \dots 1001101, & \text{if } m = 4k, \\ 0.110011001 \dots 10011010, & \text{if } m = 4k + 1, \\ 0.110011001 \dots 100110011, & \text{if } m = 4k + 2, \\ 0.110011001 \dots 1001100110, & \text{if } m = 4k + 3. \end{cases}$$

Therefore we obtain by tedious but straightforward calculations that

$$\max_{\beta \text{ } m\text{-bit}} \left| \sum_{u=0}^{m-1} \|2^u \beta\| (-1)^{\tilde{\sigma}_{u+1}} \right| = \frac{m}{10} + O(1).$$

Hence

$$\begin{aligned} \max_{2^{m-1} < N \leq 2^m} (NL_{2,N}(\gamma_{\sigma^*}))^2 &= \left(\frac{m}{20} + O(1) \right)^2 + O(m) \\ &= \left(\frac{\log N}{20 \log 2} + O(1) \right)^2 + O(\log N) \end{aligned}$$

and the result follows. \diamond

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