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ON WEAKLY CONCIRCULAR SYMMETRIC SPACES

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Abstract: The object of this paper is to establish an example of a weakly symmetric space. Some results on a $(WZS)_n$ are established and it is shown that if a $(WZS)_n$ is a decomposable space $V^r \times V^{n-r}$ $(r > 1, n-r > 1)$, then one of the decomposition spaces is flat and the other is a pseudo symmetric space. Later, the Ricci-associate of the vector field λ_i is defined and some theorems relating to it are proved.

1. Introduction

The notions of weakly symmetric space and weakly projective symmetric spaces were introduced by Tamássy and Binh, [7]. In a subsequent paper Binh, [1] studied decomposable weakly symmetric spaces.

A non-flat Riemannian space V^n $(n > 2)$ is called weakly symmetric if the curvature tensor R_{hijk} satisfies the condition

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(1.1) $R_{hijk,m} = A_m R_{hijk} + B_h R_{mijk} + D_i R_{hmjk} + E_j R_{himk} + F_k R_{hijm}$

where A, B, D, E, F are 1-forms (non-zero simultaneously) and the comma "," denotes covariant differentiation with respect to the metric tensor of the space. The 1-forms are called the associated 1-forms of the space and an n-dimensional space of this kind is denoted by $(WS)_n$. It may be mentioned in this connection that although the definition of a $(WS)_n$ is similar to that of a generalized pseudo-symmetric space studied by Chaki and Mondal, [3], the defining condition of a $(WS)_n$ is weaker than that of a generalized pseudo-symmetric space. A reduction in generalized pseudo-symmetric spaces has been obtained by Chaki and Mondal. On the analogy of $(WS)_n$, Tamássy and Binh, [8], introduced the notion of weakly Ricci symmetric spaces $(WRS)_n$. A Riemannian space V^n is called weakly Ricci symmetric if there exist 1-forms A, B, D such that $R_{ij,k} = A_k R_{ij} + B_i R_{kj} + D_j R_{ik}$, where the Ricci tensor $R_{ij} \neq 0$.

The present paper deals with non-concircular-flat Riemannian spaces V^n whose concircular curvature tensor Z_{hijk} satisfies the condition:

(1.2) $$Z_{hijk,l} = A_l Z_{hijk} + B_h Z_{lijk} + D_i Z_{hljk} + E_j Z_{hilk} + F_k Z_{hijl},$$

where

(1.3) $$Z_{hijk} = R_{hijk} - \frac{R}{n(n-1)}(g_{hk}g_{ij} - g_{hj}g_{ik}).$$

R is the scalar curvature and A, B, D, E, F are 1-forms (non-zero simultaneously). Such a space will be called a *weakly concircular symmetric space* and denoted by $(WZS)_n$. In Sect. 2, it is shown that the 1-forms B and D are identical with E and F, respectively. Then, the defining condition of a $(WZS)_n$ can always be expressed in the following form:

(1.4) $$Z_{hijk,l} = A_l Z_{hijk} + B_h Z_{lijk} + B_i Z_{hljk} + D_j Z_{hilk} + D_k Z_{hijl}.$$

2. Associated 1-forms of a (WZS_n)

In this section, first of all, it will be shown that the five associated 1-forms A, B, D, E, F of a $(WZS)_n$ can not be all different. Interchanging h and i in (1.2), we get

$$(2.1)\qquad Z_{ihjk,l} = A_l Z_{ihjk} + B_i Z_{lhjk} + D_h Z_{iljk} + E_j Z_{ihlk} + F_k Z_{ihjl}.$$

Now, adding (1.2) in (2.1) and using (1.3), we obtain

$$(2.2)\qquad (B_h - D_h) Z_{lijk} + (B_i - D_i) Z_{lhjk} = 0$$

or

$$(2.3)\qquad C_h Z_{lijk} + C_i Z_{lhjk} = 0,$$

where $C_h = B_h - D_h$. We want to show that $C_h = 0$ $(h = 1, 2, \ldots, n)$. Suppose on the contrary that there exists a fixed index q for which $C_q \neq 0$. Putting $h = l = q$ in (2.3), we get $C_q Z_{qijk} + C_i R_{qqjk} = 0$. Remembering that $R_{lhjk} = -R_{hljk}$, we find

$$(2.4)\qquad C_q Z_{qijk} = 0$$

which implies that $Z_{qijk} = 0$ for all i, j, k. Next, putting $i = q$ in (2.3), we obtain

$$C_h Z_{lqjk} + C_q Z_{hljk} = 0$$

which implies that $Z_{hljk} = 0$ for all l, h, j, k, since $Z_{qijk} = 0$ for all i, j, k and $C_q \neq 0$. Then, this space is flat contradicting our hypothesis. Hence, $C_h = 0$ for all h, which implies that

$$(2.5)\qquad B_h = D_h \qquad \text{for all } h.$$

Similarly, interchanging j and k in (1.2), and proceeding as before, we get

$$(F_k - E_k) Z_{hijl} + (F_j - E_j) Z_{hikl} = 0$$

or

$$C_k Z_{hijl} + C_j Z_{hikl} = 0,$$

where $C_k = F_k - E_k$. Finally, we obtain

$$(2.6)\qquad F_k = E_k \qquad \text{for all } k.$$

From (2.5) and (2.6), we see that the associated 1-forms A, B, D, E, F are not all different, because $B = D$, $F = E$. In virtue of this, we can state the following theorem

Theorem 1. *The defining equation of* $(WZS)_n$ *can always be expressed in the following form*

$$(2.7)\qquad Z_{hijk,l} = A_l Z_{hijk} + B_h Z_{lijk} + B_i Z_{hljk} + D_j Z_{hilk} + D_k Z_{hijl}.$$

For $(WS)_n$, similar theorems are proved in [4] and [5]. In this section, we shall obtain some formulas which will be required in our study of $(WZS)_n$.

Let R_{ij} and R denote the Ricci tensor and the scalar curvature, respectively. Then, from (1.4), we get

(2.8) $g^{hk}Z_{hijk,l} = A_l G_{ij} + B_h g^{hk} Z_{lijk} + B_i G_{lj} + D_j G_{il} + D_k g^{hk} Z_{hijl}$,

where $G_{ij} = R_{ij} - \frac{R}{n} g_{ij}$. Transvecting (2.8) with g^{ij}, we get

$$(B_h + D_h) g^{hl} G_{lk} = 0. \tag{2.9}$$

Moreover, multiplying (1.4) by g^{hl}, g^{ij} and using the equation $R^l_{k,l} = = \frac{R_{,k}}{2}$, we obtain

$$R_{,k} = \frac{2n}{n-2}(A_h + B_h - D_h) g^{hl} G_{lk} \ (n \neq 2). \tag{2.10}$$

On the other hand, transvecting (1.4) with g^{jl}, g^{ik} and using (2.9), we find that

$$B_h g^{hl} G_{lk} = 0 \ D_h g^{hl} G_{lk} = 0. \tag{2.11}$$

By the aid of (2.10) and (2.11), we can easily obtain that

$$R_{,k} = \frac{2n}{n-2} A_h g^{hl} G_{lk} \ (n \neq 2). \tag{2.12}$$

An example of a $(WZS)_n$. Now, we want to construct concrete a $(WZS)_n$ space. On the coordinate space V^n (with coordinates x^1, x^2, ... , x^n), we define a Riemannian space V^n. We calculate the components of the curvature tensor, the Ricci tensor, the concircular curvature tensor and its covariant derivative.

Let each Latin index run over $1, 2, \ldots, n$ and each Greek index over $2, 3, \ldots, n-1$. We define a Riemannian metric on R^n $(n > 2)$ by the formula

$$ds^2 = \phi (dx^1)^2 + K_{\alpha\beta} dx^\alpha dx^\beta + 2dx^1 dx^n, \tag{2.13}$$

where $[K_{\alpha\beta}]$ is a symmetric and non-singular matrix consisting of constants and ϕ is a function of $(x^1, x^2, \ldots, x^{n-1})$ independent of x^n. The only non-zero components of the Christoffel's symbols, the curvature tensor and the Ricci tensor are (see [4])

$$\begin{gathered} \Gamma^\beta_{11} = -\frac{1}{2} K^{\alpha\beta} \phi_{.\alpha}, \ \Gamma^n_{11} = \frac{1}{2}\phi_{.1}, \ \Gamma^n_{1\alpha} = \frac{1}{2}\phi_{.\alpha} \\ R_{1\alpha\beta 1} = \frac{1}{2}\phi_{.\alpha\beta}, \ R_{11} = \frac{1}{2} K^{\alpha\beta} \phi_{.\alpha\beta} \end{gathered} \tag{2.14}$$

and the components which can be obtained from these by the symmetry properties. Here "." denotes the partial differentiation with respect to the coordinates and $K^{\alpha\beta}$ are the elements of the matrix inverse to $[K_{\alpha\beta}]$.

We consider $K_{\alpha\beta}$ as the Kronecker deltas $\delta_{\alpha\beta}$ and $\phi = (1 + 2K_{\alpha\beta}x^{\alpha}x^{\beta})e^{2x^1}$. Hence, we obtain the following relation:

$$[K_{\alpha\beta}] = \begin{pmatrix} 1\ 0\ 0\ \dots\ 0 \\ 0\ 1\ 0\ \dots\ 0 \\ \dots\dots\dots\dots \\ \dots\dots\dots\dots \\ 0\ 0\ 0\ \dots\ 1 \end{pmatrix}.$$

Hence, we get $[K_{\alpha\beta}]^{-1} = [K_{\alpha\beta}]$. Furthermore, we find

$$(2.15)\quad K_{\alpha\beta}K^{\alpha\beta} = (n-2)\ \phi_{.\alpha\beta} = 4K_{\alpha\beta}e^{2x^1}\ K^{\alpha\beta}\phi_{.\alpha\beta} = 4(n-2)e^{2x^1}.$$

By using (2.14) and (2.15), we obtain

$$(2.16)\qquad R_{1\alpha\beta 1} = 2K_{\alpha\beta}e^{2x^1}\ R_{11} = 2(n-2)e^{2x^1}.$$

Hence $R = g^{ij}R_{ij} = g^{11}R_{11}$. Again from (2.15), we find $g_{in} = g_{ni} = 0$ for $i \neq 1$ which implies $g^{11} = 0$. So, we have

$$(2.17)\qquad R = 0.$$

With the help of (1.3), (2.15), (2.16) and (2.17), it is shown that in this space, the only non-zero components of Z_{hijk} are

$$(2.18)\qquad Z_{1\alpha\alpha 1} = R_{1\alpha\alpha 1} = \frac{\phi_{.\alpha\alpha}}{2} = 2e^{2x^1}.$$

The only non-zero components of $Z_{hijk,m}$ are

$$(2.19)\qquad Z_{1\alpha\alpha 1,1} = 4e^{2x^1} = 2Z_{1\alpha\alpha 1} \neq 0.$$

Hence, V^n is neither concircularly flat nor concircularly symmetric.

We want to show that our V_n is a $(WZS)_n$, that is, it satisfies (1.2). Let us consider the 1-forms

(2.20)

$$A_i = \begin{Bmatrix} 1 \text{ for } i=1 \\ 0 \text{ otherwise} \end{Bmatrix},\ B_i = \begin{Bmatrix} \frac{1}{2} \text{ for } i=1 \\ 0 \text{ otherwise} \end{Bmatrix},\ D_i = \begin{Bmatrix} \frac{1}{2} \text{ for } i=1 \\ 0 \text{ otherwise} \end{Bmatrix}$$

In order to verify the relation (2.7) in V_n, it is sufficient to check the following relations:

(i) $Z_{1\alpha\alpha 1,1} = A_1Z_{1\alpha\alpha 1} + B_1Z_{1\alpha\alpha 1} + B_{\alpha}Z_{11\alpha 1} + D_{\alpha}Z_{1\alpha 11} + D_1Z_{1\alpha\alpha 1}$

(ii) $Z_{11\alpha 1,\alpha} = A_{\alpha}Z_{11\alpha 1} + B_1Z_{\alpha 1\alpha 1} + B_1Z_{1\alpha\alpha 1} + D_{\alpha}Z_{11\alpha 1} + D_1Z_{11\alpha\alpha}$

(iii) $Z_{1\alpha 11,\alpha} = A_\alpha Z_{1\alpha 11} + B_1 Z_{\alpha\alpha 11} + B_\alpha Z_{1\alpha 11} + D_1 Z_{1\alpha\alpha 1} + D_1 Z_{1\alpha 1\alpha}$.

As for any case other than (i), (ii) and (iii), the components of Z_{hijk} and $Z_{hijk,l}$ vanish identically and the relation (2.7) holds trivially.

From (2.18), we get the following relation for the right-hand side (r.h.s) and the left-hand side (l.h.s) of (i):

$$\text{r.h.s of (i)} = (A_1 + B_1 + D_1) Z_{1\alpha\alpha 1} = 4e^{2x^1} = Z_{1\alpha\alpha 1,1} = \text{l.h.s of (i)}.$$

Now, the r.h.s of (ii)$= B_1(Z_{\alpha 1\alpha 1} + Z_{1\alpha\alpha 1}) = 0$ (by the antisymmetric property of $Z_{hijk} =$ the l.h.s of (ii)). By similar argument as in (ii), it can be shown that the relation (iii) is also true. Hence, R^n equipped with the metric g given in (2.13) is a weakly concircular symmetric space.

In Sect. 3, it is shown that a decomposable $(WZS)_n$ is of zero scalar curvature.

3. Decomposable (WZS_n) manifolds

An n-dimensional Riemannian space V^n is said to be decomposable if in some coordinates its metric is given by

$$(3.1)\qquad ds^2 = g_{ij}dx^i dx^j = \sum_{a,b=1}^{r} \bar{g}_{ab} dx^a dx^b + \sum_{a',b'=r+1}^{n} g^*{}_{a'b'} dx^{a'} dx^{b'},$$

where $\bar{g}_{ab}$ are functions of $x^1, x^2, \ldots, x^r$ $(r < n)$ denoted by $\bar{x}$ and $g^*{}_{a'b'}$ are functions of $x^{r+1}, x^{r+2}, \ldots, x^n$ denoted by x^*; $a, b, c, \ldots$ run from 1 to r and $a', b', c', \ldots,$ run from $r+1$ to n. The two parts of (3.1) are the metrics of a V^r $(r > 1)$ and a V^{n-r} $(n - r > 1)$ which are called the decomposition spaces of $V^n = V^r \times V^{n-r}$. Throughout this paper each object denoted by a bar is assumed to be from $\bar{g}_{ab}$ and of V^r, and each object denoted by a star is formed from $g^*{}_{a'b'}$ and of V^{n-r}. From (3.1), we have

$$(3.2)\qquad \begin{gathered} g_{ab} = \bar{g}_{ab},\ g_{a'b'} = g^*{}_{a'b'},\ g^{ab} = \bar{g}^{ab}, \\ g^{a'b'} = g^{*a'b'},\ g_{aa'} = g^{aa'} = 0. \end{gathered}$$

The only non-zero Christoffel symbols of the second kind are as follows: A comma and a dot shall denote covariant differentiation in V^n, V^r, respectively. Hence, we obtain the following relations, [5]:

$$\begin{aligned} &R_{a'bcd} = R_{ab'cd'} = R_{ab'c'd'} = 0 \\ (3.3) \qquad &R_{abcd,a'} = R_{ab'cd',k} = R_{ab'cd',k'} = 0. \\ &R_{abcd} = \bar{R}_{abcd}, \; R_{a'b'c'd'} = R^*{}_{a'b'c'd'} \end{aligned}$$

(3.4) $R_{ab} = \bar{R}_{ab}$ $R_{a'b'} = R^*{}_{a'b'}$ $R_{ab,c} = \bar{R}_{ab,c}$ $R_{a'b',c'} = R^*{}_{a'b',c'}$.

(3.5) $$R = g^{ij}R_{ij} = \bar{g}^{ab}\bar{R}_{ab} + g^{*a'b'}R^*{}_{a'b'} = \bar{R} + R^*.$$

We suppose that $(WZS)_n$ $(n > 2)$ is decomposable with V^r and V^{n-r} as decomposition spaces. From (2.7), we get

(3.6) $$Z_{abcd,a'} = A_{a'}Z_{abcd} + B_a Z_{a'bcd} + B_b Z_{aa'cd} + D_c Z_{aba'd} + D_d Z_{abca'}.$$

In view of the fact that the curvature tensor and its covariant derivative are product tensors, the above equation takes the form

(3.7) $$A_{a'}Z_{abcd} = 0.$$

Similarly, we get

(3.8) $$A_a Z_{a'b'c'd'} = 0.$$

Since A_i is a non-zero vector, all its components can not vanish. Suppose $A_{a'} \neq 0$ for some $a' = a'_0$. Then, from (3.7), we get $Z_{abcd} = 0$ which means that the decomposition space V^r is concircular flat. If $A_a \neq 0$ for some $a = a_0$, then by similar argument, we get $Z_{a'b'c'd'} = 0$ which means that the decomposition space V^{n-r} is concircular flat.

We suppose that $Z_{a'b'c'd'} = 0$. Then, $Z_{abcd} \neq 0$ for some a, b, c, d, because by hypothesis $(WZS)_n$ is not concircular flat. Hence, from (3.7), we get $A_{a'} = 0$. Since A_i is non-zero vector, all its components cannot vanish. In this case, $A_a \neq 0$ for some $a = a_0$. Therefore, using the equation (1.4), we get

(3.9) $$Z_{abcd,e} = A_e Z_{abcd} + B_a Z_{abcd} + B_b Z_{aecd} + D_c Z_{abed} + D_d Z_{abce}.$$

These show that the part V^r is a $(WZS)_n$. Therefore, we can state the following:

Theorem 2. *If a $(WZS)_n$ is a decomposable space $V^r \times V^{n-r}$ $(r > 1, n - r > 1)$, then one of the decomposition spaces is concircular flat and the other is a weakly symmetric space.*

By the aid of the expression (1.4), we obtain

(3.10) $$Z_{a'bcd,a} = A_a Z_{a'bcd} + B_{a'} Z_{abcd} + B_b Z_{a'acd} + D_c Z_{a'bad} + D_d Z_{a'bca}$$

and also $Z_{a'bcd,a} = 0$. Hence, (3.10) reduces to

$$B_{a'} Z_{abcd} = 0. \tag{3.11}$$

Similarly, we get

$$B_a Z_{a'b'c'd'} = 0. \tag{3.12}$$

Since $B_i \neq 0$, these all components cannot vanish. Hence, we consider the following two cases:

CASE 1. Suppose $B_{a'} \neq 0$ for a fixed a'. Then, from (3.11), the relation $Z_{abcd} = 0$ holds for all a, b, c, d. That is

$$R_{abcd} - \frac{R}{n(n-1)}(g_{ad}g_{bc} - g_{ac}g_{bd}) = 0.$$

By (3.2), (3.3) and (3.5), this equation takes the form

$$\bar{R}_{abcd} - \frac{(\bar{R} + R^*)}{n(n-1)}(\bar{g}_{ad}\bar{g}_{bc} - \bar{g}_{ac}\bar{g}_{bd}) = 0. \tag{3.13}$$

Transvecting (3.13) by $\bar{g}^{bc}$ and $\bar{g}^{ad}$, we obtain

$$\bar{R} = kR^*, \ k = \frac{r(r-1)}{(n-r)(n+r-1)}. \tag{3.14}$$

Therefore, by (3.5), we get

$$R = (1 + k)R^*. \tag{3.15}$$

CASE 2. Suppose that $B_a \neq 0$ for a fixed a. Then, from (3.12), it follows that $Z_{a'b'c'd'} = 0$ for all a', b', c', d'. Similarly, as in Case 1, we obtain

$$Z_{a'b'c'd'} = {R^*}_{a'b'c'd'} - \frac{R}{n(n-1)}({g^*}_{a'd'}{g^*}_{b'c'} - {g^*}_{a'c'}{g^*}_{b'd'}) = 0. \tag{3.16}$$

Multiplying the equation (3.16) by $g^{*a'd'}, g^{*b'c'}$, we find

$$R^* = \frac{(\bar{R} + R^*)}{n(n-1)}(n - r - 1)(n - r). \tag{3.17}$$

From (3.13), we have

$$\bar{R} = \frac{(\bar{R} + R^*)}{n(n-1)} r(r-1). \tag{3.18}$$

By using (3.17) and (3.18), we get

$$\bar{R} = \frac{r(r-1)R^*}{(n-r)(n-r-1)}. \tag{3.19}$$

Then, with the help of (3.14) and (3.19), we obtain $R^* = 0$. By the aid of (3.5) and (3.13), we get $R = 0$ for $D \neq 0$. We can state the following:
Theorem 3. *For $B \neq 0$ or $D \neq 0$, a decomposable $(WZS)_n$ is of zero scalar curvature.*

4. (WZS_n) admitting a concurrent or a recurrent vector field

This section consists of two parts, the first deals with a $(WZS)_n$ admitting a concurrent vector field u^i given by, [6]

$$u^i_{,j} = \delta^i_j p, \tag{4.1}$$

where p is non-zero constant. We have from [2]

$$u^h R_{hijk} = 0 \quad u^h R_{hk} = 0 \quad pR_{lijk} + u^h R_{hijk,l} = 0. \tag{4.2}$$

By the aid of (4.2) and the expression

$$R^h_{l,h} = \frac{R_{,l}}{2} \tag{4.3}$$

where $R^h_l = g^{hm} R_{ml}$, we obtain

$$u^h R_{,h} = -2pR. \tag{4.4}$$

Let us suppose that R is constant, by using (4.4), we get

$$R = 0 \tag{4.5}$$

Using (1.3), (1.4) and (4.2), we get that u^h is not orthogonal to both D_h and B_h. On the other hand, from (1.3), (1.4) and (4.5) and the second Bianchi Identity, we find that u^h is not also orthogonal to A_h. Hence, we can state the following:
Theorem 4. *If a $(WZS)_n$ admits a non-null recurrent vector field u^i given by* (4.1) *and the scalar curvature of $(WZS)_n$ is constant, then the vector u_l is not orthogonal to the vectors A_l, B_l, D_l.*

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