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ZERO PRODUCT PRESERVERS AND ORTHOGONALITY PRESERVERS ON ALGEBRAS OF UNBOUNDED OPERATORS

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Abstract: Applying a result of abstract ring theory we get that bijective additive mappings on standard algebras of unbounded operators preserving zero products are multiples of ring isomorphisms.
The structure of additive bijective mappings on certain classes of standard algebras of unbounded operators preserving orthogonality in both directions is also investigated. The results are quite similar to those for algebras of bounded operators.

Introduction

Linear preserver problems concern the characterization of linear maps between algebras that roughly speaking preserve certain properties of some elements of the algebras. Such problems were studied in matrix theory during the last century starting with the paper of Frobenius [4].

In the last decades there can be observed a growing interest in similar questions on abstract algebras or rings and on operator algebras over infinite dimensional spaces. These investigations led in a natural

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way to the study of mappings which are not linear but are merely additive. One of the first paper in this spirit is the classical work of Kaplansky [6]. One of the striking features is the application of results of abstract ring theory to problems on algebras of bounded operators on Banach or Hilbert spaces.

It turned out that several of these results are valid with necessary modifications also for algebras of unbounded operators on Hilbert spaces (see for example [9–12]).

The present paper is exactly in this spirit. It deals with two special preserver problems: preservers of zero products and preservers of orthogonality.

We fix the necessary notions and notation. Let $\mathcal{A}, \mathcal{B}$ be rings. A mapping $\Phi : \mathcal{A} \to \mathcal{B}$ is said to preserve zero products if $XY = 0$ $(X, Y \in \mathcal{A})$ implies $\Phi(X)\Phi(Y) = 0$. Let $\mathcal{A}, \mathcal{B}$ be $*$-algebras. A mapping $\Phi : \mathcal{A} \to \mathcal{B}$ preserves orthogonality if $A^*B = AB^* = 0$ $(A, B \in \mathcal{A})$ implies $\Phi(A)^*\Phi(B) = \Phi(A)\Phi(B)^* = 0$. If Φ is bijective then Φ is said to preserve orthogonality in both directions if

$$A^*B = AB^* = 0 \quad \Longleftrightarrow \quad \Phi(A)^*\Phi(B) = \Phi(A)\Phi(B)^* = 0.$$

Further we need some notions on algebras of unbounded operators. A standard reference is [8].

Let $\mathcal{D}$ be a dense linear manifold in an infinite dimensional, separable Hilbert space $\mathcal{H}$ with scalar product $\langle\ ,\ \rangle$ (which is supposed to be conjugate linear in the first and linear in the second component). Separability is assumed only for simplicity. The set of linear operators $\mathcal{L}^+(\mathcal{D}) = \{A : A\mathcal{D} \subset \mathcal{D}, A^*\mathcal{D} \subset \mathcal{D}\}$ is a $*$-algebra with respect to the natural operations and the involution $A \to A^+ = A^*|\mathcal{D}$. The graph topology t on $\mathcal{D}$ induced by $\mathcal{L}^+(\mathcal{D})$ is generated by the directed family of seminorms $\phi \to ||\phi||_A = ||A\phi||$, $\forall A \in \mathcal{L}^+(\mathcal{D})$, $\phi \in \mathcal{D}$. $\mathcal{D}$ is called an (F)-domain, if $(\mathcal{D}, t)$ is an (F)-space.

Remark that in this case the graph topology t can be given by a system of seminorms $\{||\cdot||_n = ||A_n \cdot ||, n \in \mathbb{N}, A_n \in \mathcal{L}^+(\mathcal{D})\}$ with:

$$(1) \quad A_1 = I,\ A_n = A^+{}_n,\ ||A_n\phi|| \le ||A_{n+1}\phi|| \quad \text{for all } \phi \in \mathcal{D}, n \in \mathbb{N}.$$

A standard $(*)$-operator algebra (on $\mathcal{D}$) is a $(*)$-subalgebra $\mathcal{A}(\mathcal{D}) \subset \mathcal{L}^+$ $+(\mathcal{D})$ containing the ideal $\mathcal{F}(\mathcal{D})$ of all finite rank operators from $\mathcal{L}^+(\mathcal{D})$. Note that every rank-one operator $F \in \mathcal{F}(\mathcal{D})$ has the form $F = \psi \otimes \phi$, $\phi, \psi \in \mathcal{D}$, where $F(\chi) = \langle \phi, \chi \rangle \psi$.

Every standard operator algebra $\mathcal{A}(\mathcal{D})$ is prime. Remember that an algebra (or a ring) $\mathcal{A}$ is prime if $X\mathcal{A}Y = 0$ implies $X = 0$ or $Y = 0$.

The paper is organized as follows. In Sect. 2 we deal with additive bijective mappings between standard operator algebras preserving zero products. The corresponding result is a simple application of an abstract ring theoretical result [2]. It appears that such mappings are scalar multiples of ring isomorphisms.

Sect. 3 is devoted to additive bijective mappings preserving orthogonality. The structure of such mappings is clarified. The results are quite similar to those for algebras of bounded operators given in [5].

Mappings preserving zero products

For the extensive literature on this topic see for example [2, 3] and the references therein. In this section we prove the following theorem.

Theorem 2.1. *Let $\mathcal{D} \subset \mathcal{H}$ be an (F)-domain and let $\mathcal{A}, \mathcal{B}$ be standard operator algebras on $\mathcal{D}$. If $\Phi : \mathcal{A} \to \mathcal{B}$ is an additive, bijective mapping that preserves zero products, then there are a number $c \in \mathbb{C}$ and a bijective either linear or conjugate linear operator $T : \mathcal{D} \to \mathcal{D}$ such that*

$$\Phi(A) = cTAT^{-1} \qquad (A \in \mathcal{A}).$$

If T is linear, then $T \in \mathcal{L}^+(\mathcal{D})$.

The main part of the proof is contained in the following theorem from ring theory (Th. 1 in [3]).

Theorem A. *Let $\mathcal{A}$ and $\mathcal{B}$ be prime rings and $\Phi : \mathcal{A} \to \mathcal{B}$ a bijective additive mapping such that $\Phi(A)\Phi(B) = 0$ for all $A, B \in \mathcal{A}$ with $AB = 0$. Suppose that the maximal right quotient ring $Q(\mathcal{A})$ of $\mathcal{A}$ contains a nontrivial idempotent E such that $E\mathcal{A} \cup \mathcal{A}E \subset \mathcal{A}$.*

i) *If $1 \in \mathcal{A}$, then $\Phi(AB) = \lambda\Phi(A)\Phi(B)$ for all $A, B \in \mathcal{A}$ where $\lambda = \frac{1}{\Phi(1)} \in Z(\mathcal{B})$ the center of $\mathcal{B}$. In particular, if $\Phi(1) = 1$, the Φ is a ring isomorphism from $\mathcal{A}$ onto $\mathcal{B}$.*

ii) *If $\deg(\mathcal{B}) \geq 3$, then there exists $\lambda \in C(\mathcal{B})$, the extended centroid of $\mathcal{B}$, such that $\Phi(AB) = \lambda\Phi(A)\Phi(B)$ for all $A, B \in \mathcal{A}$.*

In Th. A the condition $\deg(\mathcal{B}) \geq 3$ means that $\mathcal{B}$ cannot be embedded in the ring of 2×2-matrices over a field. So, this conditions is fulfilled for every standard operator algebra on $\mathcal{D}$.

The definitions and basic properties of the maximal quotient ring and the extended centroid can be found in [1]. A prime ring $\mathcal{A}$ is called centrally closed if $C(\mathcal{A})$ is trivial. We need the following characterization of centrally closed prime algebras to prove that every standard operator algebra on $\mathcal{D}$ is centrally closed (thanks to M. Brešar for this information). Let $\mathcal{A}$ be a prime algebra over $\mathbb{C}$. Then $\mathcal{A}$ is centrally closed if and only if the following holds:

If $\mathcal{I} \subset \mathcal{A}$ is a nonzero ideal and if there is an additive mapping $F : \mathcal{I} \to \mathcal{A}$ such that

$$F(UX) = F(U)X \text{ and } F(XU) = XF(U) \tag{2}$$

for all $U \in \mathcal{I}, X \in \mathcal{A}$, then there is a $\lambda \in \mathbb{C}$ such that $F(U) = \lambda U$ for all $U \in \mathcal{I}$.

Lemma 2.2. *Every standard operator algebra on $\mathcal{D}$ is centrally closed.*

Proof. Let $\mathcal{I} \subset \mathcal{A}$ be an ideal and $F : \mathcal{I} \to \mathcal{A}$ an additive mapping such that (2) is satisfied. Note that $\mathcal{F}(\mathcal{D}) \subset \mathcal{I} \subset \mathcal{A}$. We describe the structure of the mapping F. This can be done quite similar to the proof of the structure of double centralizers given in [10], Prop. 3.4. We sketch the main steps. Let $U \in \mathcal{I}, \psi \otimes \phi = X \in \mathcal{A}$. Then (2) gives $F(U \cdot \psi \otimes \phi) = F(U)\psi \otimes \phi$. The mappings T, S defined by $T(U\psi) = = F(U)\psi$, $S(U^+)\phi = F(U)^+\phi$, $\phi, \psi \in D$ are correctly defined operators on $\mathcal{D}$. Moreover, (2) implies that $S = T^+$, i.e. $T \in \mathcal{L}^+(\mathcal{D})$ and consequently $F(U) = TU = UT$ for all $U \in \mathcal{I}$. So, T commutes with all operators from $\mathcal{I}$, in particular, with all operators from $\mathcal{F}(\mathcal{D})$. But this implies $T = \lambda I$ for some $\lambda \in \mathbb{C}$. $\Diamond$

Proof of Theorem 2.1. Apply Th. A, ii) to get $\Phi = \frac{1}{\lambda}\Psi$ with a ring isomorphism Ψ. The structure of ring isomorphisms between standard operator algebras on (F)-domains follows directly from Th. 3.1 in [10]. Namely, there exists a bijective either linear or conjugate linear $T : \mathcal{D} \to \mathcal{D}$ such that $\Psi(A) = TAT^{-1}$ $(A \in \mathcal{A})$. This concludes the proof. $\Diamond$

Mappings preserving orthogonality

In this section we describe the structure of orthogonality preserving mappings on several standard operator algebras. As a corollary we obtain an unbounded version of a result of L. Molnár [7].

Theorem 3.1. *Let $\mathcal{D} \subset \mathcal{H}$ be an (F)-domain and let $\mathcal{A} \subset \mathcal{L}^+(\mathcal{D})$ be one of the following standard operator algebras:*

a) $\mathcal{A} = \mathcal{F}(\mathcal{D})$,

b) $\mathcal{A}$ *is a unital standard* $*$*-operator algebra,*

c) $\mathcal{A} \neq \mathcal{F}(\mathcal{D})$, $\mathcal{A}$ *a* $*$*-ideal of* $\mathcal{L}^+(\mathcal{D})$.

Assume that $\Phi : \mathcal{A} \to \mathcal{A}$ *is an additive bijection preserving orthogonality in both directions. Then* Φ *has one of the following forms:*

i) *There exist a nonzero constant* c *and operators* $U, V : \mathcal{D} \to \mathcal{D}$, *both either unitary or antiunitary such that*

$$\Phi(T) = cUTV \qquad (T \in \mathcal{A}) \tag{3}$$

or

ii) *There exist a nonzero constant* c *and operators* $U, V : \mathcal{D} \to \mathcal{D}$, *both either unitary or antiunitary such that*

$$\Phi(T) = cUT^+V \qquad (T \in \mathcal{A}). \tag{4}$$

Proof. The main step consists in proving that Φ preserves rank-one operators in both directions. For this we treat cases a)–c) separately.

Case a). This can be done as in [5] using the polar decomposition of $A \in \mathcal{F}(\mathcal{D})$. Remark that in the unbounded case the situation is more complicated. For $\mathcal{A} \neq \mathcal{F}(\mathcal{D})$ one can not be sure that the polar decomposition can be performed within $\mathcal{A}$ (even not within $\mathcal{L}^+(\mathcal{D})$).

Case b). Let $F = \psi \otimes \phi$ be a rank-one operator and let $\mathcal{H}_0 := \mathrm{lin}\{\phi, \psi\}$. Now we define an operator $Q \in \mathcal{A}$, which has corank one as follows.

If $\dim \mathcal{H}_0 = 1$ put

$$Q\chi = \begin{cases} \chi & \text{for all } \chi \in \mathcal{H}_0^\perp \cap \mathcal{D} \\ 0 & \text{for all } \chi \in \mathcal{H}_0. \end{cases}$$

If $\dim \mathcal{H}_0 = 2$ put

$$Q\chi = \begin{cases} \chi & \text{for all } \chi \in \mathcal{H}_0^\perp \cap \mathcal{D} \\ \langle \phi_1, \chi \rangle \psi_1 & \text{for all } \chi \in \mathcal{H}_0, \end{cases}$$

where ϕ_1, ψ_1 are nonzero elements from $\mathcal{H}_0$ such that $\phi_1 \perp \phi, \psi_1 \perp \psi$. To get $Q \in \mathcal{A}$ it is used that $I \in \mathcal{A}$. Then $F^+Q = FQ^+ = 0$ and therefore $\Phi(F)^+\Phi(Q) = \Phi(F)\Phi(Q)^+ = 0$. That means $\mathrm{ran}\,\Phi(F) \perp \mathrm{ran}\,\Phi(Q)$ and $\mathrm{ran}\Phi(F)^+ \perp \mathrm{ran}\Phi(Q)^+$.

Suppose that $\Phi(F)$ has rank larger than one. Then also $\Phi(F)^+$ has rank larger than one. Choose nonzero $\rho_1, \rho_2 \in \mathrm{ran}\,\Phi(F), \rho_1 \perp \rho_2$ and nonzero $\chi_1, \chi_2 \in \mathrm{ran}\,\Phi(F)^+, \chi_1 \perp \chi_2$. Put $S_i := \rho_i \otimes \chi_i$. Then

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