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ON THE MULTIPLICATIVE SEMIGROUP OF NEAR-RINGS

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Abstract: We give a structure theorem for the multiplicative semigroup of an arbitrary zero symmetric near-ring with DCCN and we will see that the multiplicative semigroup is of a very special nature. After that we study some classes of near-rings with special multiplicative semigroups. In particular, we study near-rings with 0-simple, regular and semisimple multiplicative semigroups.

1. Introduction and preliminary definitions

What concerns the notation and basic results used in this paper, we are refering to [7] for near-rings, in particular we use right near-rings, and to [3] and [5] for semigroups.

In the first part of the paper we study the multiplicative semigroups of zero symmetric near-rings N with descending chain condition on N-subgroups of ${}_NN$ in full generality. Usually we write N for near-rings and $(N,*)$ for the multiplicative semigroup of N. We will see that $(N,*)$ is always the union of two disjoint sets $N^{\#}$ and Z_r, $N^{\#}$ (if not empty) being regular and a union of isomorphic groups and Z_r being an

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ideal of $(N, *)$. Based on this result we then study zero symmetric near-rings whose multiplicative semigroup is 0-simple, so has no ideals other than $\{0\}$ and N. We also try to generalize results we obtain in this area and we will be in a position to prove results on regular near-rings and on near-rings whose multiplicative semigroup is semisimple.

The study whether a semigroup admits a near-ring structure or not is by far not so well developed as the study which near-rings can be defined on certain types of groups. A good survey for what has been already done from the semigroup point of view can be found in [4] and this paper also should contribute to this theme. Note that when dealing with zero symmetric near-rings N, with $|N| \geq 2$, our semigroups are always semigroups with zero.

Before starting our work we recall some definitions. If a semigroup S with at least two elements contains an element 0 such that for all $s \in \in S$, $0s = s0 = 0$, then we call S a semigroup with zero 0. If we would not want S to have more than two elements, then the trivial semigroup $\{e\}$, in which $e^2 = e$, would be a semigroup with zero. We do not want that. Furthermore, note that in a semigroup with zero 0, 0 is unique.

A non-empty subset A of a semigroup S is called a left ideal if $SA \subseteq A$, a right ideal if $AS \subseteq A$ and an ideal if it is both a left and a right ideal.

Following the notation of [3] we define:

Definition 1.1. A semigroup S is said to be 0-simple if $S^2 \neq \{0\}$ and $\{0\}$ and S are the only ideals of S. Similary a semigroup S is left 0-simple if $\{0\}$ and S are the only left ideals. For semigroups without zero, we use the terminology simple and left simple, respectively. A left (right) ideal M ($\neq \{0\}$) of a semigroup S with zero is said to be 0-minimal if M and $\{0\}$ are the only left (right) ideals of S properly contained in M. A 0-simple semigroup is said to be completely 0-simple if it contains at least one 0-minimal left ideal and at least one 0-minimal right ideal.

Definition 1.2. Let E be the set of idempotents of a semigroup S. If $e, f \in E$ we define $e \leq f$ if $ef = fe = e$.

Note that $\leq$ is a partial order on E (see [3] or [5] for more on that subject).

Definition 1.3. A non-zero idempotent e of a semigroup S is said to be primitive if e is minimal w.r.t. $\leq$ in the set of non-zero idempotents of S.

For later use we need the following lemma:

Lemma 1.4. [5, Chapter 3, Th. 3.1.] *A* 0*-simple semigroup is completely* 0*-simple if and only if it contains a primitive idempotent.*

Finally, an element a of a semigroup S is called regular if there exists $x \in S$ such that $axa = a$. S is called regular if all its elements are regular. A near-ring is called regular if its multiplicative semigroup is regular.

2. The multiplicative semigroup of a zero symmetric near-ring

In this section we take a look at the multiplicative structure of an arbitrary zero symmetric near-ring with descending chain condition on N-subgroups of ${}_NN$ (abbreviated by DCCN). Although we are in a very general situation, we will see that we can give some very detailed results. First we study the role that zero divisors play. Let N be a zero symmetric near-ring and let $Z_r := \{n \in N \mid \exists x \in N \setminus \{0\} : xn = 0\}$ be the set of right zero divisors.

Definition 2.1. $N^{\#} := N \setminus Z_r$.

In the following, we always assume $N \neq \{0\}$, so $0 \in Z_r$. We will now show that Z_r is an ideal of the multiplicative semigroup of N, in case N has the DCCN.

Theorem 2.2. *Let N be a zero symmetric near-ring with DCCN. Then Z_r is an ideal of $(N, *)$ and any proper ideal of $(N, *)$ is contained in Z_r.*

Proof. If $N^{\#} = \varnothing$, there is nothing to prove. So, let $m \in N^{\#}$. Then any m^k, k a natural number, must be in $N^{\#}$. The descending chain condition on N-subgroups of N guarantees that the chain $Nm \supseteq \supseteq Nm^2 \supseteq Nm^3 \dots$ terminates. So there is some natural number l such that $Nm^l = Nm^{l+1} = Nm(m^l)$. Consequently, for any $n \in N$ there exists $k \in N$ such that $nm^l = (km)m^l$. Since m^l is not a right zero divisor, we get $n = km$, so $N \subseteq Nm$. Clearly, $Nm \subseteq N$ and hence, $N = Nm$ (see [6, Lemma 2]).

Let $z \in Z_r$ and $n \in N$. If $n \in Z_r$, then clearly $nz \in Z_r$. If $n \in N^{\#}$, then $Nn = N$ and since $z \in Z_r$, there is an element $j \in N \setminus \setminus \{0\}$ such that $jz = 0$. Now $j = mn$ for some non-zero $m \in N$ and consequently, $jz = m(nz) = 0$. This shows that $nz \in Z_r$.

Clearly $Z_rN \subseteq Z_r$, so Z_r is an ideal of $(N, *)$.

Let I be an ideal of $(N, *)$ and suppose $I \not\subseteq Z_r$. Then there is an element $i \in I \setminus Z_r$. By the above, $Ni = N \subseteq I$ so $I = N$. Consequently, each proper ideal of $(N, *)$ is contained in Z_r. $\Diamond$

From the proof of the theorem we also have that for $n \in N^{\#}$, $Nn = N$. We will need this a few lines later.

The theorem we just have proved shows that only such finite semigroups S with zero admit a right near-ring structure where the set of right zero divisors Z_r form an ideal. In case Z_r is a proper ideal, it must be the greatest proper ideal of S.

In order to prove our main theorem of this section, we need some more facts which we collect in the next lemma.

Lemma 2.3. *Let N be a zero symmetric near-ring with DCCN such that $N^{\#} \neq \varnothing$. Then $(N^{\#}, *)$ is a semigroup and for each $a \in N^{\#}$ and each $b \in N$ the equation $xa = b$ has a unique solution s in N. If $b \in N^{\#}$, then also $s \in N^{\#}$.*

Proof. We first show that $(N^{\#}, *)$ is a semigroup. Let $n_1, n_2 \in N^{\#}$. Suppose $n_1 n_2 \in Z_r$, so there is an element $m \in N \setminus \{0\}$ such that $m(n_1 n_2) = 0$. Since $n_2 \notin Z_r$, $mn_1 = 0$ and similary, $m = 0$ which is a contradiction. So, $N^{\#}$ is closed under multiplication.

Let $a \in N^{\#}$ and $b \in N$. Consequently, $Na = N$. So, there exists an element $s \in N$ such that $sa = b$. Suppose also $s_1 a = b$. Then we have $(s - s_1)a = 0$. Since $a \in N^{\#}$, $(0 : a) = \{n \in N \mid na = 0\} = \{0\}$ and therefore, $s = s_1$. So s is the unique solution of the equation $xa = b$. By Th. 2.2, $s \in N^{\#}$ if $b \in N^{\#}$. $\Diamond$

Now we are in a position to prove our main structure theorem on the multiplicative semigroup of an arbitrary zero symmetric near-ring with DCCN.

Theorem 2.4. *Let N be a zero-symmetric near-ring with DCCN and $N^{\#} \neq \varnothing$. For $a \in N^{\#}$ let 1_a be the unique solution of $xa = a$ and let $B_a := \{x \in N^{\#} \mid 1_a x = x\}$. Then the following hold:*

(1) *Each $(B_a, *)$ is a group with identity 1_a and 1_a is a right identity of N.*

(2) *Z_r and the sets B_a ($a \in N^{\#}$) form a partition of N.*

(3) *For $a, b \in N^{\#}$, $(B_a, *)$ and $(B_b, *)$ are isomorphic groups.*

(4) *$(N^{\#}, *)$ is a regular left simple semigroup which is a union of isomorphic groups.*

Proof. Let $a \in N^{\#}$ and $b, b_1 \in B_a$. Then, $bb_1 \in N^{\#}$ by Lemma 2.3. Furthermore, $1_a(bb_1) = (1_a b)b_1 = bb_1$, so $bb_1 \in B_a$. This shows that B_a is closed under multiplication. Since $(1_a 1_a)a = 1_a(1_a a) = 1_a a$, we have that $1_a 1_a = 1_a$ by Lemma 2.3 and 1_a is a left identity in B_a (note that $1_a \in N^{\#}$ by Lemma 2.3). Since for every $m \in N$, $m1_a$ and m is a solution of $x1_a = m1_a$, we have $m = m1_a$ and 1_a is a right identity of N. In particular, 1_a is the identity in $(B_a, *)$.

Let $b \in B_a$. Let $\bar{b}$ be the (unique) solution of $xb = 1_a$. Since $(1_a\bar{b})b = 1_a(\bar{b}b) = 1_a 1_a = 1_a$, we get $1_a\bar{b} = \bar{b}$ and consequently $\bar{b} \in B_a$. On the other hand, $b\bar{b} = b1_a\bar{b} = b\bar{b}b\bar{b}$ and $1_a b\bar{b} = b\bar{b}$ and hence $b\bar{b} = 1_a$. So $\bar{b}$ is the multiplicative inverse of b in B_a.

The proof of part (1) is now complete. For part (2) it suffices to show that $\forall a, b \in N^{\#}$ either $B_a \cap B_b = \varnothing$ or $B_a = B_b$: Let $m \in B_a \cap B_b$. Then $1_a m = m = 1_b m$ and hence, 1_a and 1_b are solutions of $xm = m$. So $1_a = 1_b$ and $B_a = B_b$.

For proving (3), consider the map $\phi : B_a \longrightarrow B_b, x \mapsto 1_b x$. Note that $1_b x \in B_b$ for any $x \in N^{\#}$. Let $\phi(x_1) = 1_b x_1 = 1_b x_2 = \phi(x_2)$. By part (a) we have $x_1 = 1_a x_1 = 1_a(1_b x_1) = 1_a(1_b x_2) = 1_a x_2 = x_2$, so ϕ is injective. Let $y \in B_b$. Then, $1_a y \in B_a$ and $\phi(1_a y) = 1_b(1_a y) = = 1_b y = y \in B_b$, so ϕ is injective. Furthermore, for $x_1, x_2 \in B_a$, $\phi(x_1 x_2) = 1_b(x_1 x_2) = 1_b((x_1 1_b)x_2) = (1_b x_1)(1_b x_2) = \phi(x_1)\phi(x_2)$. We finally have proved that ϕ is a group isomorphism.

To prove part (4), it follows immediately from (2) and (3) that $(N^{\#}, *)$ is a union of isomorphic groups. Hence, $(N^{\#}, *)$ is a regular semigroup. Since for each $a \in N^{\#}$ and each $b \in N^{\#}$ the equation $xa = b$ has a unique solution s in $N^{\#}$ by Lemma 2.3, $(N^{\#}, *)$ is left simple. $\diamondsuit$

Note that it may happen that $N = Z_r$ for a zero symmetric near-ring N. Then Th. 2.4 does not give us any information. To the authors knowledge there does not exist a meaningful description of near-rings with $N = Z_r$, if possible at all. Note that Th. 2.4 has a high similarity to the "Main structure theorem of planar near-rings" (see [2, Th. 4.9]). Planar near-rings (see Def. 5.3 and [7] or [2] for a good survey) however do have some very special properties. It therefore seems to be a big surprise that our Th. 2.4 can be proved in the general setting of arbitray zero symmetric near-rings with DCCN.

Observe also that any idempotent in N which is not a right zero divisor is a right identity. This shows that a zero symmetric near-ring with DCCN and $N^{\#} \neq \varnothing$ must have a right identity (see [7, Rem. 1.112] in case of finite N). We now also can give an interesting corollary on near-rings with identity.

Corollary 2.5. *Let N be a zero symmetric near-ring with identity and DCCN. Then, $(N^{\#}, *)$ is a group.*

Proof. $N^{\#} \neq \varnothing$ since $1 \in N^{\#}$. By Th. 2.4, $N^{\#}$ is a disjoint union of groups and each idempotent in $N^{\#}$ is a multiplicative right identity of the near-ring. Since N has an identity 1, 1 is the unique idempotent in $N^{\#}$. So, $(N^{\#}, *)$ is a group by Th. 2.4. ◊

Remark 2.6. Note that by setting $Z_r = \{0\}$ in Th. 2.4 we immediately get that a zero symmetric integral near-ring N with DCCN is a union of disjoint isomorphic groups and zero. However, integral near-rings are very well studied (see [7] and [4]) and the author does not claim the originality of this result.

In the following section we use the observation that the set Z_r is an ideal of $(N, *)$ to study near-rings N which do not have proper ideals in $(N, *)$. This will imply that the multiplicative semigroup of N must be of a very special nature, since we will see that such near-rings turn out to be integral.

3. Near-rings whose multiplicative semigroup is 0-simple

According to Th. 2.2, we already see that zero symmetric near-rings with DCCN whose multiplicative semigroup is 0-simple are integral or $Z_r = N$. In the following, we will see that $Z_r = N$ cannot happen.

This study can also be seen as a kind of counterpart to the study which near-rings can be defined on simple groups (see [7] for a survey). However, note that 0-simple semigroups do not have to be congruence-free as this is the case with simple groups.

Theorem 3.1. *Let N be a zero-symmetric near-ring with DCCN such that $(N, *)$ is a 0-simple semigroup. Then N is an integral near-ring which acts 2-primitively on ${}_N N$.*

Proof. Suppose that N is a zero symmetric near-ring where the multiplicative semigroup $(N, *)$ is 0-simple and N has DCCN. Hence $N^2 \neq$ $\neq \{0\}$. If N has no right zero divisors, then it is integral, and due to the DCCN it is 2-primitive on ${}_N N$ (see [7, Rem. 9.48 d]).

Suppose $Z_r \neq \{0\}$ and $N^{\#} \neq \varnothing$. Then Th. 2.2 shows that $(N, *)$ is not 0-simple. Hence, this case cannot happen.

Suppose $N = Z_r$. We consider the left ideal $\mathbf{J}_{1/2}(N)$ of the near-ring N (see [7, Def. 5.5]). By [7, Th. 5.40] $\mathbf{J}_{1/2}(N)$ is nilpotent, so $\mathbf{J}_{1/2}(N)^k = \{0\}$, k a natural number. Now consider the set $I := \mathbf{J}_{1/2}(N) *$

$* N \cup \mathbf{J}_{1/2}(N)$. It is clear that I is an ideal of $(N, *)$. We proceed to show that $I^k = \{0\}$. If $k = 1$, this is clear, so suppose $k \geq 2$. Let $i \in I$, then either $i \in \mathbf{J}_{1/2}(N)$ and then clearly $i^k = 0$ or $i \in \mathbf{J}_{1/2}(N) * N$. In the latter case $i = jn$ for some $j \in \mathbf{J}_{1/2}(N)$ and $n \in N$. Consider $(jn)^k = \underbrace{(jn) * \ldots * (jn)}_{\text{k-times}}$. Since $\mathbf{J}_{1/2}(N)$ is a left ideal of the zero symmetric near-ring N, $N * \mathbf{J}_{1/2}(N) \subseteq \mathbf{J}_{1/2}(N)$. So, by changing the parentheses in the product $(jn) * \ldots * (jn)$, we get $(jn)^k = (j * h_1 * \ldots * h_{k-1}) * n$, where $h_i \in \mathbf{J}_{1/2}(N)$ for $i \in \{1, \ldots, k-1\}$. Since $\mathbf{J}_{1/2}(N)^k = \{0\}$, we now see that $(jn)^k = 0$ and consequently, $I^k = \{0\}$. Since $(N, *)$ is assumed to be 0-simple, $I = \{0\}$ or $I = N$. Suppose $I = N$. This means that N is nilpotent. But N^2 is an ideal of $(N, *)$, so $N^2 = N$. Therefore, N cannot be nilpotent and we arrive at a contradiction. Consequently, $I = \{0\}$ and therefore, $\mathbf{J}_{1/2}(N) = \{0\}$. Then, by [7. Th. 5.39], N has a multiplicative right identity 1_r and $1_r \in N^{\#}$. This finally shows that $N = Z_r$ cannot happen in case $(N, *)$ is 0-simple and our theorem is proved. ◊

Corollary 3.2. *Let N be a zero-symmetric near-ring with DCCN such that $(N, *)$ is a 0-simple semigroup. Then $(N, *)$ is completely 0-simple.*

Proof. By Th. 3.1, N is a 2-primitive near-ring with DCCN. So, by [7, Th. 4.46] N has a right identity 1_r. Since N is integral and 2-primitive on ${}_N N$, $Nn = N$ for any non-zero $n \in N$. So, any non-zero idempotent e in N is a right identity and therefore, any non-zero idempotent is a primitive idempotent. By Lemma 1.4, N is completely 0-simple. ◊

At least for the class of finite 0-simple (hence, completely 0-simple) semigroups we can say that most of them do not admit a near-ring structure. If a finite completely 0-simple semigroup $(S, *)$ is not a union of isomorphic groups (observe Rem. 2.6), then there is no zero symmetric near-ring N with $(N, *) = (S, *)$.

In the following we sort of extend Cor. 3.2 to a larger class of semigroups which we can use later for proving a result on regular near-rings.

Definition 3.3. A semigroup with zero 0 is said to be a 0-direct union of semigroups S_i ($i \in I$, I an index set) if $S = \cup_{i \in I} S_i$ and $S_i \cap S_j = S_i S_j = S_j S_i = \{0\}$ for $i \neq j$.

Definition 3.4. A regular semigroup is called primitive if each of its non-zero idempotents is primitive.

Lemma 3.5. [3, Th. 6.39.] *Let S be a semigroup with zero. Then S*

is a primitive regular semigroup if and only if S is a 0-direct union of completely 0-simple semigroups.

We now describe near-rings with such multiplicative semigroups:

Theorem 3.6. *Let N be a zero-symmetric near-ring with DCCN such that $(N,*)$ is a 0-direct union of completely 0-simple semigroups. Then N is integral and acts 2-primitively on ${}_N N$.*

Proof. Clearly, $N^2 \neq \{0\}$. Since N is regular by Lemma 3.5 and has the DCCN, N is a 2-semisimple near-ring by [7, Th. 9.164 c] (see also Cor. 6.3). By [7, Th. 5.32], N has a right identity 1_r and ${}_N N = \sum_{i=1}^{k} L_i$ is the (finite) direct sum of minimal left ideals of the near-ring N, all being N-groups of type 2. Suppose $k \geq 1$. By [7, Th. 3.43] each left ideal L_i has a right identity. Since ${}_N N$ is supposed to be the direct sum of at least two different left ideals, say L_1 and L_2, there exist two different idempotents $e_1 \in L_1$ and $e_2 \in L_2$, being right identities of L_1 and L_2, respectively. Now, since the sum is direct and hence also distributive (see [7, Th. 2.30]), $e_1 e_2 = e_2 e_1 = 0$ (see [7, Th. 3.43]) and $e_1(e_1+e_2) = = e_1 e_1 + e_1 e_2 = e_1$. This shows that $e_1(e_1+e_2) = (e_1+e_2)e_1 = e_1$ and consequently $e_1 < e_1 + e_2$, where $\leq$ is the partial order relation of Def. 1.2. Hence, $e_1 + e_2$ is a non-primitive idempotent, in contradiction to Lemma 3.5. Consequently, $k = 1$ and N is a minimal left ideal of the near-ring and ${}_N N$ is an N-group of type 2. Consequently, N is a simple near-ring, and therefore, $(0 : N) = \{0\}$. So, N acts 2-primitively on ${}_N N$. It is now easy to show that N is integral: Suppose $ab = 0$, $a, b \in N$ and $b \neq 0$. Since N is regular, there exists $x \in N$ such that $bxb = b$. bx cannot be zero, since otherwise $0 = b$. Hence, bx is a non-zero idempotent. Therefore, $N(bx) \neq \{0\}$ and consequently, $N(bx) = N$ since ${}_N N$ is of type 2. It follows that bx is a multiplicative right identity of N. This means that $0 = (ab)x = a(bx) = a$, which shows that $\forall a, b \in N : (ab = 0 \Rightarrow (a = 0 \vee b = 0))$ and N is integral. $\Diamond$

Finally, we can establish the following equivalences:

Corollary 3.7. *Let N be a zero symmetric near-ring with DCCN. Then the following properties are equivalent:*

(1) *$(N,*)$ is 0-simple.*
(2) *$(N,*)$ is completely 0-simple.*
(3) *$(N,*)$ is a 0-direct union of completely 0-simple semigroups.*
(4) *N is integral, acting 2-primitively on ${}_N N$.*
(5) *$(N,*)$ is a left 0-simple semigroup.*

Proof. (1) $\Rightarrow$ (2) follows by Cor. 3.2. (2) $\Rightarrow$ (3) follows by definition. (3) $\Rightarrow$ (4) follows by Th. 3.6. For establishing (4) $\Rightarrow$ (5) note that by integrality and 2-primitivity on ${}_N N$ we have $Nn = N$ for any non-zero $n \in N$. Hence the semigroup clearly must be left 0-simple. (5) $\Rightarrow$ (1) is clear. $\Diamond$

The results in the next two sections now will be easy consequences of our observations made so far. We can generalize a well known ring theoretic result to near-rings and we can also generalize a well known result on regular near-rings.

4. Near-rings with distributive elements on 0-simple semigroups

Rings on 0-simple semigroups have been studied for example in [8], where the author studies when a 0-simple multiplicative semigroup of a ring R is completely 0-simple. It is well known (see [8] for references) that a completely 0-simple multiplicative semigroup of a ring is a group with zero, which means that such a ring is a skew field.

Since our results in the last section are also valid in the ring case (a ring is a near-ring, respectively), we can contribute to this theme:

Theorem 4.1. *Let N be a zero-symmetric near-ring with DCCN, having a non-zero distributive element, where $(N, *)$ is a 0-simple semigroup. Then N is a near-field and in the case that N is a ring, N is a skew field.*

Proof. By Th. 3.1, N is an integral near-ring acting 2-primitively on ${}_N N$. Hence, for all $0 \neq n \in N$, $Nn = N$. Since N has a non-zero distributive element, N is a near-field by [7, Th. 8.3]. The rest is clear. $\Diamond$

5. Integral regular near-rings

We will apply Th. 3.6 to get some information on regular near-rings. A near-ring N is called regular if $(N, *)$ is a regular semigroup. In particular, in case of near-rings with DCCN we can generalize the following theorem of [1] to near-rings without identity.

Theorem 5.1. [1] *A zero symmetric regular near-ring with identity is integral iff it is a near-field.*

If we do not require an identity, we get the following result:

Theorem 5.2. *Let N be a zero symmetric regular near-ring with DCCN. Then the following are equivalent:*

(1) *N is integral.*

(2) *Every non-zero idempotent is primitive.*

Proof. If N is a regular zero symmetric near-ring with DCCN such that every non-zero idempotent is primitive, then Lemma 3.5 and Th. 3.6 show that N is integral. Conversely, suppose N is a zero symmetric regular and integral near-ring with DCCN. Then, as shown in the proof of Cor. 3.2, every non-zero idempotent in N is a primitive idempotent. ◊

Note that Th. 5.2 is indeed a generalization of Th. 5.1 in the case of near-rings with DCCN. Let N be a zero symmetric near-ring with DCCN and with identity which is integral and regular. Then every non-zero idempotent is primitive by Th. 5.2. Suppose e is a non-zero idempotent. Then $e1 = 1e = e$, showing that $e \leq 1$ and therefore $e = = 1$. Hence, 1 is the only non-zero idempotent. So, $N \setminus \{0\}$ is a regular semigroup with just one idempotent and hence a group by ([3], Chapter 1.9, Exercise 4). Consequently, N is a near-field.

For any near-ring N we can define an equivalence relation $\equiv$ on N as follows: Let $a, b \in N$. Then $a \equiv b :\Leftrightarrow \forall n \in N : na = nb$. Using this notation we can introduce an important class of near-rings.

Definition 5.3. A near-ring N is called planar if $|N/\equiv| \geq 3$ and if for all $a, b, c \in N$ with $a \not\equiv b$ the equation $xa = xb + c$ has a unique solution in N.

By [7, Th. 9.50] a finite zero symmetric and integral near-ring N is planar as long as $|N/\equiv| \geq 3$. So, Th. 5.2 gives us a source for planar near-rings:

Corollary 5.4. *Let N be a finite zero symmetric regular near-ring with $|N/\equiv| \geq 3$. If every non-zero idempotent is primitive, then N is a planar near-ring.*

6. Near-rings whose multiplicative semigroup is semisimple

With methods very similar to that used in the proof of Th. 3.1 we can generalize results on regular near-rings and on near-rings with 0-simple multiplicative semigroups to a larger class of near-rings (however, observe Rem. 6.4).

Regular semigroups R and also 0-simple semigroups, for example, have the property that given an ideal I of R, then $I^2 = I$. Such semigroups are called semisimple.

Definition 6.1. [3, Chap. 2.6, Exercise 7.] A semigroup S is semisimple iff $I^2 = I$ for every ideal I of S.

It is well known that a regular near-ring is semisimple in the near-ring theoretic sense. This means that given a regular zero symmetric near-ring N with DCCN, then $\mathbf{J_2}(N)=\{0\}$, i.e. N is 2-semisimple (see [7, Th. 9.164 c]). By Th. 3.1, the same holds for near-rings with 0-simple multiplicative semigroups since they are 2-primitive near-rings. In the following we will see that this result extends to near-rings whose multiplicative semigroup is not necessarily regular or 0-simple but semisimple.

Theorem 6.2. *Let N be a zero symmetric near-ring with DCCN. Suppose $(N,*)$ is a semisimple semigroup. Then $\mathbf{J_2}(N)=\{0\}$.*

Proof. As in the proof of Th. 3.1 we consider the left ideal $\mathbf{J_{1/2}}(N)$ of the near-ring N and we can show that $I := \mathbf{J_{1/2}}(N) * N \cup \mathbf{J_{1/2}}(N)$ is a nilpotent ideal of $(N,*)$, say $I^k=\{0\}$. Since $(N,*)$ is assumed to be semisimple, $I^2=I$, so we must have $I=\{0\}$ and $\mathbf{J_{1/2}}(N)=\{0\}$. By [7, Th. 5.39] we get that N has a multiplicative right identity.

Suppose M is a nilpotent N-subgroup of ${}_N N$, so $M^k=\{0\}$ for some natural number k. As an N-subgroup, M is also a left ideal of $(N,*)$. Using the same arguments as in the proof of Th. 3.1, one can show that $S := M * N \cup M$ is a non-zero ideal of $(N,*)$ and $S^k=\{0\}$. Again by the semisimplicicty of $(N,*)$ we get $S=\{0\}$ and consequently, $M=\{0\}$. Hence, N is a near-ring with right identity having no non-zero nilpotent N-subgroups. The result now follows from [7, Th. 5.49]. ◊

We now can easily re-prove that a zero symmetric near-ring with DCCN which is regular is a 2-semisimple near-ring (see [7, Th. 9.164c]).

Corollary 6.3. *Let N be a zero symmetric near-ring with DCCN. Suppose $(N,*)$ is regular. Then $\mathbf{J_2}(N)=\{0\}$.*

Proof. By Th. 3.2 it suffices to show that $(N,*)$ is semisimple. Let I be an ideal of $(N,*)$. Clearly, $I^2 \subseteq I$. Let $i \in I$. Since N is regular, there exists $x \in N$ such that $i = i(xi) \in I^2$. ◊

A short note should be in order whether the converse is also true, that is, if semisimplicity of a near-ring implies that the multiplicative semigroup of the near-ring is semisimple. The answer is no: Take a planar near-ring N for example and consider the set $A := \{n \in N \mid \mid \forall m \in N : mn = 0\}$. Suppose N is not integral, that means $A \neq \{0\}$ (see [7, Cor. 8.92]). It is easy to see that any proper N-subgroup of ${}_N N$ must be contained in A and that ${}_N N$ is a faithful, strongly monogenic N-group. So, if A contains no non-trivial subgroup of $(N,+)$ (see the example below), then N acts 2-primitively on ${}_N N$, so $\mathbf{J_2}(N)=\{0\}$ but

$A^2 = \{0\}$, so $(N, *)$ is not semisimple (it is easy to see that A is an ideal of $(N, *)$).

We give an example of a planar near-ring N on the cyclic group of order 9, where $A \neq \{0\}$ and A does not contain non-trivial subgroups of $(N, +)$. So $\mathbf{J_2}(N) = \{0\}$ but $(N, *)$ is not semisimple. The near-ring was constructed using the construction method for planar near-rings which can be found in [2, Chap. 4.1] (we use the group of fixedpointfree automorphisms $\Phi = \{id, -id\}$ acting on $(N, +)$). We only show the multiplication table of N (note that $A = \{0, 1, 2, 4, 5, 7, 8\}$):

*	0	1	2	3	4	5	6	7	8
0	0	0	0	0	0	0	0	0	0
1	0	0	0	8	0	0	1	0	0
2	0	0	0	7	0	0	2	0	0
3	0	0	0	6	0	0	3	0	0
4	0	0	0	5	0	0	4	0	0
5	0	0	0	4	0	0	5	0	0
6	0	0	0	3	0	0	6	0	0
7	0	0	0	2	0	0	7	0	0
8	0	0	0	1	0	0	8	0	0

Remark 6.4. The following question remains open: Let N be a zero symmetric near-ring with DCCN such that $(N, *)$ is a semisimple semigroup. Is $(N, *)$ regular?

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