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LEFT IDEALS IN 1-PRIMITIVE NEAR-RINGS

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Abstract: In this article we consider the structure of left ideals of zero symmetric and 1-primitive near-rings N. In particular, we show that each such left ideal (under suitable finiteness conditions) has a multiplicative right identity and we also describe the ideal structure of the left ideals when considered as sub-near-rings of N. In order to do this we also need some preliminary results on near-rings N which have strongly monogenic N-groups.

1. Introduction

Throughout this article we use right near-rings and a notation according to [3]. Let N be a near-ring and Γ be an N-group. Γ is said to be strongly monogenic if $N\Gamma \neq \{0\}$ and for all $\gamma \in \Gamma$ either $N\gamma = \Gamma$ or $N\gamma = \{0\}$. So Γ is the disjoint union of two non-empty sets $\theta_1 =: \{\gamma \in \Gamma \mid N\gamma = \Gamma\}$ and $\theta_0 =: \{\gamma \in \Gamma \mid N\gamma = \{0\}\}$. Strongly monogenic N-groups arise naturally within the class of 1-primitive near-rings N. A near-ring N is said to be 1-primitive if there exists a faithful and strongly monogenic N-group Γ which is simple, that means Γ has no non-trivial N-ideals. Strongly monogenic N-groups Γ which are simple are called N-groups of type 1. So, a 1-primitive near-ring N

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has a faithful N-group of type 1. Primitive near-rings play (similar to primitive rings) an important role in the general structure theory of near-rings. They are sort of the smallest building blocks near-rings are made of (see [3] for more on that subject). Also many interesting types of near-rings are primitive. Just to give some examples, let $(G,+)$ be a group and 0 its group zero. Let $(S,\circ)$ be a fixedpointfree automorphism group of G. Then $M_0(G) := \{f : G \longrightarrow G \mid f(0) = 0\}$ as well as $M_S(G) := \{f : G \longrightarrow G \mid f(0) = 0 \text{ and } \forall s \in S : f \circ s = s \circ f\}$ are zero symmetric and 1-primitive near-rings (see [3]) w.r.t. pointwise addition of functions and function composition. More general, any finite zero symmetric simple near-ring with identity is 1-primitive (see [3, Th. 4.47]).

The aim of this work is to shed some light on the structure of the left ideals of such near-rings. We show that each such left ideal has a multiplicative right identity and we describe their ideal structure when considered as sub-near-rings. Our discussion does not consider special cases of left ideals, for example minimal or maximal ones. Here it might very well be possible to still get more results but this does not lie within the scope of this article. (The reader interested in minimal and maximal left ideals of near-rings of the type $M_0(G)$ may consult [3].)

We first discuss near-rings which have strongly monogenic N-groups. The reason for doing so is that left ideals of zero symmetric 1-primitive near-rings are of that type, as we will see. After this discussion we restrict our considerations to the class of left ideals L of zero symmetric 1-primitive near-rings where L satisfies the descending chain condition on L-subgroups of L and we will prove our main results.

2. Strongly monogenic N-groups

A normal subgroup S of an N-group Γ is called an N-ideal if $\forall n \in N \,\forall s \in S \,\forall \gamma \in \Gamma : n(\gamma + s) - n\gamma \in S$. A reference for the following lemma can be found in [2, Lemma 2.1]. For the sake of completeness we include our own proof.

Lemma 2.1. *Let N be a zero symmetric near-ring with a strongly monogenic N-group Γ. Then there exists a greatest proper N-ideal in Γ.*

Proof. Let L be a proper N-ideal of Γ. Since N is zero symmetric, L is an N-subgroup of Γ. Hence $NL \subseteq L \neq \Gamma$. Consequently, $L \cap \theta_1 = \varnothing$

so any proper N-ideal L of Γ is contained in θ_0.

Note that the sum of N-ideals is again an N-ideal ([3, Cor. 2.3]). We now show that a finite sum of proper N-ideals is again a proper N-ideal. We use induction on the number n of N-ideals in the sum:

$n = 1$: This is clear, since we only consider proper N-ideals. So suppose each sum of $n-1$ proper N-ideals is again a proper N-ideal of Γ. We have to show that the sum of n proper N-ideals is a proper N-ideal. Let $\sum_{k=1}^{n} L_k$ be a sum of n proper N-ideals. Let $l_1+\ldots+l_n \in \sum_{k=1}^{n} L_k$. Then, for all $m \in N$, $m(l_1 + (l_2 + \ldots + l_n)) - ml_1 \in \sum_{k=2}^{n} L_k \subseteq \theta_0$. Since $ml_1 = 0$, $m(l_1 + l_2 + \ldots + l_n) \in \theta_0$, for all $m \in N$. Consequently, $(l_1 + \ldots + l_n) \in \theta_0$. Hence, $\sum_{k=1}^{n} L_k$ is a proper N-ideal.

Now let $\sum$ be the sum of all proper N-ideals of Γ. If $s \in \sum$, then s can be written as a finite sum of elements of some proper N-ideals. Therefore, $s \in \theta_0$ as we have seen. This finishes our proof. $\Diamond$

In the following the greatest proper N-ideal of a strongly monogenic N-group Γ will be denoted as $\triangle$. Note that $\Gamma/\triangle$ is again an N-group by defining $n(\gamma + \triangle) := n\gamma + \triangle$ for all $n \in N$ and $\gamma \in \Gamma$. If $\gamma \in \theta_1$, then $N(\gamma+\triangle) = \Gamma/\triangle$ and if $\gamma \in \theta_0$ then $N(\gamma + \triangle) = \{0 + \triangle\}$, so $\Gamma/\triangle$ is a strongly monogenic N-group. The fact that $\triangle$ is the greatest proper N-ideal of Γ implies that $\Gamma/\triangle$ is a simple N-group. As in [3], $\mathbf{J}_1(N)$ is the 1-radical of a near-ring N. Since there exists an N-group of type 1 we must have $\mathbf{J}_1(N) \neq N$. We just have established the following corollary:

Corollary 2.2. *Let Γ be a strongly monogenic N-group of a zero symmetric near-ring N. Let $\triangle$ be the greatest proper N-ideal of Γ. Then $\Gamma/\triangle$ is an N-group of type 1 and $\mathbf{J}_1(N) \neq N$.*

We now have:

Theorem 2.3. *Let N be a zero symmetric near-ring which has a faithful strongly monogenic N-group Γ. Then $N/\mathbf{J}_1(N)$ is a 1-primitive near-ring.*

Proof. Since $\Gamma/\triangle$ is an N-group of type 1, we have $\mathbf{J}_1(N) \subseteq (0 : \Gamma/\triangle)$. So, $\Gamma/\triangle$ is an $N/\mathbf{J}_1(N)$-group of type 1 in a natural way (see [3, Prop. 3.14]), by defining $(n + \mathbf{J}_1(N))(\gamma + \triangle) := n\gamma + \triangle$.

Let A be the annihilator of $\Gamma/\triangle$ in $N/\mathbf{J}_1(N)$. A is an ideal in $N/\mathbf{J}_1(N)$, so $A = I + \mathbf{J}_1(N)$ where $I \trianglelefteq N$ (see [3, Th. 1.30]). This means that $I\Gamma \subseteq \triangle \subseteq \theta_0$. Consequently, $I^2\Gamma = \{0\}$. Since Γ is faithful, $I^2 = \{0\}$ and therefore $I \subseteq \mathbf{J}_1(N)$ by [3, Th. 5.37 and Prop. 5.3]. This means that $\Gamma/\triangle$ is a faithful $N/\mathbf{J}_1(N)$-group of type 1. Hence, $N/\mathbf{J}_1(N)$ is a 1-primitive near-ring. $\Diamond$

Certainly well known and easy to establish is the following proposition:

Proposition 2.4. *Let $\{0\} \neq L$ be a left ideal of a zero symmetric near-ring N which is 1-primitive on an N-group Γ. Then Γ is a faithful and strongly monogenic L-group.*

Proof. Γ is a faithful L-group. If $\gamma \in \theta_0$, then clearly $L\gamma = \{0\}$. Let $\gamma \in \theta_1$. Then, $L\gamma$ is an N-ideal of Γ by [3, Prop. 3.4] and therefore $L\gamma = \{0\}$ or $L\gamma = \Gamma$. Since $(0 : \Gamma) = \{0\}$ and $L \neq \{0\}$, there must exist an element $\gamma \in \Gamma$ such that $L\gamma \neq \{0\}$, so $L\gamma = \Gamma$ and this shows that Γ is a strongly monogenic L-group. $\Diamond$

Th. 2.3 and Prop. 2.4 now prove the following lemma:

Lemma 2.5. *Let N be a zero symmetric and 1-primitive near-ring and let L be a non-zero left ideal of N. Then $L/\mathbf{J}_1(L)$ is a 1-primitive near-ring.*

As a by-product we get the following result on ideals in 1-primitive near-rings (see also [3, Rem. 4.50]):

Theorem 2.6. *Let I be an ideal of a 1-primitive near-ring N. Then I is also a 1-primitive near-ring.*

Proof. By [3, Th. 5.33] $\mathbf{J}_1(I) \subseteq \mathbf{J}_1(N) \cap I$. Since $\mathbf{J}_1(N) = \{0\}$, $\mathbf{J}_1(I) = \{0\}$. By Lemma 2.5, $I/\mathbf{J}_1(I) \cong I$ is a 1-primitive near-ring. $\Diamond$

We will end this section by proving two lemmas we will need as a tool in the next section but which are interesting in their own right.

Lemma 2.7. *Let N be a zero symmetric near-ring which has a faithful and strongly monogenic N-group Γ. Then $N\mathbf{J}_1(N) = \{0\}$ and $\mathbf{J}_1(N)$ is a proper ideal.*

Proof. Since we have that $\mathbf{J}_1(N) \subseteq (0 : \Gamma/\Delta)$, $\mathbf{J}_1(N)\Gamma \subseteq \Delta \subseteq$ $\subseteq \theta_0$. This shows that $N \neq \mathbf{J}_1(N)$ since Γ is strongly monogenic. By faithfulness of Γ we now also have that $N\mathbf{J}_1(N) = \{0\}$. $\Diamond$

Now we have to switch to near-rings with chain condition for the first time.

Lemma 2.8. *Let N be a zero symmetric near-ring with descending chain condition on N-subgroups of N. Suppose N has a faithful and strongly monogenic N-group Γ and suppose N has a multiplicative right identity. Then $NI = \{0\}$ for any proper ideal I of N and $\mathbf{J}_1(N)$ is the greatest proper ideal in N.*

Proof. Let $\mathbf{J}_1(N) := Q$. By Lemma 2.7 we have $NQ = \{0\}$. Furthermore, by Th. 2.3, N/Q is a 1-primitive near-ring. By [3, Th. 2.35] also N/Q has the descending chain condition on N/Q-subgroups of N/Q. So, by [3, Th. 4.46], N/Q is a simple near-ring. Consequently, Q is a maximal ideal. Let I be an ideal of N and suppose $I \not\subseteq Q$. By maximality of Q we have $Q + I = N$. Since I is an ideal, we have $\forall n \in N \forall i \in I \forall q \in Q : n(q+i) - nq = n(q+i) \in I$. Therefore, $N^2 \subseteq I$. By assumption N has a right identity, so $N \subseteq I$. Consequently, each proper ideal of N must be contained in Q. $\Diamond$

3. Left ideals

Before we can prove our main results on left ideals in 1-primitive near-rings we need another lemma which holds for arbitrary near-rings with a suitable chain condition.

Lemma 3.1. *Let N be a zero symmetric near-ring with descending chain condition on N-subgroups of N. If there exists an element $e \in N$ with $(0 : e) = \{0\}$, then N has a multiplicative right identity.*

Proof. Let $e \in N$ be such that $(0 : e) = \{0\}$. Then for each natural number k, $(0 : e^k) = \{0\}$. The descending chain condition on N-subgroups of N guarantees that the chain of N-subgroups $Ne \supseteq Ne^2 \supseteq$ $\supseteq Ne^3$.... terminates. So there is some natural number l such that $Ne^l = Ne^{l+1} = Ne(e^l)$. Consequently, for any $n \in N$ there exists $m \in N$ such that $ne^l = (me)e^l$. Since $(0 : e^l) = \{0\}$, we get $n = me$, so $N \subseteq Ne$. Clearly, $Ne \subseteq N$ and hence, $N = Ne$. It follows that there exists $i \in N$ such that $e = ie$. Therefore, for each $n \in N$ we have $(ni - n)e = n(ie) - ne = 0$. So $ni - n \in (0 : e) = \{0\}$. Consequently, $ni = n$ for each $n \in N$. So, i is a multiplicative right identity. $\Diamond$

It is well known (see [3, Th. 4.46]) that zero symmetric and 1-primitive near-rings N with descending chain condition on N-subgroups of N have a multiplicative right identity. The next theorem shows that this result can be extended to the left ideals of zero symmetric and 1-primitive near-rings N, as long as the left ideals satisfy a chain condition (N itself need not satisfy a chain condition).

Theorem 3.2. *Let $\{0\} \neq L$ be a left ideal of a zero symmetric and 1-primitive near-ring N. Suppose L has the descending chain condition on L-subgroups of L. Then L has a multiplicative right identity.*

Proof. By Lemma 2.5, $L/\mathbf{J}_1(L)$ is a 1-primitive near-ring and by [3, Th. 2.35] $L/\mathbf{J}_1(L)$ has the descending chain condition on $L/\mathbf{J}_1(L)$-subgroups of $L/\mathbf{J}_1(L)$. By [3, Th. 4.46] $L/\mathbf{J}_1(L)$ has a multiplicative right identity $e+\mathbf{J}_1(L)$ with $e \in L$. Hence, for all $l \in L$ there exists $l' \in$ $\in L$ such that, $le+\mathbf{J}_1(L) = l'+\mathbf{J}_1(L)$ and $l'-l \in \mathbf{J}_1(L)$. Consequently,

$le - l \in \mathbf{J}_1(L)$. Therefore, $(0 : e)_L := (0 : e) \cap L \subseteq \mathbf{J}_1(L)$. Note that $(0 : e)_L$ is a left ideal of N. Since $\mathbf{J}_1(L)^2 = \{0\}$ by Lemma 2.7, $(0 : e)_L$ is a nilpotent left ideal of N. By [3, Th. 5.37 and Prop. 5.3], $(0 : e)_L \subseteq \mathbf{J}_1(N)$. Because of 1-primitivity of N, $\mathbf{J}_1(N) = \{0\}$, so $(0 : e)_L = \{0\}$. By Lemma 3.1 there exists a multiplicative right identity in L. ◊

By taking $L = N$ in Th. 3.2 we get the result of [3, Th. 4.46] mentioned above. Note also that if given a finite zero symmetric near-ring N, then Th. 3.2 enables us to find all the left ideals of N amongst the sets of the form Ne, e an idempotent of the multiplicative semigroup of N. Maybe it could be an interesting task to characterize for which idempotents e the sets of the form Ne indeed are left ideals.

Now we can easily prove our main result on the ideal structure of left ideals in 1-primitive near-rings.

Theorem 3.3. *Let $\{0\} \neq L$ be a left ideal of a zero symmetric and 1-primitive near-ring. Suppose L has the descending chain condition on L-subgroups of L. Let I be any proper ideal of L. Then $LI = \{0\}$. Furthermore, there exists a greatest proper ideal Q in L, L/Q is a 1-primitive near-ring and $Q = \mathbf{J}_1(L)$.*

Proof. This result now follows immediately from Prop. 2.4, from Lemma 2.5, from Lemma 2.7, from Lemma 2.8 and from Th. 3.2. ◊

In particular, we have for finite near-rings:

Corollary 3.4. *Let N be a finite zero symmetric and 1-primitive near-ring. Let $\{0\} \neq L$ be a left ideal of N. Then, L has a multiplicative right identity and a greatest proper ideal Q. Furthermore, $LQ = \{0\}$.*

The following example shows that Q as in Cor. 3.4 need not be zero, so may be non-trivial.

Example 3.5. Let K be a finite field. Consider the (near)-ring R of all 3×3 matrices over K, so $R = \left\{ \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \mid a,b,c,d,e,f,g,h,i \in K \right\}$.

The ring R acts 2-primitively on the R-group $K_n = \left\{ \begin{pmatrix} j \\ k \\ l \end{pmatrix} \mid j,k,l \in K \right\}$.

It is easy to see that $L = \left\{ \begin{pmatrix} a & 0 & 0 \\ d & 0 & 0 \\ g & 0 & 0 \end{pmatrix} \mid a,d,g \in K \right\}$ is a left ideal of R. $Q = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ d & 0 & 0 \\ g & 0 & 0 \end{pmatrix} \mid d, g \in K \right\}$ is easily seen to be the greatest ideal of L and $LQ = \{0\}$. $I_1 = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ d & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid d \in K \right\}$ and $I_2 = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ g & 0 & 0 \end{pmatrix} \mid g \in K \right\}$ are the other ideals of L which both are contained in Q.

Note that the result of Cor. 3.4 applies, in particular, to near-rings of the form $M_0(G)$, $(G, +)$ a finite group. It is well known that any finite zero symmetric near-ring embeds into a finite near-ring of type $M_0(G)$. Cor. 3.4 however shows that only a small class of near-rings may be embeddable as left ideals into a finite near-ring of type $M_0(G)$. Similar questions have been considered in [1] where, for example, the author studies when a near-ring can be embedded as a left ideal into a near-ring with identity.

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