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ON A THEOREM OF ŠAPIROVSKIĬ ON THE SIZE OF A SEQUENTIAL SPACE

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This paper is dedicated to Gino Tironi on the occasion of his 65th birthday.

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Abstract: We are presenting a generalization of a theorem of Šapirovskiĭ on the cardinality of a Lindelöf sequential space. Then, we will discuss the possibility to extend this result to pseudoradial spaces.

1. Introduction and main result

At the end of a survey paper on the Souslin number [11], B. E. Šapirovskiĭ stated without proof the following:

Proposition 1 ([11], Th. 5.23). *Let X be a Lindelöf regular sequential space. If an uncountable cardinal $\lambda \leq \mathfrak{c}^+$ is a caliber of X, then $|X| \leq \mathfrak{c}$.*

Later, A. V. Arhangel'skiĭ [3] published a detailed proof of this result in the special case $\lambda = \aleph_1$. His proof is based on the following:

Proposition 2 ([3], Th. 5.1). *Let X be a Lindelöf regular space. If $T(X) = \aleph_0$ and $\aleph_1$ is a caliber of X, then $d(X) \leq \mathfrak{c}$.*

The main purpose of the present note is to establish a simultaneous generalization of these two propositions. It turns out that

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Lindelöfness in the hypotheses of Props. 1 and 2 can be weakened by assuming that the spaces have only short free sequences.

Our proof of Th. 1 was actually inspired by the argument in Arhangel'skiĭ's proof of Prop. 2.

Next, we focus on the possibility to extend Prop. 1 to spaces more general than sequential.

For notations and undefined notions we refer to [9]. In the sequel compact means compact Hausdorff. The continuum is indicated either with $\mathfrak{c}$ or with $2^{\aleph_0}$.

As usual, $c(X)$, $d(X)$, $L(X)$ and $\psi(X)$ denote cellularity, density, Lindelöf number and pseudocharacter of the space X.

A cardinal κ is a caliber of the space X if every family of open sets of cardinality κ has a subfamily of cardinality κ with a non-empty intersection.

Of course, if κ^+ is a caliber of X, then $c(X) \leq \kappa$. So, caliber $\aleph_1$ is stronger than countable cellularity. Moreover, every regular cardinal $\kappa > d(X)$ is a caliber of X.

A set $\{x_\alpha : \alpha < \kappa\}$ in the space X is said to be a free sequence provided that for any $\gamma < \kappa$ we have $\overline{\{x_\alpha : \alpha < \gamma\}} \cap \overline{\{x_\alpha : \gamma \leq \alpha < \kappa\}} = \emptyset$. The supremum of the sizes of the free sequences in X is denoted by $F(X)$. When $F(X) = \aleph_0$, we say that the space X is countably free.

Recall that a space X is said to be almost regular if it has a π-net consisting of non-empty regular closed sets, i.e. for any non-empty open set U there exists a non-empty open set V such that $\overline{V} \subseteq U$.

Theorem 1. *Let X be an almost regular space. If a cardinal λ satisfying $F(X) < \lambda \leq \left(2^{F(X)}\right)^+$ is a caliber of X, then $d(X) \leq 2^{F(X)}$.*

Proof. Let $F(X) = \kappa$ and assume by contradiction that the density of X is bigger than 2^κ. Fix a choice function $\eta : \mathcal{P}(X) \setminus \{\emptyset\} \to X$. We will define by induction an increasing family $\{A_\alpha : \alpha < \lambda\}$ of subsets of X of cardinality not exceeding 2^κ and a family $\{U_\alpha : \alpha < \lambda\}$ of non-empty open subsets of X in such a way that:

(1) $\overline{A_\alpha} \cap \overline{U_\alpha} = \emptyset$;

(2) if $\mathcal{V} \subseteq \{U_\beta : \beta < \alpha\}$ satisfies $|\mathcal{V}| \leq \kappa$ and $\bigcap \mathcal{V} \neq \emptyset$, then $\eta(\bigcap \mathcal{V}) \in A_\alpha$.

To justify the inductive construction, let us assume to have already defined the sets $\{A_\beta : \beta < \alpha\}$ and $\{U_\beta : \beta < \alpha\}$. Since $\alpha < \lambda$ and $\lambda \leq \left(2^{F(X)}\right)^+$, we have $|\{U_\beta : \beta < \alpha\}| \leq |\alpha| \leq 2^\kappa$. Consequently, the set $B = \{\eta(\bigcap \mathcal{V}) : \mathcal{V} \subseteq \{U_\beta : \beta < \alpha\}, |\mathcal{V}| \leq \kappa \text{ and } \bigcap \mathcal{V} \neq \emptyset\}$ has cardinality not exceeding 2^κ. Then, let $A_\alpha = B \cup \bigcup\{A_\beta : \beta < \alpha\}$. As we are assuming that the density of X is bigger than 2^κ, we may find a non-empty open set U_α such that $\overline{A_\alpha} \cap \overline{U_\alpha} = \emptyset$.

Since λ is a caliber of X, there exists a set $S \subseteq \lambda$ such that $|S| = \lambda$ and $\bigcap\{U_\alpha : \alpha \in S\} \neq \emptyset$. We may fix an increasing mapping $f : \lambda \to S$. Observe now that we are assuming $\kappa^+ \leq \lambda$. For any $\alpha < \kappa^+$ let $x_\alpha = \eta(\bigcap\{U_{f(\xi)} : \xi \leq \alpha\})$. We claim that the set $\{x_\alpha : \alpha < \kappa^+\}$ so obtained is a free sequence in X. To check this, fix $\alpha < \kappa^+$ and observe that for each $\beta < \alpha$ we have $x_\beta \in A_{f(\beta)+1} \subseteq A_{f(\alpha)}$. Moreover, for each $\beta \geq \alpha$ the set $U_{f(\alpha)}$ occurs in the definition of x_β and consequently $x_\beta \in U_{f(\alpha)}$. This means that $\{x_\beta : \beta < \alpha\} \subseteq A_{f(\alpha)}$ and $\{x_\beta : \alpha \leq \beta < \kappa^+\} \subseteq U_{f(\alpha)}$. Therefore $\overline{\{x_\beta : \beta < \alpha\}} \cap \overline{\{x_\beta : \alpha \leq \beta < \kappa^+\}} \subseteq \overline{A_{f(\alpha)}} \cap \overline{U_{f(\alpha)}} = \emptyset$. The validity of the claim contradicts the hypothesis and the proof is then complete. ◊

Now, we may easily derive the result of Šapirovskiĭ.

Proof of Proposition 1. In a Lindelöf sequential space the free sequences are at most countable. Applying Th. 1 in case $F(X) = \aleph_0$, we see that the density of X is at most $\mathfrak{c}$. Then, by the sequentiality we also have $|X| \leq \mathfrak{c}$. ◊

Given a space X, the cardinal $T(X)$ is the smallest κ such that whenever $\{F_\alpha : \alpha < \lambda\}$ is an increasing family of closed sets and $cf(\lambda) > \kappa$, then also $\bigcup\{F_\alpha : \alpha < \lambda\}$ is a closed set (see [10]).

Since the inequality $F(X) \leq L(X)T(X)$ holds for any space X, we get Arhangel'skiĭ Th. in [3] (Prop. 2 above) as an immediate consequence.

Notice that Th. 1 is actually more general than Props. 1 and 2 even in another respect, as there are sequential non-Lindelöf spaces having no uncountable free sequences (for instance, this is the case of every sequential space of countable spread). So, we may formulate a result a bit stronger than Prop. 1 and 2:

Corollary 1. *If X is a regular countably free (sequential) space having an uncountable cardinal $\lambda \leq \mathfrak{c}^+$ as a caliber, then $d(X) \leq \mathfrak{c}$ ($|X| \leq \mathfrak{c}$).*

2. From sequential to pseudoradial spaces

A topological space X is said to be pseudo radial provided that for any non closed set $A \subset X$ there exists a well-ordered net $\{x_\alpha : \alpha < \kappa\} \subseteq A$ which converges to a point belonging to $\overline{A} \setminus A$.

Clearly every sequential space is pseudo radial (sequentiality is just pseudoradiality restricted to countable well-ordered nets only).

A topological space X is said to be radial provided that for any set $A \subset X$ and any point $p \in \overline{A}$ there exists a well ordered net in A converging to p.

Finally, the radial character $\chi_R(X)$ of the pseudoradial space X is the smallest cardinal κ such that the definition of pseudoradiality works by taking well-ordered nets of size not exceeding κ.

All the basic and updated information on pseudoradial and related spaces are collected in [7]. A further recent result on the semiradiality of $\{0,1\}^{\omega_1}$ can be found in [5].

In this part of the paper we are primarely concerned with estimates on the cardinality of certain kinds of pseudoradial spaces. Known relevant facts in this direction are the following:

(i) ([2], Th. 11) If X is a compact radial space, then $|X| \leq 2^{c(X)}$.

(ii) ([1], Corollary 1) If X is a compact sequential space, then $|X| \leq$ $\leq 2^{c(X)}$.

The previous assertions cannot be extended (at least consistently) to the whole class of pseudoradial compact spaces (see the example described below). In particular, the failure of formula (ii) may occur [6] even for a R-monolithic space, a kind of pseudoradial space somehow very closed to be sequential.

Notice that, the estimates so far presented are formulated for compact spaces only. Taking this into account, the fact that, by Šapirovskiĭ's result (Prop. 1), Assertion (ii) can be somewhat extended to Lindelöf spaces appears quite remarkable.

Now, the natural question arises whether it is true that the cardinality of a Lindelöf pseudoradial space having $\aleph_1$ as a caliber does not exceed the continuum.

Again the answer is at least consistently in the negative, even for compact spaces. To see this, recall that it is consistent with ZFC to have $2^{\aleph_1} > 2^{\aleph_0} = \aleph_2 = \mathfrak{s}$ [8], Th. 5.4. Therefore, if in such model we consider the space $\{0,1\}^{\omega_1}$, then we get a compact pseudoradial space having $\aleph_1$ as a caliber whose cardinality is bigger than the continuum.

Taking into account Assertion (i), it seems much more reasonable the possibility to extend the previous proposition to the whole class of radial spaces. More precisely, we have the following:

Question 1. Let X be a Lindelöf radial space of caliber $\aleph_1$. Is it true that $|X| \leq 2^{\aleph_0}$?

A general version of Prop. 1 is:

Theorem 2. *If X is a pseudo radial Lindelöf regular space and λ is a caliber of X satisfying $\chi_R(X) < \lambda \leq \left(2^{\chi_R(X)}\right)^+$, then $|X| \leq 2^{\chi_R(X)}$.*

Proof. Let $\chi_R(X) = \kappa$. Since we have $F(X) \leq L(X)\chi_R(X)$, it follows in our case that $F(X) \leq \chi_R(X)$. By repeating the proof of Th. 1 with only minor modifications, we get $d(X) \leq 2^\kappa$. Since in any pseudoradial space X we always have $|X| \leq d(X)^{\chi_R(X)}$, the result clearly follows. $\Diamond$

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