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NUMERICAL ISOTONIES OF PRE-ORDERED SEMIGROUPS THROUGH THE CONCEPT OF A SCALE

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Abstract: Necessary and sufficient conditions are presented for the existence of a continuous and additive real-valued function representing a (not necessarily total) preorder defined on a semigroup. The main purpose of this paper is that of providing a characterization of the existence of a continuous and order-preserving real-valued function defined on a preordered topological semigroup such that it also preserves the binary operation. The approach followed to obtain this characterization is based on the existence of particular scales that behave well with respect to the semigroup operation and that we call additive scales. A scale is a device that consists of a nested family of sets indexed by a dense subset of a suitable part of the reals.

1. Introduction

The main purpose of this paper is that of providing a characterization of the existence of a continuous and order-preserving real-valued function F defined on a preordered topological semigroup $(S, *, \tau, \precsim)$ such that F also preserves the binary operation (i.e.: $F : S \to \mathbb{R}$ is a semigroup homomorphism into the additive real numbers). The approach followed to obtain this characterization is based on the existence of particular scales that behave well with respect to the semigroup operation and that we call additive scales. A scale is a device that consists of a nested family of sets indexed by a dense subset (e.g.: the dyadic numbers, the positive rationals, the whole set $\mathbb{Q}$ of rationals, etc.) of a suitable part of the reals (e.g.: $\mathbb{I} = [0, 1]$; $(0, +\infty)$; the whole $\mathbb{R}$, etc.). These kinds of devices occur in one form or another in almost every metrisation theory of topological spaces and topological groups. For instance, there is a clear correspondence between continuous functions into $(0, +\infty)$ and scales (see Lemma 2.2 below). The use of a scale can be understood as a generalization of the Urysohn approach (see [19]) to get continuous functions on a topological space. This fruitful idea was already used by Nachbin (see [16]) to provide results about the existence of continuous order-preserving functions on preordered topological spaces (see also [3] and [12]). We go further by extending the previous approach to the algebraical context of preordered topological semigroups. It should be also noted that the existing literature concerning this representability problem deals with totally preordered topological semigroups (see e.g. [11] or [7]). Here, we drop the assumption of the preorder being total.

The mathematical statement of the underlying problem, namely the numerical representation of preordered structures, is quite simple: We are given a nonempty set X with a preorder $\precsim$ (i.e.: a transitive and reflexive binary relation). We are looking for order-preserving functions $F : X \to \mathbb{R}$. If additional topological or algebraic structures, or both are given then one hopes to find functions which also preserve the additional structure. Surveys on the numerical representations of ordinal structures continue to appear (see e.g. [2], or else [15]). It is well understood that there are intimate relations between order and topology (at the latest since Nachbin's monograph [16]). There is some remarkable interdisciplinary aspect in this issue. On the one hand, this kind of results is of particular importance in mathematical economics related to constant returns to scale economies (see, e.g., [22], Ch. 2). Also, measurements that are often encountered in the social or biological sciences (see e.g. [17], [15] or [18]) are usually based on data that can be "compared" but not a priori "quantified". One obtains a scale but not a yardstick. All quantification a posteriori is based on the hypothetical possibility to map the set of data into the set $\mathbb{R}$ of real numbers under preservation of order and, where appropriate, of topology. In the economic and social science contexts such a mapping is called a utility function. Additional steps toward a more sophisticated quantification are possible if the supply of data has more structure, for instance if data can be "added" (like objects scaled by "mass" or "length"); then the question arises whether a function into $\mathbb{R}$ can be found in such a fashion that addition of data is mapped to addition of numbers as well.

2. Notation and preliminaries

A preorder $\precsim$ on a nonempty set X is a reflexive and transitive binary relation on X. If, in addition, $\precsim$ is antisymmetric, then it is said to be an order.

The asymmetric part $\prec$ of a preorder $\precsim$ is defined as $x \prec y \iff \iff (x \precsim y) \wedge (\neg(y \precsim x))$ $(x, y \in X)$ and the symmetric part $\sim$ is defined by $x \sim y \iff x \precsim y,\ y \precsim x$ $(x, y \in X)$. A preorder $\precsim$ on X is said to be total if for any two elements $x, y \in X$ either $x \precsim y$ or $y \precsim x$. A pair $(X, \precsim)$ consisting of a nonempty set X endowed with a preorder $\precsim$ will be referred to as a preordered set.

If in addition X is endowed with a topology τ then the triple $(X, \tau, \precsim)$ is said to be a topological preordered space.

Suppose now that there is also a binary operation $*$ defined on X. We shall use the notation $(X, *, \tau, \precsim)$.

Remark 2.1. Notice at this point that for a topological preordered space we just understand the mere juxtaposition of a preordered set $(X, \precsim)$ and a topology τ. The reason is that we will deal with preorders that are not necessarily total. However, we must point out that in the classical "topology and order" framework a "topological preordered space" it is usually understood to be something more restrictive, specially when the preorder considered is total. Thus, a preordered topological space it is often asked to satisfy the condition that the graph $\{(x, y) : x \precsim y \ (x, y \in X)\}$ is closed in the product topology of $X \times X$. When the preorder X is total, this is equivalent to say that the contour sets $U(a) = \{x \in X : a \prec x\}$ $(a \in X)$ and $L(a) = \{z \in X : z \prec a\}$ $(a \in X)$ are τ-open. In such case, this is also equivalent to say that the sets $\{x \in X : b \precsim x\}$ $(b \in X)$ and $\{z \in X : z \precsim b\}$ $(b \in X)$ are τ-closed (see e.g. [2], pp. 19–20).

The order topology can be defined on any kind of preordered set (even if the preorder is not total) as the topology whose subbasis is given by all sets $U(a)$ $(a \in X)$ and $L(b)$ $(b \in X)$. Then, a natural topology τ on $(X, \precsim)$ is defined (see [8]) as a topology that is finer than the order topology on $(X, \precsim)$. In other words, most people in "topology and order" usually start from a natural topology.

Unless otherwise stated, we will not ask a priori the topology τ to be natural. □

A semigroup is a nonempty set S together with an associative binary operation $*$. A topological semigroup $(S, *, \tau)$ is a semigroup together with a topology τ on S such that the function $\Phi : S \times S \longrightarrow S$ defined by $\Phi(x, y) = x * y$ $(x, y \in S)$ is continuous with respect to the topology τ on S and the corresponding product topology $\tau \times \tau$ on $S \times S$.

Let $(X, \precsim)$ be a preordered set. A real-valued function u is said to be order-preserving if it satisfies the following two conditions:

(i) $x \precsim y \Longrightarrow u(x) \leq u(y) \quad (x, y \in X)$.

(ii) $x \prec y \Longrightarrow u(x) < u(y) (x, y \in X)$.

It is clear that if the preorder on X is total, then a real-valued function u on the preordered set $(X, \precsim)$ is order-preserving if and only if $x \precsim y \iff u(x) \leq u(y)$ $(x, y \in X)$. In this case u is called a numerical isotony (also known as "utility function" in the economics and social sciences literature) representing the total preorder $\precsim$ on X.

A real-valued function u on a set X endowed with a binary operation $*$ is said to be additive if $u(x*y) = u(x)+u(y)$ $(x, y \in X)$. In the sequel we shall be interested in the existence of a continuous additive order-preserving function u on a structure $(X, *, \tau, \precsim)$.

When $(X, \precsim)$ is a preordered set endowed with some binary relation $*$ it is usual to ask the binary relation $*$ to satisfy some additional condition of compatibility with the ordering $\precsim$. In this direction, $(X, *)$ is said to be monotone (or, equivalently, $*$ is said to satisfy the property of monotonicity of the translations) if $x \precsim y \iff x * z \precsim y * z \iff$ $\iff z * x \precsim z * y$ $(x, y, z \in X)$.

Let X be a nonempty set endowed with a topology τ. Let T be a dense subset of the Euclidean real line $\mathbb{R}$ (respectively: of the set $(0, +\infty) \subset \mathbb{R}$). A family $\mathcal{F} = \{X_t : t \in T\}$ of subsets of X is said to be a scale (respectively: a positive scale) on the topological space (X, τ) if the following conditions hold:

(i) X_t is a τ-open subset of X for every $t \in T$.

(ii) $\bar{X}_s \subseteq X_t$ for every $s, t \in T$ such that $s < t$, where $\bar{Y}$ stands for the τ-closure of a subset $Y \subseteq X$.

(iii) $\bigcup_{t \in T} X_t = X$ and $\bigcap_{t \in T} X_t = \emptyset$.

Following [16], given a preordered set $(X, \precsim)$ a subset $A \subseteq X$ is called decreasing if for every $x, z \in X$ it holds that $(z \precsim x) \wedge (x \in A) \Longrightarrow$ $\Longrightarrow z \in A$.

Leaning on the concept of a decreasing set, a powerful tool to deal with numerical representations of preordered sets was introduced in [3]. This is the key concept of a decreasing scale (also called separable system in [12]). Throughout the paper we shall use a particular case of decreasing scales. Thus, we say that a family $\mathcal{G} = \{G_r : r \in \mathbb{Q}\}$ is a countable decreasing scale on a topological preordered space $(X, \tau, \precsim)$ if the following conditions hold:

(i) G_r is a decreasing subset of X for every $r \in \mathbb{Q}$.

(ii) $\mathcal{G} = \{G_r : r \in \mathbb{Q}\}$ is a scale on (X, τ).

The concept of a positive countable decreasing scale is defined in the obvious analogous way. In the particular case when τ is the discrete topology on X, then a countable decreasing scale $\mathcal{G} = \{G_r : r \in \mathbb{Q}\}$ will be referred to as a countable decreasing pseudoscale on the preordered set $(X, \precsim)$.

If X is any nonempty set endowed with a binary operation $*$, and A and B are two nonempty subsets of X, then define $A * B = \{a * b : : a \in A, b \in B\}$.

A countable decreasing scale $\mathcal{G} = \{G_r : r \in \mathbb{Q}\}$ on a topological preordered structure $(X, \tau, \precsim)$ is said to be separating if for every $x, y \in \in X$ with $x \prec y$, there exist $r_1, r_2 \in \mathbb{Q}$ such that $r_1 < r_2$ and $x \in \in G_{r_1}, y \notin G_{r_2}$.

Moreover $\mathcal{G}$ is said to be additive if it satisfies the following two conditions:

1. $G_q * G_r \subseteq G_{q+r} \quad (q, r \in \mathbb{Q})$.
2. $(X \setminus G_q) * (X \setminus G_r) \subseteq (X \setminus G_{q+r}) \quad (q, r \in \mathbb{Q})$.

Similar definitions are given for the case of positive countable decreasing scales.

To conclude this section we furnish the following useful lemma that interpretes scales as continuous real-valued functions, and vice versa. (An analogous result would also furnish the equivalence between the concept of a positive scale and that of a continuous real-valued function taking values in $(0, +\infty)$.)

Lemma 2.2. *Let X be a nonempty set endowed with a topology τ. Then the following conditions hold:*

(i) *Given a continuous function $u : X \longrightarrow \mathbb{R}$, the family $\mathcal{F} = = \{u^{-1}(-\infty, q) : q \in \mathbb{Q}\}$ is a scale on (X, τ).*

(ii) *Given a scale $\mathcal{F} = \{X_t : t \in T\}$ defined on (X, τ), where T is a dense subset of $\mathbb{R}$, it holds that the map $u : X \longrightarrow \mathbb{R}$ defined by $u(x) = \inf\{t \in T : x \in X_t\}$ $(x \in X)$, is a continuous function.*

Proof. (i) This follows from direct checking.

(ii) By definition of the concept of a scale, it is clear that the map $u : X \longrightarrow \mathbb{R}$ is well-defined. It is now straightforward to prove the continuity of u. (See [10], pp. 43–44 for details.) $\Diamond$

3. Continuous order-preserving maps through the concept of a decreasing scale

In this section we make use of the concept of a decreasing scale to obtain in a straightforward manner a characterization of the existence of a continuous order-preserving function on a preordered topological space (see also [3] or [12]). This result will be used in the next section.

Theorem 3.1 *Let (X, τ) be a nonempty topological space, endowed* *with a preorder $\precsim$ (not necessarily total). The following conditions are equivalent:*

(i) *There exists a continuous order-preserving function $u:(X,\tau,\precsim)\rightarrow \longrightarrow \mathbb{R}$.*

(ii) *There exists a separating countable decreasing scale $\mathcal{G}=\{G_r : r \in \in \mathbb{Q}\}$ on $(X,\tau,\precsim)$.*

Proof. (i)$\Rightarrow$(ii) Let $u : X \longrightarrow \mathbb{R}$ be a continuous order-preserving map.

Define, for every $q \in \mathbb{Q}$, $G_q = u^{-1}(-\infty, q)$. It follows that G_q is decreasing because u is order-preserving. Furthermore, it is also τ-open since u is continuous. In addition, for every $p, q \in \mathbb{Q}$ such that $p < q$, it follows that $\bar{G}_p \subseteq u^{-1}(-\infty, p]$, because by continuity of u, the set $u^{-1}(-\infty, p]$ is closed. Hence $\bar{G}_p \subseteq u^{-1}(-\infty, p] \subseteq u^{-1}(-\infty, q) \subseteq G_q$. Also, $X = \bigcup_{p\in\mathbb{Q}} G_p$ since u is defined on the whole X, and $\emptyset = \bigcap_{p\in\mathbb{Q}} G_p$ because $x \notin G_q$ for any $q \in \mathbb{Q}$ such that $q < u(x)$.

Let now $x, y \in X$ be such that $x \prec y$. Since u is order-preserving, it follows that $u(x) < u(y)$. Hence we can find $r_1, r_2 \in \mathbb{Q}$ such that $u(x) < r_1 < r_2 < u(y)$. Thus $x \in G_{r_1}$, $y \notin G_{r_2}$. Therefore the family $\mathcal{G} = \{G_p : p \in \mathbb{Q}\}$ is a separating countable decreasing scale.

(ii)$\Rightarrow$(i) Now suppose that $\mathcal{G} = \{G_r : r \in \mathbb{Q}\}$ is a separating countable decreasing scale on $(X, \precsim)$. Given $x \in X$, set $u(x) = \inf\{q \in \in \mathbb{Q} : x \in G_q\}$. By Lemma 2.2, u is well-defined and continuous.

To conclude, let us prove that the map u just defined is order-preserving: For every $x, y \in X$ with $x \precsim y$ and every $q \in \mathbb{Q}$ such that $y \in G_q$ it follows that $x \in G_q$ because G_q is a decreasing set. Consequently, $u(x) \leq u(y)$. Also, given $x, y \in X$ such that $x \prec y$, and $r_1, r_2 \in \mathbb{Q}$ such that $r_1 < r_2$; $x \in G_{r_1}$, $y \notin G_{r_2}$, we have that $u(x) \leq r_1$ and also $u(y) \geq r_2$ because $u(y) < r_2$ would imply that $y \in G_p$ for some $p \in \mathbb{Q}$ with $p < r_2$, and thus $G_p \subseteq G_{r_2}$ by hypothesis, so that $y \in G_{r_2}$, which is a contradiction. Therefore $u(x) \leq r_1 < r_2 \leq u(y)$ and, in particular, $u(x) < u(y)$. $\Diamond$

Remark 3.2. We do not ask the topology τ to be a priori a natural topology. However, it is important to notice that the existence of a continuous order-preserving function u on $(X, \tau, \precsim)$ implies that the topology τ is a fortiori natural, because given $x \in X$ we have that the contour sets $L(x) = u^{-1}(-\infty, u(x))$ and $U(x) = u^{-1}(u(x), +\infty)$ are τ-open by continuity of u. $\square$

If we drop the continuity condition, we obtain the following result as an immediate corollary of Th. 3.1.

Corollary 3.3. *Let X be a nonempty set, endowed with a preorder $\precsim$ (not necessarily total). The following conditions are equivalent:*

(i) *There exists an order-preserving function $u : (X, \precsim) \longrightarrow \mathbb{R}$.*

(ii) *There exists a separating countable decreasing pseudoscale $\mathcal{G} = \{G_r : r \in \mathbb{Q}\}$ on $(X, \precsim)$.*

4. Continuous additive representations of preordered semigroups

The study of the existence of additive numerical representations of ordered groups and semigroups has been considered since long in the specialized literature. Thus, in a paper (see [13]) published early in 1901, O. Hölder proved that a totally ordered group $(G, *, \precsim)$ with $*$ monotone admits an additive numerical isotony if and only if it is Archimedean (i.e.: it holds that for every $x, y \in G$ with $x \prec y$, $x \prec x * x$, and $y \prec y * y$, there exists a strictly positive natural number n such that $y \prec n \cdot x$, where $n \cdot x$ stands for $x * \ldots (n \text{ times}) \ldots * x$ $(x \in G)$).

In what concerns a totally ordered semigroup $(S, *, \precsim)$ with the binary operation $*$ being monotone, an Archimedean-like condition that also characterizes the existence of an additive numerical isotony was obtained in 1950 by N. S. Alimov. (See [1]. See also [9] or [7].) Such condition is the following one, that we call super-Archimedeanness:

$\ll$ *For every $x, y \in S$ with $x \prec x * x; y \prec y * y$ and $x \prec y$ there exists a strictly positive natural number n such that $(n+1) \cdot x \prec n \cdot y$. Also, for every $z, t \in S$ with $z * z \prec z; t * t \prec t$ and $z \prec t$ there exists $k > 0$ $(k \in \mathbb{N})$, such that $k \cdot z \prec (k+1) \cdot t$.* $\gg$

Observe that as a direct consequence of the existence of a numerical isotony it holds that an Archimedean totally ordered group $(G, *, \precsim)$ with $*$ monotone is Abelian (i.e.: $x * y = y * x$ $(x, y \in G)$). Also, any super-Archimedean totally ordered semigroup $(S, *, \precsim)$ with $*$ monotone is commutative (i.e.: $z * t = t * z$ $(z, t \in S)$). These properties can actually be proved directly, without using the existence of an additive numerical isotony representation (see, e.g. [5] or [6]).

For the case of a totally ordered group $(G, *, \precsim)$ with $*$ monotone, the conditions of Archimedeanness and super-Archimedeanness are equivalent. However, this is no longer true for the case of a totally ordered semigroup $(S, *, \precsim)$ with $*$ monotone. An example is $\mathbb{N} \setminus \{0\} \times \mathbb{N} \setminus \{0\}$ endowed with the sum $+$ defined coordinatewise, and the lexicographic ordering. Observe that $(1,0) \prec (1,1)$ but $(n,n) \prec (n+$ $+1,0)$ for any positive $n \in \mathbb{N}$. In this general case of a totally ordered semigroup $(S,*,\precsim)$ with $*$ monotone, super-Archimedeanness always implies Archimedeanness, but the converse is not true as last example shows. (See [7] for more details.)

The existence of continuous and additive numerical isotonies representing a totally ordered group or semigroup structure, with $*$ monotone and endowed with a topology τ, has also been analyzed in the literature. (See e.g. [4].)

However, in those classical results concerning additive representability of groups or semigroups, the concept of a scale has not been used yet.

A semigroup $(S,*)$ endowed with a total preorder $\precsim$ is said to be positive if $x \prec x*x$ for every $x \in S$. It is obvious that if a totally ordered positive semigroup $(S,*,\precsim)$ admits an additive numerical isotony u, then, a fortiori, it holds that $u(x) > 0$ for every $x \in S$.

The following theorem improves Th. 3.1, now working with an algebraic structure of which a semigroup is a particular case.

Theorem 4.1. *Let S be a nonempty set endowed with a binary operation $*$, a topology τ and a preorder $\precsim$ (not necessarily total). Then the following conditions are equivalent:*

(i) *There exists a continuous and additive order-preserving function u on $(S,*,\tau,\precsim)$,*

(ii) *There exists an additive separating countable decreasing scale $\mathcal{G} =$ $= \{G_r : r \in \mathbb{Q}\}$ on $(S,*,\tau,\precsim)$.*

Proof. (i)⇒(ii) As in Th. 3.1, assume that there exists a continuous and additive order-preserving function $u : S \longrightarrow \mathbb{R}$ on the structure $(S,*,\tau,\precsim)$. Define $G_r = u^{-1}((-\infty, r))$ for every $r \in \mathbb{Q}$.

Let us show that the scale $\mathcal{G} = \{G_r : r \in \mathbb{Q}\}$ is indeed additive: First notice that being $x \in G_p; y \in G_q$ $(p,q \in \mathbb{Q})$, we have that $u(x) <$ $< p; u(y) < q$. Thus $u(x) + u(y) < p + q$. But u is additive, so we have $u(x*y) < p+q$ or equivalently $x*y \in G_{p+q}$. Hence $G_p * G_q \subseteq G_{p+q}$ $(p,q \in \mathbb{Q})$. Also, we observe that being $x \notin G_p; y \notin G_q$ $(p,q \in \mathbb{Q})$, it follows by definition of u, that $u(x) \geq p; u(y) \geq q$. Thus $u(x) +$ $+ u(y) \geq p+q$. Then, by additivity of u, we obtain $u(x*y) \geq p+q$ or equivalently $x*y \notin G_{p+q}$.

(ii)⇒(i) Given the scale $\mathcal{G}$ define $u(x) = \inf\{p \in \mathbb{Q} : x \in G_p\}$, so that u is a continuous order-preserving function. To conclude, it is enough to prove that u is additive. Thus, being $x, y \in S$ observe that $x \in G_p$, $y \in G_q \Rightarrow x * y \in G_{p+q}$ $(x, y \in S,\ p, q \in \mathbb{Q})$. Hence, by definition of u, it is plain that $u(x * y) \leq u(x) + u(y)$. Suppose now that $u(x * y) < u(x) + u(y)$ and let $p, q \in \mathbb{Q}$ be chosen such that $p < u(x)$, $q < u(y)$, $u(x * y) < p + q$. It follows then that $x \notin G_p$, $y \notin$ $\notin G_q$, $x * y \in G_{p+q}$, in contradiction with the additivity of $\mathcal{G}$. Therefore $u(x * y) = u(x) + u(y)$ $(x, y \in S)$. $\Diamond$

Remark 4.2. Observe that in the proof of Th. 4.1 we do not ask $*$ to be associative or commutative. Neither we need $\precsim$ to be total or monotone with respect to $*$.

In the particular case when S is a positive semigroup, we can replace "scale" by "positive scale" in the statement of Th. 4.1. The proof is straightforward. $\Box$

We have already mentioned that if $(S, *, \precsim)$ is a totally ordered semigroup with $*$ monotone, super-Archimedeanness is a necessary and sufficient condition for the existence of an additive numerical isotony (see e.g. [1]).

Consequently, by Th. 4.1 it follows that in this case the existence of an additive separating countable decreasing pseudoscale $\mathcal{G} = \{G_r :$ $: r \in \mathbb{Q}\}$ on $(S, *, \precsim)$ is equivalent to super-Archimedeanness. Let us give now a direct proof of this fact for the particular case of a positive semigroup. The proof for the general case is similar.

Proposition 4.3. *Let $(S, *, \precsim)$ be a positive totally ordered semigroup, with the operation $*$ being monotone. Then the following conditions are equivalent:*

(i) *There exists a positive additive separating countable decreasing pseudoscale $\mathcal{G} = \{G_r : r \in \mathbb{Q} \cap (0, +\infty)\}$ on $(S, *, \precsim)$.*

(ii) *The structure $(S, *, \precsim)$ satisfies the condition of super-Archimedeanness.*

Proof. (i)$\Rightarrow$(ii) Let $\mathcal{G} = \{G_r : r \in \mathbb{Q} \cap (0, +\infty)\}$ be a positive additive separating countable decreasing pseudoscale on $(S, *, \precsim)$. Let $x, y \in S$ such that $x \prec y$, $x \prec x * x$ and $y \prec y * y$. By hypothesis there exist $p, q \in \mathbb{Q} \cap (0, +\infty)$ with $p < q$, such that $x \in G_p$, $y \notin G_q$. Hence, there exist $n > 0$, $n \in \mathbb{N}$ such that $(n+1)p < nq$. By additivity of the pseudoscale $\mathcal{G}$, it is clear that $x \in G_p \Rightarrow (n+1) \cdot x \in G_{(n+1)p}$; $y \notin G_q \Rightarrow$ $\Rightarrow n \cdot y \notin G_{nq}$. Consequently $(n+1) \cdot x \prec n \cdot y$ since otherwise, $G_{(n+1)p}$ being a decreasing set, we would have that $n \cdot y \precsim (n+1) \cdot x \Rightarrow n \cdot y \in$ $\in G_{(n+1)p} \subseteq G_{nq}$, which is a contradiction.

(ii)$\Rightarrow$(i) Suppose that $(S, *, \precsim)$ is super-Archimedean Choose an element $x_0 \in S$. For every $p \in \mathbb{Q} \cap (0, +\infty)$, set $p = \frac{m}{n}$ with $m, n \in \mathbb{N}$

and define $G_p = \{x \in S : n \cdot x \prec m \cdot x_0\}$. Observe that G_p is well-defined, since by monotonicity of the operation, being $a, b \in S$ it holds that $a \prec b \iff n \cdot a \prec n \cdot b$ for any positive natural number n. Let us see now that $p < q \Rightarrow G_p \subseteq G_q$. To prove this, take $a, b, c \in \mathbb{N}$ such that $p = \frac{a}{b}$ and $q = \frac{c}{b}$. Obviously $a < c$, so that $a \cdot x_0 \prec c \cdot x_0$. We have also that $x \in G_p \iff b \cdot x \prec a \cdot x_0$ and $x \in G_q \iff b \cdot x \prec c \cdot x_0$, whence it is plain that $G_p \subseteq G_q$. Each element G_p is a decreasing set because being $x \in S$ with $x \in G_p$ and $y \precsim x$ it follows that $n \cdot y \precsim n \cdot x \prec m \cdot x_0$ where $p = \frac{m}{n}$ with $m, n \in \mathbb{N}$. Hence we have $n \cdot y \prec m \cdot x_0$ by transitivity, and this is equivalent to say that y belongs to G_p.

At this point, let us prove the additivity of the family $\mathcal{G} = \{G_p : p \in \mathbb{Q} \cap (0, +\infty)\}$ corresponding to the set of all subsets G_p just defined. To this task, let p, q be positive rational numbers that we can write as $p = \frac{a}{b}, q = \frac{c}{b}$ for suitable $a, b, c \in \mathbb{N}$. If $x \in G_p$ and $y \in G_q$ it follows, equivalently, that $b \cdot x \prec a \cdot x_0$ and $b \cdot y \prec c \cdot x_0$. Since $(S, *)$ is commutative because it is super-Archimedean, we have that $b(x * y) = (b \cdot x) * (b \cdot y) \prec (a + c) \cdot x_0$. Thus $x * y \in G_{p+q}$. Therefore $G_p * G_q \subseteq G_{p+q}$. With similar arguments we can prove that $(S \setminus G_p) * (S \setminus G_q) \subseteq (S \setminus G_{p+q})$.

Given any element $x \in S$, three possibilities may occur: either $x \prec x_0$, or $x = x_0$ or else $x_0 \prec x$. If $x \prec x_0$ it is plain that $x \in G_1$. If $x = x_0$, since $x_0 \prec x_0 * x_0$ by the hypothesis made on S, it follows that $x \in G_2$. Finally, if $x_0 \prec x$, by the Archimedean property we can find $n > 0, n \in \mathbb{N}$ such that $x \prec n \cdot x_0$ or, equivalently, $x \in G_n$. So we have that $S = \bigcup_{p \in \mathbb{Q} \cap (0,+\infty)} G_p$.

With an analogous reasoning we can also prove that $\bigcap_{p \in \mathbb{Q} \cap (0,+\infty)} G_p = \emptyset$: Indeed, if $x_0 \precsim x$ we have that $x \notin G_1$ and if $x \prec x_0$, by the Archimedean property there exists $n \in \mathbb{N}, n > 0$, such that $x_0 \prec n \cdot x$. Thus $x \notin G_{\frac{1}{n}}$. Observe, in addition, that for every $x \in S$ it follows that $\inf\{r \in \mathbb{Q} \cap (0, +\infty) : x \in G_r\} > 0$.

At this point we have that $\mathcal{G}$ is a positive and additive countable decreasing pseudoscale.

To conclude, let us prove that $\mathcal{G}$ is separating: Let $x, y \in S$ be such that $x \prec y$. By the super-Archimedean property, there exists $n \in \mathbb{N} \setminus \{0\}$ such that $(n+1) \cdot x \prec n \cdot y$. Let $\lambda(y) > 0$ be defined as $\lambda(y) = \inf\{r \in \mathbb{Q} \cap (0, +\infty) : y \in G_r\}$. Now choose rational numbers $p_1, p_2 \in \mathbb{Q}$ such that $n\lambda(y) < p_1 < p_2 < (n+1)\lambda(y)$. It is clear that

$\lambda(y) < \frac{p_1}{n} \Rightarrow y \in G_{\frac{p_1}{n}} \Rightarrow n \cdot y \in nG_{\frac{p_1}{n}} \subseteq G_{p_1} \Rightarrow (n+1)\cdot x \in G_{p_1}$. Also, $\frac{p_2}{n+1} < \lambda(y) \Rightarrow y \notin G_{\frac{p_2}{n+1}}$. So we have $(n+1)\cdot x \in G_{p_1}$, which easily implies $x \in G_{\frac{p_1}{n+1}}$, and $y \notin G_{\frac{p_2}{n+1}}$, and obviously $\frac{p_1}{n+1} < \frac{p_2}{n+1}$. $\Diamond$

To conclude this section, we pay attention to the continuity of additive numerical isotonies in the case of a totally ordered semigroup $(S, *, \precsim)$, with $*$ monotone, endowed with a natural topology.

When, in addition, the binary operation $* : S \times S \longrightarrow S$, that maps the pair $(x, y) \in S \times S$ to the element $x * y \in S$ $(x, y \in S)$, is continuous (considering on $S \times S$ the product topology $\tau \times \tau$), then S is said to be a totally ordered topological semigroup.

In the particular case of a totally ordered group $(G, *, \precsim)$ with $*$ monotone, endowed with a natural topology τ, we say that the structure is a totally ordered topological group if the binary operations $* : G \times \times G \longrightarrow G$ defined as above, and $I : G \longrightarrow G$ defined by $I(x) = -x$ $(x \in G)$, where "$-x$" stands for the opposite element of x in $(G, *)$, are both continuous.

In the main case of semigroups, the following remarkable result arises.

Proposition 4.4. *Every super-Archimedean totally ordered topological semigroup $(S, *, \tau, \precsim)$ where $*$ is monotone and τ is a natural topology, is representable through a continuous and additive numerical isotony.*

Proof. See Th. 6 in [4], where it is proved a much stronger result, that states that every additive numerical isotony representing a monotone topological totally ordered semigroup $(S, *, \tau, \precsim)$ must indeed be continuous. $\Diamond$

Remark 4.5. Despite every additive numerical isotony representing a monotone topological totally ordered semigroup $(S, *, \tau, \precsim)$ must be continuous, the analogous result for additive separating countable decreasing pseudoscales is not true. Actually on such structures we have that an additive separating countable decreasing pseudoscale $\mathcal{G}$ may or may not be an scale. An example is the additive set $(\mathbb{R}, +, \leq)$ of the real numbers endowed with the usual ordering, on which we consider the additive pseudoscale $\mathcal{G} = \{(-\infty, q] : q \in \mathbb{Q}\}$ of decreasing (but not open) subsets. $\square$

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