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A RESULT ON VANISHING DERIVATIONS FOR COMMUTATORS ON RIGHT IDEALS

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Abstract: Let R be a prime ring of characteristic different from 2, δ and d non-zero derivations of R, I a non-zero right ideal of R, $S_4(x_1, \dots, x_4)$ the standard polynomial in 4 variables. Suppose that, for any $x, y \in I$, $\delta[d([x,y]), [x,y]] = 0$. If $S_4(I,I,I,I)I \neq 0$, then both δ and d are inner derivations induced respectively by the elements a, b, that is $\delta(x) = [a,x]$ and $d(x) = [b,x]$ for all $x \in R$, such that $aI = bI = 0$ and $ba = 0$.

This paper is included in a line of investigation concerning the relationship between the structure of a prime ring R and the behaviour of some derivation defined on R. The well known Posner's second theorem states that if R is a prime ring and d is a non-zero derivation of R such that $[d(r), r] \in Z(R)$, the center of R, for all $r \in R$, then R is commutative [17]. Later many authors obtained a number of results in this context by considering appropriate conditions on the subset $P(d,k,S) = \{[d(s), s]_k \ / s \in S\}$ for a suitable subset S of R. For example, if I is a non-zero two-sided ideal of R and $f(x_1, \dots, x_n)$ a multilinear polynomial, Lee and Lee proved that the condition $P(d,k,f(I)) = 0$ implies

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that either $f(x_1,\ldots,x_n)$ in central valued on R or $\mathrm{char}(R)=2$ and R satisfies the standard identity $S_4(x_1,\ldots,x_4)$ [11]. In a recent paper we proved that if I is a non-zero right ideal of R, $f(x_1,x_2)=[x_1,x_2]=$ $=x_1x_2-x_2x_1$ and $\mathrm{char}(R)\neq 2$, then the condition $aP(d,1,f(I))=0$, for $a\in R$, forces $aI=ad(I)=0$, unless the case when $S_4(I,I,I,I)I=$ $=0$ [7]. Let now d and δ be non-zero derivations of R and, as above, I a non-zero right ideal of R, $f(x_1,x_2)=[x_1,x_2]$. Here we will consider the situation when $\delta(P(d,1,f(I)))=0$ and we will prove the following

Theorem. *Let R be a prime ring of characteristic different from 2, d and δ non-zero derivations of R, I a non-zero right ideal of R. If, for any $x,y\in I$, $\delta([d([x,y]),[x,y]])=0$ then both δ and d are inner derivations induced respectively by the elements a,b, that is $\delta(r)=[a,r]$ and $d(r)=[b,r]$ for all $r\in R$, such that $aI=bI=ba=0$, unless the case when $S_4(I,I,I,I)I=0$.*

Before beginning the proof of our result, for the sake of completeness, we prefer to recall some basic notations, definitions and some easy consequences of the result of Kharchenko [10] about the differential identities on a prime ring R. We refer to [1, Ch. 7] for a complete and detailed description of the theory of generalized polynomial identities involving derivations.

Fact 1. We denote by Q the Martindale quotients ring of R and let $C=Z(Q)$ be the extended centroid of R ([1, Ch. 2]). It is well known that any derivation of a prime ring R can be uniquely extended to a derivation of its Martindale quotients ring Q, and so any derivation of R can be defined on the whole Q [1, p. 87]. Moreover, if R is a K-algebra we can assume that K is a subring of C.

Now, we denote by $\mathrm{Der}(Q)$ the set of all derivations on Q. By a derivation word we mean an additive map Δ of the form $\Delta=d_1d_2\ldots d_m$, with each $d_i\in\mathrm{Der}(Q)$. Then a differential polynomial is a generalized polynomial, with coefficients in Q, of the form $\Phi({}^{\Delta_j}x_i)$ involving non-commutative indeterminates x_i on which the derivations words Δ_j act as unary operations. The differential polynomial $\Phi({}^{\Delta_j}x_i)$ is said a differential identity on a subset T of Q if it vanishes for any assignment of values from T to its indeterminates x_i.

Fact 2. Let D_{int} be the C-subspace of $\mathrm{Der}(Q)$ consisting of all inner derivations on Q and let d and δ be two non-zero derivations on R. By [10, Th. 2] we have the following result (see also [15, Th. 1]):

Let R be a prime ring of characteristic different from 2, if d and δ are $C-$linearly indipendent modulo D_{int} and $\Phi({}^{\Delta_j}x_i)$ is a differential identity on R, where Δ_j are derivations words of the following form δ, d, δ^2, δd, d^2, then $\Phi(y_{ji})$ is a generalized polynomial identity on R, where y_{ji} are distinct indeterminates.

As a particular case, we have:

If d is a non-zero derivation on R and $\Phi(x_1,\ldots,x_n,{}^{d}x_1,\ldots,{}^{d}x_n,{}^{d^2}x_1,\ldots,{}^{d^2}x_n)$ is a differential identity on R, then one of the following holds:

(i) either $d \in D_{\text{int}}$

(ii) or R satisfies the generalized polynomial identity

$$\Phi(x_1,\ldots,x_n,y_1,\ldots,y_n,z_1,\ldots,z_n).$$

Fact 3. Denote by $T = Q *_C C\{X\}$ the free product over C of the C-algebra Q and the free C-algebra $C\{X\}$, with X a countable set consisting of non-commuting indeterminates $\{x_1,\ldots,x_n,\ldots\}$. The elements of T are called generalized polynomial with coefficients in Q. Moreover if I is a non-zero right ideal of R, then I, IR and IQ satisfy the same generalized polynomial identities with coefficients in Q. For more details about these objects we refer the reader to [1] and [5].

Fact 4. The assumption $S_4(I,I,I,I)I \neq 0$ is essential to the main result. For example, consider $R = \mathrm{End}_F(V)$, for F a field and $(V:F) \geq 3$ (possibly infinite), and let e_{ij} be the usual matrix unit in R. Let $I = (e_{11}+e_{22})R$, δ the inner derivation induced by the element $a = e_{22}$, that is $\delta(x) = [e_{22},x] = e_{22}x - xe_{22}$, d the inner derivation induced by the element e_{13}, that is $d(x) = [e_{13},x] = e_{13}x - xe_{13}$, for all $x \in R$. In this case, notice that $S_4(x_1,x_2,x_3,x_4)x_5$ is an identity for I, moreover

$$[e_{22},[e_{13},[(e_{11}+e_{22})x_1,(e_{11}+e_{22})x_2]]_2] = 0$$

for any $x_1,x_2 \in R$, but clearly $e_{22}I \neq 0$.

Lemma 1. *Suppose that δ is the inner derivation induced by $a \in R$ and d is the inner derivation induced by $b \in R$. If R does not satisfy any non-trivial generalized polynomial identity then the Theorem holds.*

Proof. Here we suppose that R does not satisfy any non-trivial generalized polynomial identity. The conclusion will be that $aI = bI = ba = 0$.

Denote $l_R(I)$ the left annihilator of I in R. Suppose first that $\{1,a,b\}$ are linearly C-independent modulo $l_R(I)$, that is $(\alpha a + \beta b + \gamma)I = 0$ if and only if $\alpha = \beta = \gamma = 0$. Since R is not a GPI-ring, a fortiori it cannot be a PI-ring. Thus, by [14, Lemma 3] there exists $x_0 \in I$ such that $\{x_0, ax_0, bx_0\}$ are linearly C-independent. In this case we have that

$$[a,[b,[x_0x_1,x_0x_2]]_2]$$

is a non-trivial generalized polynomial identity for R, a contradiction.

Therefore $\{1,a,b\}$ are linearly C-dependent modulo $l_R(I)$, that is there exist $\alpha,\beta,\gamma\in C$, not all zero, such that $(\alpha a+\beta b+\gamma)I=0$.

Note that if $\alpha=\beta=0$ then also $\gamma=0$. Hence either $\beta\neq 0$ or $\alpha\neq 0$. From now we divide the proof in three cases:

CASE 1. $\alpha=0$ and $\beta\neq 0$.

If $\gamma=0$ it follows that $\beta bI=0$, that is $bI=0$.

On the other hand, if $\gamma\neq 0$, by $(\beta b+\gamma)I=0$ we have that there exists $0\neq\lambda\in C$ such that $(b+\lambda)I=0$. Since b and $b'=b+\lambda$ induce the same inner derivation, we may replace b by b' in the basic hypothesis. Therefore, in any case we may suppose $bI=0$. Thus I satisfies the identity

$$[a,[b,[x_1,x_2]]_2]=[a,[x_1,x_2]^2b].$$

If $\{1,a\}$ are linearly C-independent modulo $l_R(I)$, by [14, Lemma 3], since R cannot satisfy any polynomial identity, we have that there exists $x_0\in I$ such that $\{x_0,ax_0\}$ are linearly C-independent. So the identity

$$[a,[x_0x_1,x_0x_2]^2b]$$

is a non-trivial generalized polynomial identity for R, a contradiction.

Hence there exists $\alpha'\in C$ such that $(a-\alpha')I=0$. As above, since a and $a'=a-\alpha'$ induce the same inner derivation, we may replace a by a'. Therefore, in any case we may suppose $aI=0$.

All these facts say that, for all $x_1,x_2\in I$,

$$0=[a,[b,[x_1,x_2]]_2]=-[x_1,x_2]^2ba.$$

By [6] we have that either $[x_1,x_2]x_3$ is an identity for I or $ba=0$. Since R is not GPI, the first conclusion cannot occur, therefore $aI=bI=$ $=ba=0$.

CASE 2. $\alpha\neq 0$ and $\beta=0$.

If $\gamma=0$ it follows that $\alpha aI=0$, that is $aI=0$.

On the other hand, if $\gamma\neq 0$, by $(\alpha a+\gamma)I=0$ we have that there exists $0\neq\lambda\in C$ such that $(a+\lambda)I=0$. Since a and $a'=a+\lambda$ induce the same inner derivation, again we may replace a by a'. Therefore, in any case we may suppose $aI=0$. Thus I satisfies

$$0 = [a, [b, [x_1, x_2]]_2] =$$
$$= ab[x_1, x_2]^2 - b[x_1, x_2]^2 a - [x_1, x_2]^2 ba + 2[x_1, x_2]b[x_1, x_2]a$$

and right multiplying this by $x_3 \in I$, I satisfies

$$ab[x_1, x_2]^2 x_3.$$

By using again [6], either $[x_1, x_2]^2 x_3$ is an identity for I or $abI = 0$. Since R is not GPI, the first conclusion cannot occur. So, by $abI = 0$, we have that I satisfies

$$[b, [x_1, x_2]]_2 a.$$

If $\{1, b\}$ are linearly C-independent modulo $l_R(I)$ then, again by [14, Lemma 3], since R is not PI, there exists $x_0 \in I$ such that $\{x_0, bx_0\}$ are linearly C-independent. In this case

$$[b, [x_0 x_1, x_0 x_2]]_2 a$$

is a non-trivial GPI for R, a contradiction.

If there exists $\beta' \in C$ such that $(b - \beta')I = 0$, since b and $b' = b - \beta'$ induce the same inner derivation, we may replace b by b'. So $bI = 0$, I satisfies

$$[a, [b, [x_1, x_2]]_2] = [x_1, x_2]^2 ba$$

and as above, since R is not GPI, we must have $ba = 0$.

CASE 3. $\alpha \neq 0$ and $\beta \neq 0$.

In this case there exist $\gamma', \beta' \in C$, with $\beta' \neq 0$, such that $ay = \gamma' y + \beta' by$, for all $y \in I$. Thus, for all $y_0 \in I$, R satisfies

$$\begin{aligned}[a, [b, [y_0 x_1, y_0 x_2]]_2] &= ab[y_0 x_1, y_0 x_2]^2 + \gamma'[y_0 x_1, y_0 x_2]^2 b + \\ &+ \beta' b[y_0 x_1, y_0 x_2]^2 b - 2\gamma'[y_0 x_1, y_0 x_2]b[y_0 x_1, y_0 x_2] - \\ &- 2\beta' b[y_0 x_1, y_0 x_2]b[y_0 x_1, y_0 x_2] - [b, [y_0 x_1, y_0 x_2]]_2 a. \end{aligned} \tag{A}$$

If $\{1, b\}$ are linearly C-independent modulo $l_R(I)$ then, again by [14, Lemma 3], since R is not PI, there exists $x_0 \in I$ such that $\{x_0, bx_0\}$ are linearly C-independent. In this case, for $y_0 = x_0$, the (A) is a non-trivial GPI for R, a contradiction.

If there exists $\beta'' \in C$ such that $(b - \beta'')I = 0$, since b and $b' = b - \beta''$ induce the same inner derivation, we may replace b by b'. Therefore $bI = 0$. It follows that I satisfies

$$a[x_1, x_2]^2 b - [x_1, x_2]^2 ba. \tag{B}$$

If $\{1, a\}$ are linearly C-independent modulo $l_R(I)$ then, as above, there exists $x_0 \in I$ such that $\{x_0, ax_0\}$ are linearly C-independent. In this case

$$a[x_0x_1, x_0x_2]^2b - [x_0x_1, x_0x_2]^2ba$$

is a non-trivial GPI for R, a contradiction.

On the other hand, if there exists $\alpha' \in C$ such that $(a-\alpha')I = 0$, since a and $a' = a - \alpha'$ induce the same inner derivation, we may replace a by a'. Hence $aI = 0$. It follows from (B) that I satisfies $[x_1, x_2]^2ba$ that is $ba = 0$, since $[x_1, x_2]x_3$ cannot be an identity for I (this follows again from [6]). $\Diamond$

Lemma 2. *Without loss of generality, in case δ is the inner derivation induced by the element a and d is the inner one induced by the element b, R is simple and equal to its own socle, $IR = I$ and $a, b \in I$.*

Proof. By Lemma 1, R is GPI and so Q has non-zero socle H with non-zero right ideal $J = IH$ [16]. Note that H is simple, $J = JH$ and J satisfies the same basic conditions as I, in view of [15]. Since $Ja \neq 0$ and $Jb \neq 0$, we may replace a and b respectively by $0 \neq c_1a$ and $0 \neq c_2b$, for some $c_1, c_2 \in J$. Now just replace R by H, I by J, a by c_1a, b by c_2b and we are done. $\Diamond$

Lemma 3. *Let $R = M_n(F)$ be the ring of $n \times n$ matrices over the field F of characteristic different from 2 and $n \geq 3$. Let d be a non-zero inner derivation of R, a a non-central element of R and I a non-zero right ideal of R. If $[d([r_1, r_2]), [r_1, r_2]]a = 0$, for all $r_1, r_2 \in I$, then d is induced by an element b such that $bI = ba = 0$.*

Proof. First say b an element of R which induces the derivation d. We denote e_{ij} the usual matrix unit with 1 in the (i,j)-entry and zero elsewhere and write $a = \sum a_{ij}e_{ij}$, $b = \sum b_{ij}e_{ij}$, with a_{ij} and b_{ij} elements of F. Moreover assume $I = eR$ for some $e = \sum_{i=1}^{t} e_{ii}$ and $t \geq 2$.

In what follows we first suppose that $(b - \beta)I \neq 0$, for any $\beta \in F$ to derive a contradiction.

Suppose that there exist $i \neq j$ such that $b_{ij} \neq 0$ $(j \leq t)$. Without loss of generality we replace b by $b_{ij}^{-1}(b - b_{jj})$ so that we assume $b_{ij} = 1$ and $b_{jj} = 0$.

Let $[x, y] = [e_{jj}, e_{ji}] = e_{ji} \in [I, I]$. Then $0 = [b, e_{ji}]_2a = -2e_{ji}be_{ji}a$. We have that, for all $i \neq j$ such that $b_{ij} \neq 0$, $e_{ii}a = 0$. So if $i \leq k \leq n$ such that $b_{kj} \neq 0$, for $k \neq j$, then $e_{kk}a = 0$.

Let now $k \neq j$ such that $b_{kj} = 0$. For $[x, y] = [e_{jj}, e_{ji} + e_{jk}] = e_{ji} + e_{jk} \in [I, I]$,

$$0 = [b, e_{ji} + e_{jk}]_2 a = -2(e_{ji} + e_{jk})b(e_{ji} + e_{jk})a$$

and left multiplying by $(e_{ii} + e_{ik})$

$$0 = (e_{ii} + e_{ik})(e_{ji} + e_{jk})b(e_{ji} + e_{jk})a = (e_{ii} + e_{ik})a = e_{ik}a.$$

Therefore the matrix a has just only one non-zero row, write $a = e_{jj}a$.

Let now $k < j$ and $k \neq j$. For $[x, y] = [e_{jj} + e_{kj}, e_{jk}] = e_{jk} - e_{jj}$ we have $[b, e_{jk} - e_{kj}]_2 a = 0$. Left multiplying by e_{ii} we get

$$0 = e_{ii}[b, e_{jk} - e_{kj}]_2 a = e_{ii}[b, e_{jk} - e_{kj}]_2 e_{jj} a = \\ = e_{ii} b e_{jj} a = (e_{ij} b_{ij})(e_{jj} a)$$

and from $b_{ij} \neq 0$ we have $e_{jj}a = 0$ that is $a = 0$.

Assume now $b_{ij} = 0$ for all $i \neq j$ and $j \leq t$. Since $(b - \beta)I \neq 0$, for $\beta \in F$, in this case there exist $1 \leq r, s \leq t$, with $r \neq s$, such that $b_{rr} \neq b_{ss}$. Replacing b with $(b_{rr} - b_{ss})^{-1}(b - b_{ss})$, we may assume that $b_{rr} = 1$ and $b_{ss} = 0$.

Let f be the F-automorphism of R defined by $f(x) = (1 - e_{rs})\cdot$ $\cdot x(1 + e_{rs})$. Thus we have that $f(x) \in I$, for all $x \in I$ and $[f(b), [x_1, x_2]]_2 f(a) = 0$, for all $x_1, x_2 \in I$. If $a \neq 0$ then $f(a) \neq 0$ and, as above, the (r, s)-entry of $f(b)$ is zero. On the other hand

$$f(b) = (1 - e_{rs})b(1 + e_{rs}) = b + b_{rr}e_{rs} - b_{ss}e_{rs}$$

that is $b_{rr} = b_{ss}$, a contradiction.

This means that there exists $\beta \in F$ such that $(b - \beta)I = 0$. Denote $b - \beta = p$. Since b and p induces the same inner derivation d, we have that $[p, [r_1, r_2]]_2 a = 0$ with $pI = 0$. In this case, by the assumption of this Lemma, we have

$$0 = [p, [x_1, x_2]]_2 a = [x_1, x_2]^2 pa.$$

Since $S_4(I, I, I, I)I \neq 0$, it follows that $[x_1, x_2]^2$ is not an identity for I and so, by [6], we conclude that $pa = 0$. $\Diamond$

Lemma 4. *Let R be a prime ring of characteristic different from 2, I a non-zero right ideal of R, d an inner derivation of R and $a \in R$. If $[d([r_1, r_2]), [r_1, r_2]]a = 0$, for all $r_1, r_2 \in I$, then d is induced by an element b such that $bI = ba = 0$, unless the case when $S_4(I, I, I, I)I =$ $= 0$.*

Proof. Suppose that $S_4(I, I, I, I)I \neq 0$. As a reduction of Lemma 1, we have that if R is not a GPI-ring, then we are done. Thus consider the only case when R satisfies a non-trivial generalized polynomial identity.

Thus the Martindale quotients ring Q of R is a primitive ring with non-zero socle $H = Soc(Q)$. H is a simple ring with minimal right ideals. Let D be the associated division ring of H, by [16] D is a simple central algebra finite dimensional over $C = Z(Q)$. Thus $H \otimes_C F$ is a simple ring with minimal right ideals, with F the central closure of C. Let b be an element of R which induces the derivation d. Moreover $[b, [x_1, x_2]]_2 a = 0$, for all $x_1, x_2 \in IH \otimes_C F$ (see for instance [5, Th. 2] and [12, Prop.]). Notice that if C is finite, we choose $F = C$.

Suppose that for all $\beta \in C$ there exists $c \in IH$ such that $(b - \beta)c \neq 0$. Denote $p = b - \beta$, so $pc \neq 0$, moreover b and p induce the same inner derivation. Since H is regular [9] there exists $g^2 = g \in IH$, such that $c \in IH$, and $e^2 = e \in H \otimes_C F$, such that

$$g, pg, gp, a, c, pc, cp \in e(H \otimes_C F)e \cong M_n(F) \quad \text{and} \quad n \geq 3.$$

Let $x_1, x_2 \in ge(H \otimes_C F)e$ and $a = eae \neq 0$, then

$$0 = e[p, [x_1, x_2]]_2 a = [epe, [x_1, x_2]]_2 eae.$$

By Lemma 3 $epege(H \otimes_C F)e = 0$, in particular $0 = epegc$ and hence $pc = 0$, a contradiction. This means that there exists $\beta \in C$ such that $p = b - \beta$ and $pI = 0$. Therefore I satisfies

$$0 = [p, [x_1, x_2]]_2 a = [x_1, x_2]^2 pa.$$

Since $S_4(I, I, I, I)I \neq 0$, $[x_1, x_2]^2$ is not an identity for I and we conclude (as in Lemma 3) $pa = 0$. $\Diamond$

Lemma 5. *Let $R = H$ be a regular ring, $I = IH$, and $p, q \in R$ such that $p[x_1, x_2]^2 q = 0$, for any $x_1, x_2 \in I$. If I does not satisfy $S_4(x_1, x_2, x_3, x_4)x_5$, then either $pI = 0$ or $q = 0$.*

Proof. Suppose that $pI \neq 0$. There exist $a_1, a_2, a_3, a_4, a_5, a_6 \in I$ such that $S_4(a_1, a_2, a_3, a_4)a_5 \neq 0$ and $pa_6 \neq 0$. Since $R = H$ is regular, hence there exists $g = g^2 \in R$ such that $gR = a_1R + a_2R + a_3R + a_4R + a_5R + a_6R$. Then $g \in IR = I$ and $a_i = ga_i$ for each $i = 1, \ldots, 6$. Consider now the simple Artinian ring gRg and notice that $(gRpg)[gx_1g, gx_2g]^2(gqRg) = 0$. Moreover $S_4(gR, gR, gR, gR)gR \neq 0$, because $S_4(a_1, a_2, a_3, a_4)a_5 \neq 0$, and $pg \neq 0$, because $pga_6 = pa_6 \neq 0$.

Let A be the subgroup of gRg generated by the polynomial $[gx_1g, gx_2g]^2$, then $(gRpg)x(gqRg) = 0$, for all $x \in A$. Since A is a non-central Lie ideal of gRg, it is well known that $[gRg, gRg] \subseteq A$, that is

$$(gRpg)[gx_1g, gx_2g](gqRg) = 0 \qquad \text{for all } x_1, x_2 \in R.$$

Let $U = [gx_1g, gx_2g](gqRg)$, so $(gRpg)U = 0$.

Since $(gRpg)[Ugx_1g, gx_2g](gqRg) = 0$, then $gRpgx_2gUgx_1gqRg = 0$. Moreover $pg \neq 0$ implies that either $gq = 0$ or $U = 0$. If this last case occurs, it follows that $gq = 0$, because gRg cannot be commutative. Hence in any case we have $gq = 0$.

Let $r \in R$ and $f = g + gr(1-g)$, so $f^2 = f \in I$, $fg = g$ and $gR = fR$. Since $S_4(gR, gR, gR, gR)gR \neq 0$ and $pg \neq 0$, by calculation it follows that also $S_4(fR, fR, fR, fR)fR \neq 0$ and $pf \neq 0$.

Moreover $(fRpf)[fx_1f, fx_2f]^2(fqRf) = 0$ and, following the same above argument, we have $fq = 0$, that is $0 = (g + gr(1-g))q = grq$. By the arbitrarity of $r \in R$, and $g \neq 0$, we get $q = 0$. $\diamondsuit$

Lemma 6. *Let both δ and d be inner derivations induced respectively by the elements a and b.* (i) *If $aI = 0$, then the Theorem holds.* (ii) *If $bI = 0$, then the Theorem holds.*

Proof. Suppose $S_4(I, I, I, I)I \neq 0$.

In view of Lemma 2, R is a simple GPI-ring and equal to its own socle, $R = H$, $IR = I$ and $a, b \in I$.

(i) Since $aI = 0$, then I satisfies

$$-b[x_1, x_2]^2a - [x_1, x_2]^2ba + 2[x_1, x_2]b[x_1, x_2]a$$

that is

$$[b, [x_1, x_2]]_2a = 0$$

and we end up by Lemma 4.

(ii) Since $bI = 0$, then $ba = 0$ and I satisfies $a[x_1, x_2]^2b$. In this condition, since $b \neq 0$, we are done by Lemma 5. $\diamondsuit$

Lemma 7. *Let both δ and d be inner derivations induced respectively by a and b elements of R. Then the Theorem holds.*

Proof. In view of Lemma 2 R is a simple GPI-ring and equal to its own socle, $R = H$, $IR = I$ and $a, b \in I$. Since if $aI = 0$ we conclude by Lemma 6, then we may assume $aI \neq 0$ and show that under this assumption, a contradiction occurs.

Suppose there exist $a_1, a_2, a_3, a_4, a_5, a_6 \in I$ such that

$$S_4(a_1, a_2, a_3, a_4)a_5 \neq 0 \quad \text{and} \quad aa_6 \neq 0.$$

Since $R = H$ is regular, hence there exists $e = e^2 \in R$ such that $eR = a_1R + a_2R + a_3R + a_4R + a_5R + a_6R$. Then $e \in IR = I$ and $a_i = ea_i$ for each $i = 1, \ldots, 6$.

Let $x \in R$, $[e, ex(1-e)] = ex(1-e) \in [I, I]$. Thus

$$0 = [a, [b, ex(1-e)]_2] = 2(-aex(1-e)bex(1-e) + ex(1-e)bex(1-e)a)$$

and left multiplying by $(1-e)$

$$(1-e)aex(1-e)bex(1-e)=0$$

and it follows easily that either $(1-e)ae=0$ or $(1-e)be=0$.

Here our purpose is to show that $(1-e)ae=0$ if and only if $(1-e)be=0$.

Suppose first that $(1-e)ae=0$. For any $x, y \in R$

$$0=(1-e)[a,[b,[ex,ey]]_2]e=$$

[1] $$=(1-e)ab[ex,ey]^2e-(1-e)b[ex,ey]^2ae=$$

$$=(1-e)ab[exe,eye]^2e-(1-e)b[exe,eye]^2eae.$$

Since eRe does not satisfy $S_4(x_1,x_2,x_3,x_4)$, $[exe,eye]^2$ cannot be central in eRe. Let A be the subgroup of eRe generated by the polynomial $[ex_1e,ex_2e]^2$, then $(1-e)abx-(1-e)bxeae=0$, for any $x \in A$. Since A is a non-central Lie ideal of eRe, it is well known that $[eRe,eRe] \subseteq A$, that is for all $x, y \in R$.

[2] $$(1-e)abe[ex,ey]e-(1-e)b[ex,ey]eae.$$

Let now $z \in R$:

$$0=(1-e)ab[ex,eyez]e-(1-e)b[ex,eyez]eae=$$

$$=(1-e)ab[ex,ey]eze+(1-e)abey[ex,ez]e-$$

$$-(1-e)b[ex,ey]ezeae-(1-e)bey[ex,ez]eae.$$

In particular we choose $ey=[ex,ez]$. Since from [1]

$$(1-e)ab[ex,ez]^2e-(1-e)b[ex,ez]^2ae=0$$

we have

$$(1-e)ab[ex,[ex,ez]]eze-(1-e)b[ex,[ex,ez]]ezeae=0$$

that is

$$(1-e)ab[ez,ex]_2eze-(1-e)b[ez,ex]_2ezeae=0.$$

From this, since $[eze,exe]_2$ cannot be central in eRe, the subgroup B generated by the polynomial $[eze,exe]_2$ contains the non-central Lie ideal $[eRe,eRe]$, and as above, it follows that

[3] $$0=(1-e)ab[eze,exe]eze-(1-e)b[eze,exe]ezeae.$$

Now rewrite equation [1] as follows:

[2′] $$(1-e)abe[eze,exe]e-(1-e)b[eze,exe]eae.$$

Right multiplying equation [1’] by ze we obtain

[3′] $(1-e)abe[eze, exe]eze - (1-e)b[eze, exe]eaeze.$

By equations [3] and [3'] it follows

$$(1-e)b[eze, exe][eae, eze] = 0$$

and for $x = ae$

$$0 = (1-e)b[eae, eze]^2 = (1-e)be[ae, ez]^2 e$$

and a fortiori $0 = (1-e)be[ae, ez]^3$. Denote by $h(x) = [ae, x]$, the inner derivation induced by ae, then $0 = (1-e)beh(ex)^3$, for all $x \in R$.

Notice that if $ae = 0$ then we get the contradiction $0 = aea_6 = aa_6 \neq 0$. Thus, by [4] it follows that either $[ae, eR] = 0$ or $(1-e)beh(eR) = 0$. In case $[ae, eR] = 0$, for any $x \in R$, we have $0 = [ae, ex(1-e)] = aex(1-e) - ex(1-e)ae = aex(1-e)$, and by $e \neq 1$, it follows again the contradiction $ae = 0$. Therefore must be $(1-e)beh(eR) = 0$, that is, for all $x \in R$,

[4] $0 = (1-e)be[ae, ex] = (1-e)beaex - (1-e)bexae.$

In particular, for $x = y(1-e)$, $y \in R$, $0 = (1-e)beaey(1-e)$, which implies $0 = (1-e)beae = (1-e)bae$. Again by equation [4]

$$0 = (1-e)be[ae, ex] = (1-e)beaex - (1-e)bexae = -(1-e)bexae$$

for all $x \in R$, i.e. $(1-e)be = 0$.

Suppose now that $(1-e)be = 0$. From the main assumption, for all $x, y \in R$

$$0 = (1-e)[a, [b, [ex, ey]]_2] = (1-e)a[b, [ex, ey]]_2.$$

Since $S_4(eR, eR, eR, eR)e \neq 0$, by the result in [7] we conclude that $(1-e)ae = 0$.

Therefore, if $aI \neq 0$ and $S_4(I, I, I, I)I \neq 0$, we always have $(1-e)ae = (1-e)be = 0$. Recall that δ is the inner derivation induced by a and d the inner one induced by b. Thus both δ and d are defined in I, $\delta(I) \subseteq I$ and $d(I) \subseteq I$. Let $\overline{I} = \frac{I}{I \cap l_R(I)}$, where $l_r(I)$ is the left annihilator of I in R. For all $x, y \in I$ we have $\overline{d(x)} = \overline{d}(\overline{x})$ and $\overline{\delta(x)} = \overline{\delta}(\overline{x})$. $\overline{I}$ is a prime ring of characteristic different from 2, which satisfies the main assumption:

$$\overline{\delta}[\overline{d}([\overline{x}, \overline{y}]), [\overline{x}, \overline{y}]] = 0$$

for all $\overline{x}, \overline{y} \in \overline{I}$. By [8], since $[\overline{I}, \overline{I}] \neq \overline{0}$, that is $[I, I]I \neq 0$, we have that either $\overline{\delta} = \overline{0}$ or $\overline{d} = \overline{0}$. So we have that either $\delta(I)I = 0$ or $d(I)I = 0$, that is either $[a, I]I = 0$ or $[b, I]I = 0$.

If $[a,I]I=0$, by [2, Lemma] there exists $\alpha\in C$ such that $(a-\alpha)I=0$. Denote $a-\alpha=q$, then $qI=0$ and $qb=0$. Moreover the inner derivation induced by q satisfies the same condition of the inner one induced by a, hence, for all $x,y\in I$

$$0=[q,[b,[x,y]]_2]=[b,[x,y]]_2q$$

and, by Lemma 4, there exists $\beta\in C$ such that $p=b-\beta$ and $pI==pq=0$. Thus we are done (see (ii) of Lemma 6).

In the either case, that is $[b,I]I=0$, as above there exists $\alpha\in C$ such that $(b-\alpha)I=0$. Denote $a-\alpha=p$, then $pI=0$ and $pa=0$. Moreover the inner derivation induced by p satisfies the same condition of the inner one induced by b, hence, for all $x,y\in I$

$$0=[a,[p,[x,y]]_2]=a[x,y]^2p$$

and by Lemma 5, since $aI\neq 0$, we get $p=0$, that is $b=\alpha\in C$, a contradiction. ◊

Finally we premise the following:

Lemma 8. *Let R be a prime ring of characteristic different from 2, I a non-zero right ideal of R, $f(x_1,\dots,x_n)$ a multilinear polynomial on R. If for any $i=1,\dots,n$, $[f(r_1,\dots,z_i,\dots,r_n),f(r_1,\dots,r_n)]=0$, for all $r_1,\dots,r_n,z_i\in I$, then $CI=eRC$ for some idempotent element e in the socle of RC and $f(x_1,\dots,x_n)$ is central valued on $eRCe$.*

Proof. Let $t,r_1,\dots,r_n\in I$, such that $[t,I]\neq 0$. By our assumption

$$[f(r_1,\dots,[t,r_i],\dots,r_n),f(r_1,\dots,r_n)]=0$$

for all $i=1,\dots,n$. Therefore

$$[t,f(r_1,\dots,r_n)]_2=$$

$$\Big[\sum_i f(r_1,\dots,[t,r_i],\dots,r_n),f(r_1,\dots,r_n)\Big]=0.$$

Hence, by [13, Th. 2], we get the conclusion. ◊

Proof of the Theorem. Since $S_4(I,I,I,I)I\neq 0$, the polynomial $[x_1,x_2]$ is not central in I. By our assumption, I satisfies the differential identity

$$\delta([d([x_1,x_2]),[x_1,x_2]]).$$

Let $a\in I$, then R satisfies the generalized differential identity

$$[\delta d([ax_1,ax_2]),[ax_1,ax_2]]+[d([ax_1,ax_2]),\delta([ax_1,ax_2])]=$$
$$=[[\delta d(a)x_1+d(a)\delta(x_1)+\delta(a)d(x_1)+a\delta d(x_1),ax_2],[ax_1,ax_2]]+$$
$$+[[ax_1,\delta d(a)x_2+d(a)\delta(x_2)+\delta(a)d(x_2)+a\delta d(x_2)],[ax_1,ax_2]]+$$

$$+[[d(a)x_1+ad(x_1),ax_2],[\delta(a)x_1+a\delta(x_1),ax_2]]+$$
$$+[[d(a)x_1+ad(x_1),ax_2],[ax_1,\delta(a)x_2+a\delta(x_2)]]+$$
$$+[[ax_1,d(a)x_2+ad(x_2)],[\delta(a)x_1+a\delta(x_1),ax_2]]+$$
$$+[[ax_1,d(a)x_2+ad(x_2)],[ax_1,\delta(a)x_2+a\delta(x_2)]]\,.$$

CASE 1. δ and d are linearly C-independent modulo D_{int}.

In this case, by Kharchenko's theorem, R satisfies the identity

$$[[\delta d(a)x_1+d(a)y_1+\delta(a)z_1+at_1,ax_2],[ax_1,ax_2]]+$$
$$+[[ax_1,\delta d(a)x_2+d(a)y_2+\delta(a)z_2+at_2],[ax_1,ax_2]]+$$
$$+[[d(a)x_1+az_1,ax_2],[\delta(a)x_1+ay_1,ax_2]]+$$
$$+[[d(a)x_1+az_1,ax_2],[ax_1,\delta(a)x_2+ay_2]]+$$
$$+[[ax_1,d(a)x_2+az_2],[\delta(a)x_1+ay_1,ax_2]]+$$
$$+[[ax_1,d(a)x_2+az_2],[ax_1,\delta(a)x_2+ay_2]]\,.$$

In particular R satisfies the blended components

$$[[at_1,ax_2],[ax_1,ax_2]] \quad \text{and} \quad [[ax_1,at_2],[ax_1,ax_2]].$$

Therefore R satisfies some non-trivial GPI, then Q is primitive and H is regular. As remarked in Lemma 2, when R is GPI-ring, we may replace H by R. Moreover, since R is a regular ring, we may assume $I=eR$ for some $e^2=e\in IR$. Thus R satisfies

$$[[et_1,ex_2],[ex_1,ex_2]] \quad \text{and} \quad [[ex_1,et_2],[ex_1,ex_2]]$$

that is I satisfies

$$[[t_1,x_2],[x_1,x_2]] \quad \text{and} \quad [[x_1,t_2],[x_1,x_2]].$$

By Lemma 8 we have the contradiction that $[x_1,x_2]$ is central in I.

CASE 2. δ and d are linearly C-dependent modulo D_{int}.

Now there exist γ_1 and γ_2 in C such that $\gamma_1\delta+\gamma_2 d\in D_{int}$ and by Lemma 7, at most one of the two derivations can be inner.

Suppose $\gamma_1=0$ and $\gamma_2\neq 0$. In this case for some element $q\in Q$, $d=d_q$ is the inner derivation induced by q and δ is an outer derivation. By the assumptions, $\delta([q,[x_1,x_2]]_2)$ is a differential identity for I. Thus, for $a\in I$ $\delta([q,[ax_1,ax_2]]_2)$ is a differential identity for R. We have that

$$\delta([q,[ax_1,ax_2]]_2)=[\delta(q),[ax_1,ax_2]]_2+$$
$$+[q,[\delta(a)x_1+a\delta(x_1),ax_2]+[ax_1,\delta(a)x_2+a\delta(x_2)],[ax_1,ax_2]]+$$
$$+[[q,[ax_1,ax_2]],[\delta(a)x_1+a\delta(x_1),ax_2]+[ax_1,\delta(a)x_2+a\delta(x_2)]]\,.$$

As above, by Kharchenko's result, R satisfies the GPI

$$[\delta(q),[ax_1,ax_2]]_2+$$
$$+[q,[\delta(a)x_1+ay_1,ax_2]+[ax_1,\delta(a)x_2+ay_2)],[ax_1,ax_2]]+$$
$$+[[q,[ax_1,ax_2]],[\delta(a)x_1+ay_1,ax_2]+[ax_1,\delta(a)x_2+ay_2)]].$$

In particular R satisfies the blended component

$$[[q,[ay_1,ax_2]],[ax_1,ax_2]]+[[q,[ax_1,ax_2]],[ay_1,ax_2]].$$

Hence $2[q,[ax_1,ax_2]]_2$ is an identity for R. R is again a ring satisfying a non-trivial GPI. Then $I=eR$ for $e^2=e\in I=IR$, $2[q,[ex_1,ex_2]]_2$ is an identity for R and $2[q,[x_1,x_2]]_2$ is an identity for I. Since $q\notin C$, this implies that $[x_1,x_2]$ is central in I [13], a contradiction.

Suppose now $\gamma_2=0$ and $\gamma_1\neq 0$. Then for some non-central element $q\in Q$, $\delta=d_q$ is the inner derivation induced by q and d is an outer derivation. In this case, for $a\in I$, R satisfies the differential identity

$$[q,[d([ax_1,ax_2]),[ax_1,ax_2]]]=$$
$$=[q,[[d(a)x_1+ad(x_1),ax_2]+[ax_1,d(a)x_2+ad(x_2)],[ax_1,ax_2]]]$$

and as above, using the Kharchenko's theorem, R satisfies the following generalized polynomial identities

$$[q,[[ax_1,ay_2],[ax_1,ax_2]]] \quad \text{and} \quad [q,[[ay_1,ax_2],[ax_1,ax_2]]].$$

Once again R is a non-trivial GPI-ring, then we assume $eR=I=IR$ for some idempotent $e\in IR$ and I satisfies the identities

$$[q,[[x_1,y_2],[x_1,x_2]]] \quad \text{and} \quad [q,[[y_1,x_2],[x_1,x_2]]].$$

Clearly we may assume that both the polynomials

$$[[x_1,y_2],[x_1,x_2]] \quad \text{and} \quad [[y_1,x_2],[x_1,x_2]]$$

are not central in I, because $S_4(I,I,I,I)I\neq 0$. By [3] one of the following holds:

i) either q centralizes $[I,I]$, in the case I satisfies some polynomial identities;

ii) or q centralizes $[I_0,I]$, for some $I_0\subseteq I$ right ideal of R, in case I_0 and I do not satisfy any polynomial identity.

Notice that the first case cannot occur, because if not, from [13, Th. 6] and since $q\notin C$, should follow the contradiction $[[I,I],I]=0$. On the other hand, if $[q,[I_0,I_0]]=0$, again by [13] and $q\notin C$, we have the contradiction that I_0 satisfies the polynomial identity $[[x_1,x_2],x_3]$.

Finally we assume that both γ_1 and γ_2 are not zero. So $\delta=\gamma d+$ $+d_q$, with $0\neq\gamma\in C$ and $q\in Q$. Therefore, for $a\in I$, R satisfies the differential identity

$$(\gamma\delta + d_q)[d([ax_1, ax_2]), [ax_1, ax_2]] =$$
$$= (\gamma\delta)[d([ax_1, ax_2]), [ax_1, ax_2]] + [q, [d([ax_1, ax_2]), [ax_1, ax_2]].$$

Suppose that d is an outer derivation. In this case R satisfies the differential identity

$$\gamma\big(\big[[d^2(a)x_1 + d(a)d(x_1) + d(a)d(x_1) + ad^2(x_1), ax_2], [ax_1, ax_2]\big] +$$
$$+ \big[[ax_1, d^2(a)x_2 + d(a)d(x_2) + d(a)d(x_2) + ad^2(x_2)], [ax_1, ax_2]\big] +$$
$$+ [[d(a)x_1 + ad(x_1), ax_2], [d(a)x_1 + ad(x_1), ax_2]] +$$
$$+ [[d(a)x_1 + ad(x_1), ax_2], [ax_1, d(a)x_2 + ad(x_2)]] +$$
$$+ [[ax_1, d(a)x_2 + ad(x_2)], [d(a)x_1 + ad(x_1), ax_2]] +$$
$$+ [[ax_1, d(a)x_2 + ad(x_2)], [ax_1, d(a)x_2 + ad(x_2)]]\big) +$$
$$+ [q, [[d(a)x_1 + ad(x_1), ax_2] + [ax_1, d(a)x_2 + ad(x_2)], [ax_1, ax_2]]].$$

Thus the Kharchenko's theorem provides that

$$\gamma\big(\big[[d^2(a)x_1 + d(a)y_1 + d(a)y_1 + az_1, ax_2], [ax_1, ax_2]\big] +$$
$$+ \big[[ax_1, d^2(a)x_2 + d(a)y_2 + d(a)y_2 + az_2], [ax_1, ax_2]\big] +$$
$$+ [[d(a)x_1 + ay_1, ax_2], [d(a)x_1 + ay_1, ax_2]] +$$
$$+ [[d(a)x_1 + ay_1, ax_2], [ax_1, d(a)x_2 + ay_2]] +$$
$$+ [[ax_1, d(a)x_2 + ay_2], [d(a)x_1 + ay_1, ax_2]] +$$
$$+ [[ax_1, d(a)x_2 + ay_2], [ax_1, d(a)x_2 + ay_2]]\big) +$$
$$+ [q, [[d(a)x_1 + ay_1, ax_2] + [ax_1, d(a)x_2 + ay_2], [ax_1, ax_2]]]$$

is a polynomial identity for R. Hence R satisfies the blended components

$$[[az_1, ax_2], [ax_1, ax_2]] \quad \text{and} \quad [[ax_1, az_2], [ax_1, ax_2]].$$

As remarked in Case 1, using Lemma 8, this implies the contradiction that $[x_1, x_2]$ is central in I.

Finally, if d is Q-inner, then δ is also Q-inner and we obtain the required conclusion by Lemma 7. ◊

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