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ON ISOTROPIC CONGRUENCES OF LINES IN ELLIPTIC THREE-SPACE

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Abstract: In this paper isotropic congruences of lines in elliptic three-space $\mathbb{E}^3$ are parametrized explicitly. We additionally obtain a rational parametrization of the set of oriented lines in $\mathbb{E}^3$. Assuming that a congruence C of lines in $\mathbb{E}^3$ is generated by a mapping joining the left and right spherical kinematic image of C, we are able to assign an energy to congruences and define minimal congruences. It turns out that the isotropic congruences are minimal.

1. Introduction

Two-parameter families of lines are called congruences of lines. They have been studied since the second half of the nineteenth century, see [9, 12].

The geometry of elliptic three-space $\mathbb{E}^3$ and especially the theory of line congruences in $\mathbb{E}^3$ was studied after the method of moving frames, exterior differential calculus and the quaternionic representation of points in $\mathbb{E}^3$ have been developed, see [1, 6, 25].

Elliptic three-space $\mathbb{E}^3$ is a point model of the three-parametric group of rotations about a fixed point O in Euclidean three-space $\mathbb{R}^3$. This fact was used to study this group and elliptic three-space simultanously. Some theorems on kinematics in the Euclidean bundle have counterparts in elliptic geometry, see [1, 21, 25].

H. R. Müller [18, 19, 20, 21] and W. Blaschke [2, 3] gave the foundations for Ch. Lübbert's work [13, 14, 17, 16] on ruled surfaces in elliptic three-space $\mathbb{E}^3$.

The study of congruences of lines in elliptic three-space is motivated by the following facts: Congruences of lines in $\mathbb{E}^3$ admit a simple charactrization by means of the properties of a mapping that assigns the left to the right spherical kinematic image, as we will see in Sec. 3. Some of these constructions can also be done in Euclidean three-space $\mathbb{R}^3$. Furthermore, the constructions of isotropic congruences in $\mathbb{E}^3$ admit discretizations that can be used in numerical geometry in order to solve certain boundary value problems for line congruences, see [22].

We pay attention to rational congruences of lines since they can be rewritten as tensor product Bézier (or B-spline) volumes and thus they have a geometrically favorable representation. Isotropic congruences of lines in elliptic three-space and Euclidean three-space as well appear in the context of energy minimizing congruences, see [22] and Sec. 4 of the present paper.

This paper is organized in the following way: In Sec. 2 we give a brief introduction to quaternions, define the spherical kinematic mapping and show the Klein model of line space. Afterwards we show how to treat congruences of lines in $\mathbb{E}^3$. Sec. 3 is dedicated to the computation of a rational parametrization of the set of lines in elliptic space. For that end we use the preparations from Sec. 2. Sec. 4 deals with a possibility of assigning an energy to congruences of lines. Finally Sec. 5 gives some simple examples and Sec. 6 gives ideas for future research.

2. Line geometry in elliptic three-space $\mathbb{E}^3$

2.1. Quaternions. Let $\{1, i, j, k\}$ be a basis in $\mathbb{R}^4$. Any vector $x = (x_0, x_1, x_2, x_3) \in \mathbb{R}^4$ allows the unique representation $x = x_0 + ix_1 + jx_2 + kx_3$. Leaving the addition of vectors in $\mathbb{R}^4$ the usual one and defining a multiplication for vectors in $\mathbb{R}^4$ by defining it for the basis vectors as

$$(1) \quad i^2 = j^2 = k^2 = -1, \; ij = -ji = k, \; jk = -kj = i, \; ki = -ik = j,$$

$\mathbb{R}^4$ becomes the skew field $\mathbb{H}$ of quaternions. The element $x \in \mathbb{R}^4$ is called quaternion. The quaternion $\widetilde{x} := x_0 - ix_1 - jx_2 - kx_3$ is called the conjugate quaternion to x. The norm $N(x)$ of x reads

(2) $$N(x) := x\widetilde{x} = x_0^2 + x_1^2 + x_2^2 + x_3^2.$$

A quaternion x satisfying $N(x) = 1$ is called unit quaternion. The norm is multiplicative since $N(xy) = xy\widetilde{xy} = xy\widetilde{y}\widetilde{x} = xN(y)\widetilde{x} = N(x)N(y)$. If $N(x) \neq 0$, we can compute the inverse quaternion x^{-1} of x as $x^{-1} = \widetilde{x}/N(x)$.

The one-dimensional subspaces $x\mathbb{R}$ of $\mathbb{R}^4$ are the points X in projective space $\mathbb{P}^3$. Now we coordinatize the points of $\mathbb{P}^3$ by unit quaternions and endow $\mathbb{P}^3$ with the metric

(3) $$\cos d := |x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3| =: \langle X, Y\rangle_e,$$

where x and y are unit quaternions representing the points X and Y in $\mathbb{P}^3$. The thus obtained metric space is called three-dimensional elliptic space $\mathbb{E}^3$. The real value $0 \leq d \leq \pi/2$ is called elliptic distance of X and Y. Two points X and Y are said to be *orthogonal*, if $\langle X, Y\rangle_e = 0$. A standard model of elliptic three-space $\mathbb{E}^3$ is the Euclidean unit sphere $S^3 \in \mathbb{R}^4$ where antipodal points are identified.

The elliptic metric (3) also allows a quaternionic representation

(4) $$\langle X, Y\rangle_e = \frac{1}{2}(x\widetilde{y} + y\widetilde{x}) = \frac{1}{2}(\widetilde{x}y + \widetilde{y}x).$$

Elliptic space $\mathbb{E}^3$ is a Cayley–Klein space with absolute quadric

(5) $$\Omega:\ x_0^2 + x_1^2 + x_2^2 + x_3^2 = 0.$$

Points being conjugate with respect to Ω are said to be orthogonal in the sense of elliptic geometry and have elliptic distance $\pi/2$. The group of motions in elliptic three-space isom$\mathbb{E}^3$ is induced by the group of automorphic collineations of Ω, see [1, 8, 21].

2.2. The spherical kinematic mapping. An oriented straight line L in elliptic three-space $\mathbb{E}^3$ can be spanned by an ordered pair of orthogonal points, say X and Y. Here and in the following we distinguish between the oriented lines $X \vee Y$ and $Y \vee X$. The unit quaternions x and y representing orthogonal points X and Y help to parametrize the set of points incident with L as

(6) $$L(t) = x\cos t + y\sin t,$$

where $t \in [0, \pi)$. Since X and Y are orthogonal, we have $x\widetilde{y} = -y\widetilde{x}$ and $\widetilde{x}y = -\widetilde{y}x$, see (4). Now we define the left and right spherical kinematic image L^l and L^r of L by

$$L^l = \widetilde{x}y \text{ and } L^r = y\widetilde{x}. \tag{7}$$

It can easily be verified that L^l and L^r are contained in the three-dimensional subspace $\mathrm{im}\mathbb{H} := i\mathbb{R}+j\mathbb{R}+k\mathbb{R}$ of $\mathbb{R}^4$. Furthermore, they are unit vectors which is an immediate consequence of the multiplicativity of the norm (2) for quaternions. L^l and L^r do not depend on the choice of X and Y on L: Changing X to $X' = x\cos t - y\sin t$ and Y to $Y' = x\sin t + y\cos t$, we find that L^l and L^r remain unchanged. Reversing the orientation of $L = X \vee Y$ reverses the orientation of L^l and L^r.

Thus we have a mapping from the set $\vec{\mathcal{L}}$ of oriented lines in elliptic three-space to pairs of points in the Euclidean unit sphere S^2. This mapping is called spherical kinematic mapping and was first given by W. K. Clifford in 1873, see [6]. The spherical kinematic mapping from the set $\vec{\mathcal{L}} \to S^2 \times S^2$ is one-to-one and onto.

The group $\mathrm{isom}\mathbb{E}^3$ of elliptic motions consists of elements $x' = \widetilde{q}xp$, where p and q are unit quaternions. The image L' of an oriented line L under $\beta \in \mathrm{isom}\mathbb{E}^3$ has $L'^l = \widetilde{p}L^l p$ and $L'^r = \widetilde{q}L^r q$ for its left and right image, respectively. This shows that $\beta \in \mathrm{isom}\mathbb{E}^3$ induces Euclidean motions in the spherical kinematic images. Thus we can say: The group $\mathrm{isom}\mathbb{E}^3$ of motions in elliptic three-space $\mathbb{E}^3$ is the free product of SO_3 and SO_3.

2.3. Klein model of line space. In order to represent lines in projective or elliptic three-space one can use Plücker coordinates of lines. These coordinates are assigned to lines in the folllowing way: Let a line L be spanned by two points X and Y in $\mathbb{P}^3$. We collect their coordinate vectors in a 2×4-matrix

$$\begin{bmatrix} x_0 & x_1 & x_2 & x_3 \\ y_0 & y_1 & y_2 & y_3 \end{bmatrix} \tag{8}$$

and compute the determinantes $l_{i,j}$ of the 2×2-submatrices built from the i-th and j-th columns of (8). The vector

$$(x, y) \mapsto L = (L_1, L_2, L_3; L_4, L_5, L_6) := (l_{0,1}, l_{0,2}, l_{0,3}; l_{2,3}, l_{3,1}, l_{1,2})$$

comprises the Plücker coordinates of L. In order to simplify the computations, we define the vectors $l := (L_1, L_2, L_3)$ and $\bar{l} := (L_4, L_5, L_6)$.

Obviously, the Plücker coordinates are independent on the choice of X and Y on L. Since the coordinates of X and Y are homogeneous, the Plücker coordinates of L are homogeneous too. The Plücker coordinates of L can be interpreted as coordinates of points in projective five-space $\mathbb{P}^5$.

The mapping $\gamma : \mathcal{L} \to P^5$ is called Klein mapping and maps lines L in projective three-space to points in projective five-space $\mathbb{P}^5$. The coordinates of L satisfy the relation

$$M_2^4 : \langle l, \bar{l} \rangle = L_1L_4 + L_2L_5 + L_3L_6 = 0, \quad (9)$$

where $\langle \cdot, \cdot \rangle$ denotes the canonical scalar product in Euclidean three-space $\mathbb{R}^3$. Equation (9) is the equation of a regular quadric of dimension four and index two and is called Klein quadric or Plücker quadric. It is a point model for the set $\mathcal{L}$ of lines in $\mathbb{P}^3$.

Since γ is linear in both arguments x and y, pencils of lines in $\mathbb{P}^3$ correspond to lines in M_2^4 and bundles and fields of lines correspond to the planes in M_2^4. Thus M_2^4 carries two three-parameter families of planes, where planes of the same kind share a point and planes of different kind are either skew or share a line.

Now we assume that $\mathbb{P}^3$ is endowed with the metric (3) and the points X and Y spanning L are orthogonal. With (7) we can compute the left and right spherical kinematic image L^l and L^r of L and find

$$L^l = l - \bar{l} \text{ and } L^r = l + \bar{l}. \quad (10)$$

The spherical kinematic mapping is linear in the normalized Plücker coordinates of a line, it is called a linear line mapping, see [4, 10, 15, 23, 24].

Applying the polarity with regard to Ω to L the left vector L^l reverses its orientation while the right keeps its orientation. This is easily sean since $(l, \bar{l})$ change ist roles while changing from line coordinates to dual line coordinates, i.e. axis coordinates, see [24, 26]. Consequently we have for the reciprocal polar $L^\star$ of L: $L^{\star l} = \bar{l} - l = -L^l$ and $L^{\star r} = \bar{l} + l = L^r$.

Since L^l and L^r are unit vectors contained in im$\mathbb{H}$, we additionally have $\langle L^l, L^l \rangle_e = \langle l, l \rangle - 2\langle l, \bar{l} \rangle + \langle \bar{l}, \bar{l} \rangle = 1$. Comparing with (9), we observe that

$$\langle l, l \rangle + \langle \bar{l}, \bar{l} \rangle = 1, \quad (11)$$

is valid for the left and right image of a line L in elliptic three-space. Eqs. (9) and (11) are the necessary and sufficient constraints to a vector in $\mathbb{R}^6$ to be the coordinate vector of a line in elliptic three-space $\mathbb{E}^3$.

A point model of the manifold of lines in $\mathbb{E}^3$ thus is the intersection of two quadrics in $\mathbb{R}^6$. This algebraic manifold of degree four admits a rational parametrization as we will see later.

2.4. Congruences of lines in $\mathbb{E}^3$. A congruence C of lines in elliptic space $\mathbb{E}^3$ is a two-parameter manifold of lines. We assume that x and y are unit quaternions representing orthogonal points. With (6) we can parametrize a congruence C of lines, if we assume $x = x(u^1, u^2)$ and $y = y(u^1, u^2)$ depend on two parameters u^1 and u^2, varying in some domain $D \subset \mathbb{R}^2$. The points on lines L of C now read

$$L(u^1, u^2; t) = x(u^1, u^2)\cos t + y(u^1, u^2)\sin t, \tag{12}$$

where $t \in [0, \pi)$. In the following we assume that the involved functions are smooth.

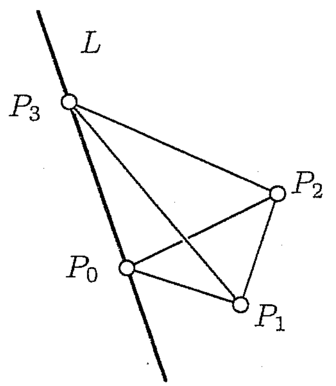

Fig. 1. Polar simplex attached to a line $L \in C$.

In order to discuss line congruences in $\mathbb{E}^3$ we attach a simplex Σ to each line L of C. The vertices P_i, $i \in \{0, 1, 2, 3,\}$ of Σ are pairwise orthogonal and we let P_0 and P_3 be the orthogonal points spanning L, see Fig. 1.

The edges $E_i := P_0 \vee P_i$ of Σ are polar with regard to Ω from (5) to the edges $E_i^\star := P_j \vee P_k$ with cyclic ordering of $i, j, k \in \{1, 2, 3\}$. Since Σ is a polar simplex of Ω from (5), we have

$$\langle P_i, P_j \rangle_e = \delta_{ij}. \tag{13}$$

Now we try to express the differentials of P_i as linear combinations of P_i. We assume

$$dP_i = \omega_i^s P_s \tag{14}$$

differentiate (13) and find that (14) is skew-symmetric: $\omega_i^j = -\omega_j^i$. The differential forms ω_i^j can be found as $\omega_i^j = \langle dP_i, P_j \rangle_e$. Exteriorly differentiating equation (14) together with (13) leads to the conditions of integrability

$$d\omega_i^j + \omega_i^s \wedge \omega_s^j = 0, \tag{15}$$

with $i, j \in \{0, 1, 2, 3\}$.

We use (7) in order to compute the left and right images of the edges E_i. The left and right spherical kinematic images of E_i build an orthonormal frame. This is easily shown, if we compute

$$(16)\qquad \langle E_i^l, E_j^l\rangle_e = \frac{1}{2}(\widetilde{p}_0 p_i \widetilde{\widetilde{p}_0 p_j} + \widetilde{p}_0 p_j \widetilde{\widetilde{p}_0 p_i}) = \widetilde{p}_0\langle P_i, P_j\rangle_e p_0 = \delta_{ij},$$

where p_k are the unit quaternions representing the points P_k. The same is true for E_i^r.

The differentials of E_i^l and E_i^r can be expressed as linear combinations of E_i^l and E_i^r, respectively and we have

$$(17)\qquad dE_i^l = \rho_i^k E_k^l \text{ and } dE_i^r = \sigma_i^k E_k^r.$$

The ρ_i^j and σ_i^j can be computed as $\rho_i^j = \langle E_j^l, dE_i^l\rangle_e$ and $\sigma_i^j = \langle E_j^r, dE_i^r\rangle_e$. And obviously, the systems of linear differential equations (17) are skew-symmetric. Exterior differentiation leads to the integrability conditions

$$(18)\qquad d\rho_i^j + \rho_k^j \wedge \rho_i^k = 0 \text{ and } d\sigma_i^j + \rho_k^j \wedge \sigma_i^k = 0,$$

with $i, j, k \in \{1,2,3\}$. The differential forms ω_i^j, ρ_i^j and σ_i^j are not independent. Besides of the integrability conditions (15) and (18) their are additional relations to be satisfied. These relations show the equivalence of (14) and (17).

In order to find these additional relations, we compute for example

$$(19)\qquad dE_1^l = d(\widetilde{p}_0 p_1) = \omega_0^2 \widetilde{p}_2 p_1 + \omega_0^3 \widetilde{p}_3 p_1 + \omega_1^2 \widetilde{p}_0 p_2 + \omega_1^3 \widetilde{p}_0 p_3 = \rho_1^2 E_2^l + \rho_1^3 E_3^l,$$

where we used the definition (2) of the norm and the fact that $\omega_0^1 = -\omega_1^0$. The middle term of (19) can now be simplified by inserting $E_i^l = \widetilde{p}_0 p_i$, $i \in \{1,2\}$. Recalling $E_i^\star = P_j \vee P_k$ and thus having $E_i^{\star l} = \widetilde{p}_k p_j$ and $E_i^{\star r} = p_j \widetilde{p}_k$, with cyclic ordering of i, j and k, equation (19) can finally be written as

$$(20)\qquad -\omega_0^3 E_3^l - \omega_0^3 E_2^l + \omega_1^2 E_2^l + \omega_1^3 E_3^l = \rho_1^2 E_2^l + \rho_1^3 E_3^l.$$

In the latter equation we used that changing E_i to $E_i^\star$ results in $E_i^{\star l} = -E_i^l$ and the right image vectors remain unchanged. Performing the above calculations for all combinations of indices and analogously for the right vectors E_i^r, we finally obtain

$$(21)\qquad \begin{aligned} \rho_2^1 &= \omega_2^1 + \omega_0^3, & \sigma_2^1 &= \omega_2^1 - \omega_0^3, \\ \rho_1^3 &= \omega_1^3 + \omega_0^2, & \sigma_1^3 &= \omega_1^3 - \omega_0^2, \\ \rho_3^2 &= \omega_3^2 + \omega_0^1, & \sigma_3^2 &= \omega_3^2 - \omega_0^1. \end{aligned}$$

Now we are going to characterize congruences C of lines in $\mathbb{E}^3$ by characterizing the correspondence between the left and right image of

C, respectively. For that it turned out to be useful to study the spherical image (in the sense of elementary differential geometry of surfaces) of the spherical kinematic image of a line L, see [1, 21].

We pay our attention to congruences of lines where the left and right spherical kinematic image is joined by a mapping $\lambda' = (\lambda'^1, \lambda'^2) : : L^l \to L^r$. For the following investigations we define:

Definition 2.1. Let $C: D \subset \mathbb{R}^2 \to \mathcal{L}$ be a congruence of lines in elliptic three-space $\mathbb{E}^3$ and $C^l: D \subset \mathbb{R}^2 \to S^2$ and $C^r: D \subset \mathbb{R}^2 \to S^2$ be its left and right image, respectively. Further, assume that there exists a mapping $\lambda': S^2 \to S^2$ with $C^r = \lambda' \circ C^l$. Then, we say C is *generated* by λ'.

We assume that the spherical kinematic mapping provides an orientation preserving conformal mapping λ' joining the left and right spherical kinematic image of a congruence C of lines. Taking the third equation of (17)

$$dE_3^l = \rho_3^1 E_1^l + \rho_3^2 E_2^l \text{ and } dE_3^r = \sigma_3^1 E_1^r + \sigma_3^2 E_2^r, \tag{22}$$

we can build the complex differential forms

$$\rho := \rho_3^2 + i\rho_3^1 \text{ and } \sigma := \sigma_3^2 + i\sigma_3^1, \tag{23}$$

where i is the imaginary unit with $i^2 = -1$. The mapping $E_3^l \to E_3^r$ is conformal, if the differential forms (23) are linearly dependent. The differential forms (23) are $\mathbb{R}$-linear dependent if and only if

$$\rho \wedge \sigma = \rho_3^2 \wedge \sigma_3^2 - \rho_3^1 \wedge \sigma_3^1 + i(\rho_2^1 \wedge \sigma_3^2 + \rho_3^2 \wedge \sigma_3^1) = 0, \tag{24}$$

which is identically equal to zero, if the real and imaginary part are equal to zero. This characterizes isotropic congruences in elliptic space $\mathbb{E}^3$, see [21]. Thus we define:

Definition 2.2. A congruence C of lines in elliptic space is called *isotropic*, if the differential form $\rho \wedge \sigma$ from (24) vanishes identically for all $u = (u^1, u^2) \in D \subset \mathbb{R}^2$.

The linear dependence of ρ and σ in case of a conformal mapping $\lambda: E_3^l \to E_3^r$ results in the existence of a complex number $f := f^1 + + if^2 \neq 0$ with $\rho := f\sigma$. In [21] it is shown that the distribution parameter of ruled surfaces in an isotropic congruence of lines C is independent on the direction in the congruence C at any line L.

The striction points of the ruled surfaces contained in an isotropic congruence C in elliptic space $\mathbb{E}^3$ are independent on the direction in

C. The two striction points of any ruled surface through a fixed line L in C coincide with the two central points of L, see [21].

At last we remark that an orientation reversing conformal mapping $\lambda: E_3^l \to E_3^r$ generates a congruence C that is obtained by applying the polarity with regard to Ω to a certain elliptic isotropic congruence C of lines. It is known that there exists a counterpart in Euclidean space $\mathbb{R}^3$, see [11].

3. Parametrization of isotropic congruences

In this section we attack the main problem of computing a parametrization for congruences of lines be means of mappings that assign the left image of the congruence lines to right image. For the convenience of the reader we briefly summarize our concept: The mapping $C: D \to \mathcal{L}$ defines the congruence of lines. Applying ^{l} and ^{r} to C according to (10) we obtain the left and right image C^r and C^l, respectively. These are actually domains in the Euclidean unit sphere S^2. The stereographic projection $\sigma: \ S^2 \setminus S \to \mathbb{R}^2$ results in two domains in $\mathbb{R}^2 \cong \mathbb{C}$. Now the generation of C can easily be described by a mapping $\lambda: \ \sigma(C^l) \to$ $\to \sigma(C^r)$. Table 1 shows the action of involved mappings.

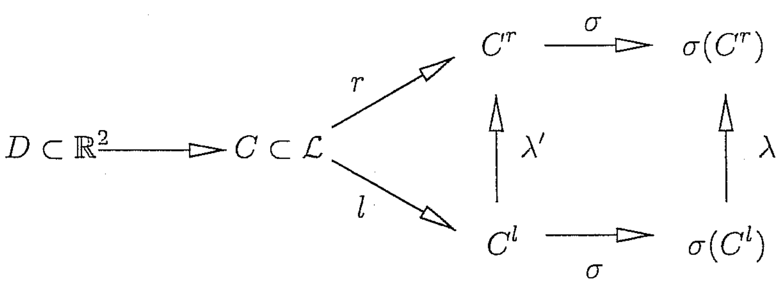

Table 1. Action of mappings C, ^{r}, ^{l}, σ, λ, λ'.

In order to find a parametrization of a congruence of lines generated by a mapping λ' joining the left and right spherical kinematic image, we apply the stereographic projection $\sigma: S^2 \setminus S \to \mathbb{C}$ and $\lambda: \ D \subset \mathbb{C} \to \mathbb{C}$ is the mapping joining $\sigma \circ C^l$ and $\sigma \circ C^r$. We let $S = (0, 0, -1)$ be the center of σ. We let $u := u_1 + iu_2$ be coordinates in $\sigma \circ C^l$. Note that u^i are not the ones mentinoed in (12). If $\sigma \circ C^r$ is the λ-image of $\sigma \circ C^l$, then $\lambda := \lambda^1 + i\lambda^2$ are coordinates in $\sigma \circ C^r$. On the other hand, with (10) we have

(25) $L^l = (L_1 - L_4, L_2 - L_5, L_3 - L_6),\ L^r = (L_1 + L_4, L_2 + L_5, L_3 + L_6)$

for the left and right image of the congruence lines. Applying the stereographic projection $\sigma: S^2 \setminus P \to \mathbb{C}$ to L^l and L^r, the left and right image of C is parametrized by

$$\begin{aligned} \sigma(L^l) &= \frac{L_1 - L_4}{1 + L_3 - L_6} + i\frac{L_2 - L_5}{1 + L_3 - L_6} = u^1 + iu^2 =: u, \\ \sigma(L^r) &= \frac{L_1 + L_4}{1 + L_3 + L_6} + i\frac{L_2 + L_5}{1 + L_3 + L_6} = \lambda^1 + i\lambda^2 =: \lambda. \end{aligned} \tag{26}$$

Eqs. (26) are linear in L_1, L_2, L_4 and L_5, if we separate real and imaginary part. The coordinates L_1, L_2, L_4 and L_5 can thus be expressed in terms of u and λ as

$$\begin{aligned} L_1 &= \frac{u^1}{2}(1 + L_3 - L_6) + \frac{\lambda^1}{2}(1 + L_3 + L_6), \\ L_2 &= \frac{u^2}{2}(1 + L_3 - L_6) + \frac{\lambda^2}{2}(1 + L_3 + L_6), \\ L_4 &= \frac{\lambda^1}{2}(1 + L_3 + L_6) - \frac{u^1}{2}(1 + L_3 - L_6), \\ L_5 &= \frac{\lambda^2}{2}(1 + L_3 + L_6) - \frac{u^2}{2}(1 + L_3 - L_6). \end{aligned} \tag{27}$$

To eliminate L_3 and L_6 from (27) we use the normalization (11) and the Plücker condition (9). First of which together with (27) reads

$$\lambda\overline{\lambda}(1 + L_3 + L_6)^2 + u\overline{u}(1 + L_3 - L_6)^2 + 2(L_3^2 + L_6^2 - 1) = 0. \tag{28}$$

Inserting (27) into the Plücker condition (9) we get

$$\lambda\overline{\lambda}(1 + L_3 + L_6)^2 - u\overline{u}(1 + L_3 - L_6)^2 + 4L_3L_6 = 0. \tag{29}$$

Eqs. (28) and (29) are linear in the squared norms of u and λ. These squared norms are

$$\lambda\overline{\lambda} = \frac{1 - L_3 - L_6}{1 + L_3 + L_6} \text{ and } u\overline{u} = \frac{1 - L_3 + L_6}{1 + L_3 - L_6}.$$

By multiplying both of these equations with the denominator of their respective right hand sides, we obtain a system of two linear equations in the two unknowns L_3 and L_6. The solutions of this system are

$$L_3 = \frac{1 - u\overline{u}\lambda\overline{\lambda}}{(1+\lambda\overline{\lambda})(1+u\overline{u})}, \quad L_6 = \frac{u\overline{u} - \lambda\overline{\lambda}}{(1+\lambda\overline{\lambda})(1+u\overline{u})}. \tag{30}$$

Finally we have a parametrization of a congruence C of lines with help of a function $\lambda: D \subset \mathbb{C} \to \mathbb{C}$ such that the left image is mapped to the right. We only have to collect the L_i in a vector and arrive at

$$L(u^1, u^2) = \begin{bmatrix} \dfrac{u^1}{1+u\overline{u}} + \dfrac{\lambda^1}{1+\lambda\overline{\lambda}} \\ \dfrac{u^2}{1+u\overline{u}} + \dfrac{\lambda^2}{1+\lambda\overline{\lambda}} \\ \dfrac{1 - u\overline{u}\lambda\overline{\lambda}}{(1+u\overline{u})(1+\lambda\overline{\lambda})} \\ \dfrac{\lambda^1}{1+\lambda\overline{\lambda}} - \dfrac{u^1}{1+u\overline{u}} \\ \dfrac{\lambda^2}{1+\lambda\overline{\lambda}} - \dfrac{u^2}{1+u\overline{u}} \\ \dfrac{u\overline{u} - \lambda\overline{\lambda}}{(1+u\overline{u})(1+\lambda\overline{\lambda})} \end{bmatrix}. \tag{31}$$

Thus we can state the following theorem:

Theorem 3.1. *Congruences C of lines in elliptic space $\mathbb{E}^3$ which are generated by a mapping $\lambda: \mathbb{C} \to \mathbb{C}$ can be parametrized by* (31).

The above computations used the normalization condition (11) and Plücker condition (9). Viewing the real-valued functions $\lambda^i: D \subset \mathbb{R}^2 \to \mathbb{R}$ ($i \in \{1,2\}$) as two independent real parameters, we can say:

Theorem 3.2. *The four-dimensional manifold of lines in elliptic three-space can be rationally parametrized by* (31).

Remark. It is not surprising that the set $\mathcal{L}$ of lines in projective three-space $\mathbb{P}^3$ admits a rational parametrization since the Klein quadric is a point model of it and quadrics always admit rational parametrizations. The set $\vec{\mathcal{L}}$ of oriented lines in elliptic three-space is described by normalized Plücker coordinates. At this point we leave the projective space $\mathbb{P}^5$ and enter the vector space $\mathbb{R}^6$. The Plücker coordinates of lines are normalized according to (11) and have to satisfy (9) and thus $\vec{\mathcal{L}}$ is the set of points lying in the intersection of two quadrics in $\mathbb{R}^6$. Such intersections are at most of algebraic degree four and do not always admit rational parametrizations.

Remark. The set of oriented lines in Euclidean three-space $\mathbb{R}^3$ does also admit a rational parametrization. It is easily found, if we assume that $l = (L_1, L_2, L_3)$ is parametrized rationally. With help of the stereographic projection from $\mathbb{R}^2 \to S^2 \setminus S$, we obtain $l = 1/U(2u^1, 2u^2, (1 - -(u^1)^2 - (u^2)^2))$, with $U = 1 + (u^1)^2 + (u^2)^2$ and $S = (0, 0, -1)$. Inserting this into the Plücker condition (9), we can express for example L_6 by means of u^1, u^2, while leaving $L_4 = u^3$ and $L_5 = u^4$ as remaining two paramters. Thus, we obtain a parametrization of $\vec{\mathcal{L}}$ in $\mathbb{R}^3$ as

$$l = \left(\frac{2u^1}{1 + (u^1)^2 + (u^2)^2}, \frac{2u^2}{1 + (u^1)^2 + (u^2)^2}, \frac{1 - (u^1)^2 - (u^2)^2}{1 + (u^1)^2 + (u^2)^2} \right)^T,$$

$$\bar{l} = \left(u^3, u^4, -\frac{2(u^1u^3 + u^2u^4)}{1 - (u^1)^2 - (u^2)^2} \right)^T,$$

where the Euclidean constraints $\langle l, l \rangle = 1$ and $\langle l, \bar{l} \rangle = 0$ are satisfied.

Furthermore, we can say that the special choice of the generating function λ results in special congruences which can now easily be studied.

Assuming that λ^i depend on two real parameters u^1 and u^2, respectively and assuming further S^2 is parametrized by isothermal coordinates, then conformal mappings $\lambda' : S^2 \to S^2$ can be characterized by the differential equations

$$\lambda'^1_{,1} = \lambda'^2_{,2} \;\&\; \lambda'^2_{,1} = -\lambda'^1_{,2} \tag{32}$$

These equations are known as the Cauchy–Riemann equations for holomorphic functions $\lambda := \lambda^1 + i\lambda^2 : D \subset \mathbb{C} \to \mathbb{C}$. Further it is known that (anti-)conformal mappings $S^2 \to S^2$ are induced by the (anti-) conformal mappings $D \subset \mathbb{C} \to \mathbb{C}$. Thus it makes no difference whether a congruence C of oriented lines is said to be generated by a mapping $D \subset \mathbb{C} \to \mathbb{C}$ or a mapping $S^2 \to S^2$.

We are able to state and prove the following result:

Theorem 3.3. *The congruences of lines in elliptic three-space $\mathbb{E}^3$ generated by holomorphic functions $\lambda : D \subset \mathbb{C} \to \mathbb{C}$ are isotropic congruences.*

Proof. We have to show that the differential form (24) vanishes identically. For that end we compare the spherical kinematic images of a congruence C parametrized by (31). With (10) we find

(33) $$L^l = \frac{1}{U}\begin{bmatrix} 2u^1 \\ 2u^2 \\ 1-(u^1)^2-(u^2)^2 \end{bmatrix} \text{ and } L^r = \frac{1}{\Lambda}\begin{bmatrix} 2\lambda^1 \\ 2\lambda^2 \\ 1-(\lambda^1)^2-(\lambda^2)^2 \end{bmatrix},$$

where $U := 1+(u^1)^2+(u^2)^2$ and $\Lambda := 1+(\lambda^1)^2+(\lambda^2)^2$. Comparing (7) and (10) we find that $L^l = E_3^l$ and $L^r = E_3^r$. Differentiating E_3^l and E_3^r from (33) we get

(34) $$dL^l = L^l_{,1}du^1 + L^l_{,2}du^2 \text{ and } dL^r = L^r_{;1}d\lambda^1 + L^r_{;2}d\lambda^2,$$

where ${}_{,i}$ indicates partial differentiation with respect to u^i and ${}_{;i}$ indicates partial differentiation with respect to λ^i.

Furthermore, we have $\lambda^1 = \lambda_1(u^1,u^2)$ and $\lambda^2 = \lambda_2(u^1,u^2)$, since we assume C is generated by $\lambda = \lambda^1 + i\lambda^2$. Consequently, we have

(35) $$d\lambda^1 = \lambda^1_{,1}du^1 + \lambda^1_{,2}du^2 \text{ and } d\lambda^2 = \lambda^2_{,1}du^1 + \lambda^2_{,2}du^2.$$

With the definition of ρ^i_j and σ^i_j from (17) and with

(36) $$\langle dL^l, dL^l\rangle = \Phi\delta_{ij}du^i du^j \text{ and } \langle dL^r, dL^r\rangle = \Psi\delta_{ij}d\lambda^i d\lambda^j,$$

where $\Phi :\ D \to \mathbb{R}$, $\Phi = 4U^{-2}$ and $\Psi :\ D \to \mathbb{R}$, $\Psi = 4\Lambda^{-2}$ are smooth real-valued functions, we have

(37) $$\rho^2_3 = -\sqrt{\Phi}du^1,\ \rho^1_3 = \sqrt{\Phi}du^2 \text{ and } \sigma^2_3 = -\sqrt{\Psi}d\lambda^1,\ \sigma^1_3 = \sqrt{\Psi}d\lambda^2.$$

Repeating the computations done in (24) and using (35) we find

(38) $$\rho\wedge\sigma = \sqrt{\Phi\Psi}(\lambda^2_{,1} + \lambda^1_{,2} + i(\lambda^1_{,1} - \lambda^2_{,2}))du^1\wedge du^2.$$

The differential form (38) is identically zero, if λ satisfies the Cauchy–Riemann equations (32) for holomorphic functions $\lambda : D \subset \mathbb{C} \to \mathbb{C}$. ◊

4. Harmonically generated congruences of lines

In Sec. 3 it turned out that a holomorphic function $\lambda :\ D \subset \mathbb{C} \to$ $\to \mathbb{C}$ generates isotropic congruences of lines. The generating function λ is a harmonic one. Its real and imaginary part satisfy the Laplace equation

(39) $$\Delta\lambda^i = \lambda^i_{,11} + \lambda^i_{,22},$$

with $i \in \{1,2\}$. This is also true for anti-holomorphic functions λ.

Now we consider $\mathbb{R}^2$ as a Riemannian manifold with the constant metric $g = \operatorname{diag}(1,1)$. Then, λ can be considered as a mapping from one Riemannian manifold into an other one: $\lambda : (\mathbb{R}^2, g) \to (\mathbb{R}^2, g)$. We can assign an energy density $e(\lambda)$ to λ. Following [7], we have

$$e(\lambda) = \frac{1}{2}((\lambda^1_{,1})^2 + (\lambda^1_{,2})^2 + (\lambda^2_{,2})^2 + (\lambda^2_{,2})^2). \tag{40}$$

The energy $E(\lambda)$ of λ then is the integral

$$E(\lambda) = \int_D e(\lambda) \star 1, \tag{41}$$

with $\star 1$ being the Riemannian volume element of $\mathbb{R}^2$, see also [7].

Eqs. (39) are the Euler–Lagrange equation for the variational problem $E(\lambda) \to \min$ and thus the solutions of (39) make $E(\lambda)$ from (41) stationary.

So we can say that the energy of the holomorphic (and anti-holomorphic) functions λ is minimal (or at least stationary) in the class of mappings $\mathbb{C} \to \mathbb{C}$. This fact could be used to define minimal congruences in elliptic three-space $\mathbb{E}^3$.

Definition 4.1. A congruence C of oriented lines in elliptic three-space $\mathbb{E}^3$ defined over a domain $D \subset \mathbb{C}$ is called *minimal congruence*, if the generating function $\lambda : D \to \mathbb{C}$ is harmonic.

In order to clarify in which sense these minimal congruences are minimal, we define:

Definition 4.2. Assume C is a congruence of oriented lines in $\mathbb{E}^3$ generated by a function $\lambda : D \subset \mathbb{C} \to \mathbb{C}$. Then, the *energy density* $e(C)$ of C is the energy density $e(\lambda)$ of λ and its *energy* $E(C)$ is the energy $E(\lambda)$ of λ.

The congruences of lines parametrized by (31) are minimal in the sense of the above definition, if they are generated by harmonic functions $\lambda : D \to \mathbb{C}$: The stereographic projection $\sigma : S^2 \setminus S \to \mathbb{C}$ is conformal and harmonicity preserving. It therefore maps the harmonically joined domains $\sigma \circ C^l$ and $\sigma \circ C^r$ to harmonically joined domains in the unit sphere. The harmonic correspondence between the left and right spherical kinematic image is not effected. So we can state:

Theorem 4.1. *The congruences of lines in elliptic three-space* $\mathbb{E}^3$ *parametrized by* (31) *are minimal congruences in the sense of* Def. 4.1, *if the generating functions are harmonic (conformal or anti-conformal) functions* $\mathbb{C} \to \mathbb{C}$.

Comparing Th. 3.3 and Def. 4.1, we have:

Theorem 4.2. *The minimal congruences in elliptic three-space* $\mathbb{E}^3$ *are isotropic, if they are generated by a holomorphic* λ.

Remark. The minimal congruences are rational congruences of lines in $\mathbb{E}^3$, if they are generated by rational functions λ.

As outlined in Subsec. 2.2 the group isom$\mathbb{E}^3$ of motions in elliptic three-space induces Euclidean motions in the spherical kinematic image. Any $\beta \in$ isom$\mathbb{E}^3$ induces a pair of rotations in the unit sphere S^2. The conformal correspondence between two domains C^l and C^r is invariant under these rotations. Therefore we have the following theorem:

Theorem 4.3. *The generation of minimal congruences in* $\mathbb{E}^3$ *is invariant under elliptic motions.*

5. Examples

This section is to give some simple examples of minimal congruences.

$\lambda(u) = 0$: Since $d\lambda = 0$, λ is not holomorphic. The lines of the congruence C generated $\lambda = 0$ are parametrized by

$$L(u^1, u^2) = \frac{1}{1 + (u^1)^2 + (u^2)^2}(u^1, u^2, 1, -u^1, -u^2, (u^1)^2 + (u^2)^2)^T.$$

Observing that $L_1 + L_4 = L_2 + L_5$, we have an elliptic linear line congruence. Its Klein image is the oval quadric

$$L_1 + L_4 = L_2 + L_5 = L_1^2 + L_2^2 - L_3 L_6 = 0.$$

It is not an isotropic congruence in the sense of Def. 2.2, since λ is not conformal ($d\lambda = 0$). It is of course minimal since it has vanishing energy.

$\lambda(u) = u = \mathrm{id}_{\mathbb{C}}$: The identity mapping defines the bundle of lines with vertex $(1, 0, 0, 0)\mathbb{R}$:

$$L(u^1, u^2) = \frac{1}{1 + (u^1)^2 + (u^2)^2}(2u^1, 2u^2, 1 - (u^1)^2 - (u^2)^2, 0, 0, 0)^T.$$

The bundle and thus any bundle is an isotropic and a minimal congruence of lines. There are no principal directions in it.

$\lambda(u) = \overline{u}$: Note that $\lambda(u) = \overline{u}$ is not holomorphic. The last simple example given here leads to the set of lines in a plane, i.e. a field of lines parametrized by

$$L(u^1, u^2) = \frac{1}{1 + (u^1)^2 + (u^2)^2}(2u^1, 0, 1 - (u^1)^2 - (u^2)^2, 0, 2u^2, 0)^T.$$

This field of lines and thus any field of lines is minimal but not an isotropic congruence.

6. Conclusion and future research

We have shown a way to parametrize the manifold of lines in elliptic three-space rationally. We also gave an explicit parametrization of isotropic congruences in $\mathbb{E}^3$. An energy is assigned to congruences. This energy is in close relation to the generation of the congruence, it comes from it.

An open problem is the geometric interpretation of the energy density and energy as defined above. Is it possible to express the energy density in terms of the coeficients of the fundamental forms of a congruence? Does the energy of a congruence C of oriented lines in elliptic three-spcae $\mathbb{E}^3$ have a geometric meaning for two-parametric motions in spherical kinematics?

Is it possible to characterize congruences which are generated by special conformal or anti-conformal mappings, especially Möbius transformations? Do these congruences have remarkable properties?

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