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PÁL-TYPE INTERPOLATION AND QUADRATURE FORMULAE ON LAGUERRE ABSCISSAS

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Abstract: The aim of this paper is to study a modified Pál-type interpolation problem on Laguerre abscissas. We prove the regularity of the problem and we give the explicit formulae of the interpolation. As an application we obtain Birkhoff-type quadrature formulae which have higher degree of precision than the precision of the interpolational quadrature formulae in general.

1. Introduction

The (0,2)-interpolation and the Pál-type interpolation, as special (lacunary) Birkhoff-interpolation problems were studied by several authors when the nodes are the zeros of the classical orthogonal polynomials. On the infinite interval $[0, \infty)$ with Laguerre abscissas the (0,2) interpolation (cf. [1], [2], [3]), and the Pál-type interpolation (cf. [4]) were of special interest.

In this paper we study the following interpolation problem: On the interval $[0, \infty)$ let $\{x_i\}_{i=0}^{n}$ and $\{x^*{}_i\}_{i=1}^{n}$ be two sets of interscaled nodal points:

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(1) $$0 \le x_0 < x^*{}_1 < x_1 < \cdots < x_{n-1} < x^*{}_n < x_n < \infty.$$

For $k \ge 1$ fixed integer, find a polynomial $R_m(x)$ of minimal degree satisfying the (0;1) interpolation conditions

(2) $$R_m(x_i) = y_i, \quad R'_m(x^*{}_i) = y'_i \qquad (i = 1, \dots, n)$$

with Hermite-type boundary conditions

(3) $$R_m^{(j)}(x_0) = y_0^{(j)} \qquad (j = 0, \dots, k),$$

where y_i, y'_i and $y_0^{(j)}$ are arbitrary real numbers.

In Sec. 2 we show that, if $\{x_i\}_{i=1}^n$ and $\{x^*{}_i\}_{i=1}^n$ are the zeros of the Laguerre polynomials $L_n^{(k)}(x)$ and $L_n^{(k-1)}(x)$, respectively, and $x_0 = 0$, then the problem is regular (there exists a unique polynomial $R_m(x)$ of degree $2n + k$ satisfying the above conditions). (Here $L_n^{(k)}(x)$ denotes the Laguerre polynomial of degree n with the parameter k.)

Using the identities (cf. (5.1.13) and (5.1.14) in [5])

(4) $$L_n^{(k)}(x) = L_n^{(k+1)}(x) - L_{n-1}^{(k+1)}(x)$$

and

(5) $$xL_n^{(k)\prime}(x) = nL_n^{(k)}(x) - (n+k)L_{n-1}^{(k)}(x)$$

we obtain

(6) $$\frac{d}{dx}\left[x^k L_n^{(k)}(x)\right] = (n+k)x^{k-1}L_n^{(k-1)}(x).$$

It is known that $L_n^{(\alpha)}(x)$ $(\alpha > -1)$ has n distinct real roots in $(0, \infty)$, hence applying the Rolle theorem and (6) we obtain that the zeros of $L_n^{(k)}(x)$ and $L_n^{(k-1)}(x)$ form the interscaled system (1). In Pál-type interpolation the function values are prescribed at the zeros of $\omega_n(x) = (x - x_1)\dots(x - x_n)$, while the derivative values are prescribed at the zeros of $\omega'_n(x)$. Hence the interpolational polynomial $R_m(x)$ is a modified Pál-type interpolational polynomial with $\omega_{n+k}(x) = x^k L_n^{(k)}(x)$.

In Sec. 2 we construct the fundamental polynomials and we prove the existence and uniqueness of the interpolational polynomial. In Sec. 3 we derive quadrature formulae for the integration of $f(x)$ on $[0, \infty)$ with respect to the weight function $\varrho(x) = e^{-x}$.

2. The fundamental polynomials

Let

$$0 = x_0 < x^*_1 < x_1 < \cdots < x_{n-1} < x^*_n < x_n < \infty. \tag{7}$$

be given, where $\{x_i\}_{i=1}^n$ and $\{x^*_i\}_{i=1}^n$ are the zeros of the Laguerre polynomials $L_n^{(k)}(x)$ and $L_n^{(k-1)}(x)$, respectively. Let us denote by $\ell_j(x)$ and $\ell^*_j(x)$ the fundamental polynomials of Lagrange interpolation on these nodal points, that is

$$\ell_j(x) = \frac{L_n^{(k)}(x)}{L_n^{(k)\prime}(x_j)(x-x_j)}, \qquad \ell^*_j(x) = \frac{L_n^{(k-1)}(x)}{L_n^{(k-1)\prime}(x^*_j)(x-x^*_j)}. \tag{8}$$

and so

$$\ell_j(x_i) = \delta_{i,j}, \qquad \ell^*_j(x^*_i) = \delta_{i,j}.$$

Lemma 1. *For k and n positive integers, on the nodal points (7) the fundamental polynomials of the interpolational problem in (1)–(3) are*

$$\begin{aligned} A_j(x) = {} & \frac{1}{x_j^{k+1} L_n^{(k-1)}(x_j)} \Bigg[x^{k+1}\ell_j(x) L_n^{(k-1)}(x) - \\ & - \frac{x^k L_n^{(k)}(x)}{L_n^{(k)\prime}(x_j)} \int_0^x \frac{t L_n^{(k-1)\prime}(t) - x_j L_n^{(k-1)}(t)}{t-x_j} dt \Bigg] \quad (j=1,\ldots,n), \end{aligned} \tag{9}$$

$$B_j(x) = \frac{x^k L_n^{(k)}(x)}{(x^*_j)^k L_n^{(k)}(x^*_j)} \int_0^x \ell^*_j(t)dt \qquad (j=1,\ldots,n), \tag{10}$$

$$\begin{aligned} C_j(x) = {} & p_j(x) x^j L_n^{(k)}(x) L_n^{(k-1)}(x) + x^k L_n^{(k)}(x) \times \\ & \times \left[c_j - \int_0^x \frac{L_n^{(k-1)\prime}(t) p_j(t) + q_j(t) L_n^{(k-1)}(t)}{t^{k-j}} dt \right] \\ & (j = 0, \ldots, k-1), \end{aligned} \tag{11}$$

and

$$C_k(x) = \frac{1}{k! L_n^{(k)}(0)} x^k L_n^{(k)}(x), \tag{12}$$

where $p_j(x)$ and $q_j(x)$ are polynomials of degree at most $k-j-1$, determined by (19) and (22), and the constants c_j are defined in (20). The polynomials $A_j(x)$, $B_j(x)$ and $C_j(x)$ are of degree at most $2n+k$.

Proof. From (4) and

$$L_n^{(k-1)'}(x) = -L_{n-1}^{(k)}(x) \tag{13}$$

(cf. (5.1.14) in [5]) we get by substituting $x = x_j$

$$L_n^{(k-1)'}(x_j) = L_n^{(k-1)}(x_j) \qquad (j = 1, \dots, n), \tag{14}$$

hence the integrand in (9) is a polynomial and $A_j(x)$ is of degree $2n+k$.

On using (6) and (8) it is easy to verify that the polynomials $A_j(x)$ $(j = 1, \dots, n)$ satisfy the equations

$$\begin{cases} A_j(x_i) &= \delta_{i,j}, \quad A_j'(x^*{}_i) = 0 \qquad (i = 1 \dots, n), \\ A_j^{(l)}(0) &= 0 \qquad (l = 0, \dots, k), \end{cases} \tag{15}$$

the polynomials $B_j(x)$ $(j = 1, \dots, n)$ satisfy the equations

$$\begin{cases} B_j(x_i) &= 0, \quad B_j'(x^*{}_i) = \delta_{i,j} \qquad (i = 1 \dots, n), \\ B_j^{(l)}(0) &= 0 \qquad (l = 0, \dots, k), \end{cases} \tag{16}$$

and the polynomial $C_k(x)$ fulfills the equations

$$\begin{cases} C_k(x_i) &= 0, \quad C_k'(x^*{}_i) = 0 \qquad (i = 1 \dots, n), \\ C_k^{(l)}(0) &= \delta_{l,k} \qquad (l = 0, \dots, k). \end{cases} \tag{17}$$

Now for fixed $j \in \{0, 1, \dots, k-1\}$ we will find the polynomial $C_j(x)$ in the form

$$C_j(x) = p_j(x) x^j L_n^{(k)}(x) L_n^{(k-1)}(x) + x^k L_n^{(k)}(x) g_n(x), \tag{18}$$

where $p_j(x)$ and $g_n(x)$ are polynomials of degree $k-j-1$ and n, respectively. It is clear that $C_j^{(l)}(0) = 0$ for $l = 0, \dots, j-1$, and because of $L_n^{(k)}(x_i) = 0$ we have $C_j(x_i) = 0$ for $i = 1, \dots, n$.

The coefficients of the polynomial $p_j(x)$ are determined by the system

$$C_j^{(l)}(0) = \frac{d^l}{dx^l}\left[p_j(x) x^j L_n^{(k)}(x) L_n^{(k-1)}(x)\right]_{x=0} = \delta_{j,l} \quad (l = j, \dots, k-1). \tag{19}$$

From the equation $C_j^{(k)}(0) = 0$ we get

$$(20) \qquad c_j := g_n(0) = \frac{-1}{k! L_n^{(k)}(0)} \frac{d^k}{dx^k} \left[p_j(x) x^j L_n^{(k)}(x) L_n^{(k-1)}(x) \right]_{x=0}.$$

Using (6) and $L_n^{(k-1)}(x^*{}_i) = 0$, from the condition $C_j'(x^*{}_i) = 0$ we get

$$g_n'(x^*{}_i) = -(x^*{}_i)^{j-k} L_n^{(k-1)'}(x^*{}_i) p_j(x^*{}_i)$$

and we can define $g_n'(x)$ as it follows

$$(21) \qquad g_n'(x) = -\frac{L_n^{(k-1)'}(x) p_j(x) + q_j(x) L_n^{(k-1)}(x)}{x^{k-j}}$$

where $q_j(x)$ is a polynomial of degree $k-j-1$. The function $g_n'(x)$ will be a polynomial if and only if

$$(22) \quad \frac{d^s}{dx^s} \left[L_n^{(k-1)'}(x) p_j(x) + q_j(x) L_n^{(k-1)}(x) \right]_{x=0} = 0 \quad (s = 0, \ldots, k-j-1).$$

The coefficients of $q_j(x)$ are determined uniquely by these equations. Now integrating (21) we get $g_n(x) = g_n(0) + \int_0^x g_n'(t)dt$, where substituting (20) we obtain (11). Hence the polynomials $C_j(x)$ $(j = 0, \ldots, k-1)$ satisfy the equations

$$(23) \qquad \begin{cases} C_j(x_i) & = 0, \quad C_j'(x^*{}_i) = 0 \qquad (i = 1 \ldots, n), \\ C_j^{(l)}(0) & = \delta_{j,l} \qquad (l = 0, \ldots, k), \end{cases}$$

which completes the proof. ◊

Theorem 1. *For k and $n \geq 1$ fixed integers, if $\{y_i\}_{i=1}^n$, $\{y_i'\}_{i=1}^n$ and $\{y_0^{(j)}\}_{j=0}^k$ are arbitrary real numbers, then on the nodal points (7) there exists a unique polynomial $R_m(x)$ of degree at most $2n+k$ satisfying the equations (2) and (3). The polynomial $R_m(x)$ can be written in the form*

$$(24) \qquad R_m(x) = \sum_{j=1}^n A_j(x) y_j + \sum_{j=1}^n B_j(x) y_j' + \sum_{j=0}^k C_j(x) y_0^{(j)},$$

where the fundamental polynomials $A_j(x)$, $B_j(x)$ and $C_j(x)$ are defined in Lemma 1.

Proof. By Lemma 1 the polynomial $R_m(x)$ in (24) satisfies the conditions (2) and (3), hence the existence part of the statement is proved.

For the uniqueness let us consider the homogeneous problem: Find a polynomial $Q_m(x)$ of degree at most $2n+k$ satisfying the conditions

$$\begin{cases} Q_m(x_i) &= 0, \quad Q'_m(x^*{}_i) = 0 \quad (i = 1 \dots, n), \\ Q_m^{(l)}(0) &= 0 \qquad (l = 0, \dots, k). \end{cases}$$

Due to these equations it is clear that

$$Q_m(x) = x^k L_n^{(k)}(x) q_n(x),$$

where $q_n(x)$ is a polynomial at most n. Furthermore by (6)

$$Q'_m(x^*{}_i) = L_n^{(k)}(x^*{}_i)(x^*{}_i)^k q'_n(x^*{}_i) = 0 \qquad (i = 1, \dots, n),$$

from which $q'_n(x^*{}_i) = 0$ for $i = 1, \dots, n$, that is $q'_n(x) \equiv 0$, hence $q_n(x) \equiv c$. So $Q_m(x) = c\, x^k L_n^{(k)}(x)$, but

$$\frac{d^k Q_m}{dx^k}(0) = c\, k! L_n^{(k)}(0) = 0.$$

As $L_n^{(k)}(0) \neq 0$ it follows $c = 0$, hence $Q_m(x) \equiv 0$, which completes the proof of the uniqueness. ◊

3. Birkhoff-type quadrature formulae with Laguerre abscissas

Theorem 2. *For $k \geq 1$ fixed integer let $\{x_i\}_{i=1}^n$ and $\{x^*{}_i\}_{i=1}^n$ be the zeros of the Laguerre polynomials $L_n^{(k)}(x)$ and $L_n^{(k-1)}(x)$, respectively. Then there exist the coefficients A_j, B_j and C_j such that the quadrature formulae*

$$(25) \quad \int_0^\infty f(x) e^{-x} dx \sim \sum_{j=1}^n A_j f(x_j) + \sum_{j=1}^n B_j f'(x^*{}_j) + \sum_{j=0}^{k-1} C_j f^{(j)}(0)$$

are exact for the polynomials of degree at most $2n + k$.

Proof. Integrating (24) on the interval $[0, \infty)$ with respect to the weight function e^{-x} we obtain

$$(26) \quad \int_0^\infty R_m(x) e^{-x} dx = \sum_{j=1}^n A_j f(x_j) + \sum_{j=1}^n B_j f'(x^*{}_j) + \sum_{j=0}^{k} C_j f^{(j)}(0),$$

with

$$\begin{aligned} A_j &= \int_0^\infty A_j(x)e^{-x}dx \qquad j=1,\dots,n, \\ B_j &= \int_0^\infty B_j(x)e^{-x}dx \qquad j=1,\dots,n, \\ C_j &= \int_0^\infty C_j(x)e^{-x}dx \qquad j=0,1,\dots,k, \end{aligned} \tag{27}$$

where the polynomials $A_j(x)$, $B_j(x)$ and $C_j(x)$ are defined in Th. 1. Hence the quadrature formula (26) is exact for the polynomials of degree at most $2n+k$.

Furthermore, by the orthogonality

$$C_k = \int_0^\infty C_k(x)e^{-x}dx = \frac{1}{k!L_n^{(k)}(0)}\int_0^\infty L_n^{(k)}(x)x^k e^{-x}dx = 0, \tag{28}$$

which completes the proof. ◊

Lemma 2. *For $k \geq 1$ fixed integer the coefficients of the quadrature formula* (25) *are*

$$A_j = \frac{(2n+k)(n+k-1)!}{x_j^{k-1}[L_n^{(k-1)}(x_j)]^2 n\, n!(n+k)}, \tag{29}$$

$$B_j = \frac{-(n+k)!}{(x^*{}_j)^k[L_n^{(k)}(x^*{}_j)]^2 n\, n!} \tag{30}$$

for $j=1,\dots,n$.

Proof. It is known that (cf. (5.1.6) in [5])

$$L_n^{(k)}(x) = \sum_{\nu=0}^{n}\binom{n+k}{n-\nu}\frac{(-x)^\nu}{\nu!} \tag{31}$$

Let

$$\frac{L_n^{(k)}(x)}{x-x_j} = a_{j,n-1}x^{n-1} + a_{j,n-2}x^{n-2} + \dots + a_{j,0}$$

and comparing the coefficients in

$$(a_{j,n-1}x^{n-1} + a_{j,n-2}x^{n-2} + \dots a_{j,0})(x-x_j) = L_n^{(k)}(x)$$

we get

$$\begin{cases} a_{j,n-1} &= \frac{(-1)^n}{n!}, \\ a_{j,n-2} &= \frac{(-1)^n}{n!}\big[x_j - n(n+k)\big]. \end{cases}$$

Comparing the coefficients of x terms in the linear combination

$$x^2 \frac{L_n^{(k)}(x)}{x - x_j} = \sum_{i=0}^{n+1} \gamma_{j,i} L_i^{(k-1)}(x)$$

we have $\gamma_{j,n+1} = -(n+1)$ and $\gamma_{j,n} = x_j + n + k$. Hence by (5.1.1) in [5]

$$\int_0^\infty x^2 \frac{L_n^{(k)}(x)}{x - x_j} L_n^{(k-1)}(x) x^{k-1} e^{-x} dx =$$

$$= \gamma_{j,n} \int_0^\infty \big[L_n^{(k-1)}(x)\big]^2 x^{k-1} e^{-x} dx = \tag{32}$$

$$= (x_j + n + k)(k-1)! \binom{n+k-1}{n}.$$

In a similar way we get

$$\int_0^\infty \left[\int_0^x \frac{t L_n^{(k-1)'}(t) - x_j L_n^{(k-1)}(t)}{t - x_j} dt\right] L_n^{(k)}(x) x^k e^{-x} dx = \tag{33}$$

$$= \left(1 - \frac{x_j}{n}\right) \frac{(n+k)!}{n!},$$

and from (9), (32) and (33) we obtain

$$A_j = \int_0^\infty A_j(x) e^{-x} dx = \frac{(2n+k)(n+k-1)!}{x_j^k L_n^{(k-1)}(x_j) L_n^{(k)'}(x_j) n\, n!} =$$

$$= \frac{(2n+k)(n+k-1)!}{x_j^{k-1} [L_n^{(k-1)}(x_j)]^2 n\, n! (n+k)},$$

where we used

$$x_j^k L_n^{(k)'}(x_j) = (n+k) x_j^{k-1} L_n^{(k-1)}(x_j), \tag{34}$$

which follows from (6).

Following the same idea we get

$$\int_0^x l^*{}_j(t) dt = \sum_{\nu=0}^n \gamma_\nu L_\nu^{(k)}(x) = \frac{1}{n L_n^{(k-1)'}(x^*{}_j)} L_n^{(k)}(x) + \ldots$$

and so

$$B_j = \int_0^\infty B_j(x)dx =$$
$$= \frac{1}{n({x^*}_j)^k L_n^{(k)}({x^*}_j) L_n^{(k-1)'}({x^*}_j)} \int_0^\infty [L_n^{(k)}(x)]^2 x^k e^{-x} dx =$$
$$= \frac{(n+k)!}{({x^*}_j)^k L_n^{(k)}({x^*}_j) L_n^{(k-1)'}({x^*}_j) n\, n!} = \frac{-(n+k)!}{({x^*}_j)^k [L_n^{(k)}({x^*}_j)]^2 n\, n!},$$

where we used $L_n^{(k-1)'}({x^*}_j) = -L_n^{(k)}({x^*}_j)$, which follows from (4) and (13). ◊

Example. For $k = 1$, if the nodes $\{x_i\}_{i=1}^n$ and $\{{x^*}_i\}_{i=1}^n$ are the zeros of the Laguerre polynomials $L_n^{(1)}(x)$ and $L_n(x)$, respectively, the quadrature formula

$$\int_0^\infty f(x)e^{-x}dx \sim \frac{2n+1}{n(n+1)} \sum_{j=1}^n \frac{1}{[L_n(x_j)]^2} f(x_j) - \tag{35}$$
$$- \frac{n+1}{n} \sum_{j=1}^n \frac{1}{x_j^* [L_n^{(1)}(x_j^*)]^2} f'({x^*}_j) - \frac{n}{n+1} f(0)$$

is exact for the polynomials of degree at most $2n+1$.

Thus, substituting $n = 1$ into (35), the quadrature formula

$$\int_0^\infty f(x)e^{-x}dx \sim \frac{3}{2}f(2) - 2f'(1) - \frac{1}{2}f(0)$$

is exact for cubic polynomials, and for $n = 2$ we obtain the quadrature formula

$$\int_0^\infty f(x)e^{-x}dx \sim \frac{5}{12}[(2+\sqrt{3})f(3-\sqrt{3}) + (2-\sqrt{3})f(3+\sqrt{3})] - \tag{36}$$
$$-\frac{3}{8}[(2+\sqrt{2})f'(2-\sqrt{2}) + (2-\sqrt{2})f'(2+\sqrt{2})] - \frac{2}{3}f(0),$$

which is exact for the polynomials of degree at most 5.

For $k = 2$ and $n = 1$ from (25) we have

$$\int_0^\infty f(x)e^{-x}dx \sim \frac{8}{9}f(3) - \frac{3}{2}f'(2) + \frac{1}{9}f(0) - \frac{1}{6}f'(0),$$

which is exact for the polynomials of degree at most 4.

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