Mathematica Pannonica
15/2 (2004), 199–207

RATIONAL APPROXIMATIONS TO TASOEV CONTINUED FRACTIONS

Takao **Komatsu**

Department of Mathematical System Science, Faculty of Science and Technology, Hirosaki University, Hirosaki, 036-8561 Japan

Received: June 2004

MSC 1991: 11 J 82, 11 J 70; 11 A 55

Keywords: Rational approximations, Tasoev continued fractions, Rogers–Ramanujan continued fractions, Hurwitz continued fractions.

Abstract: Rational approximations to Tasoev continued fractions, the exponential quasi-periodic continued fractions, are given. This general result includes the previous known results and yields some new approximations.

1. Introduction

Hurwitz continued fractions, quasi-periodic simple continued fractions, have the form

$$[a_0; a_1, \dots, a_n, \overline{Q_1(k), \dots, Q_p(k)}]_{k=1}^{\infty} =$$
$$= [a_0; a_1, \dots, a_n, Q_1(1), \dots, Q_p(1), Q_1(2), \dots, Q_p(2), Q_1(3), \dots],$$

where a_0 is an integer, $a_1, \dots, a_n$ are positive integers, $Q_1, \dots, Q_p$ are polynomials with rational coefficients which take positive integral values for $k = 1, 2, \dots$ and at least one of the polynomials is not constant.

Tasoev continued fractions ([8], [10]) are also quasi-periodic but $Q_j(k)$ includes exponentials in k instead of polynomials. The author obtained the closed form of $[0; \overline{\underbrace{a^k, \dots, a^k}_{m}}]_{k=1}^{\infty}$ in [2], and found some

E-mail address: komatsu@cc.hirosaki-u.ac.jp
This research was supported in part by the Grant-in-Aid for Scientific research (C) (No. 15540021), the Japan Society for the Promotion of Science.

more general forms by applying the similar method in [3]. Namely,

$$[0;\overline{ua^k}]_{k=1}^\infty=\frac{\sum_{n=0}^\infty u^{-2n-1}a^{-(n+1)^2}\prod_{i=1}^n(a^{2i}-1)^{-1}}{\sum_{n=0}^\infty u^{-2n}a^{-n^2}\prod_{i=1}^n(a^{2i}-1)^{-1}},$$

$$[0;ua-1,\overline{1,ua^{k+1}-2}]_{k=1}^\infty=\frac{\sum_{n=0}^\infty(-1)^n u^{-2n-1}a^{-(n+1)^2}\prod_{i=1}^n(a^{2i}-1)^{-1}}{\sum_{n=0}^\infty(-1)^n u^{-2n}a^{-n^2}\prod_{i=1}^n(a^{2i}-1)^{-1}},$$

$$[0;\overline{ua^k,va^k}]_{k=1}^\infty=\frac{\sum_{n=0}^\infty u^{-n-1}v^{-n}a^{-(n+1)(n+2)/2}\prod_{i=1}^n(a^i-1)^{-1}}{\sum_{n=0}^\infty u^{-n}v^{-n}a^{-n(n+1)/2}\prod_{i=1}^n(a^i-1)^{-1}}$$

and

$$[0;ua-1,1,va-2,\overline{1,ua^{k+1}-2,1,va^{k+1}-2}]_{k=1}^\infty=$$
$$=\frac{\sum_{n=0}^\infty(-1)^n u^{-n-1}v^{-n}a^{-(n+1)(n+2)/2}\prod_{i=1}^n(a^i-1)^{-1}}{\sum_{n=0}^\infty(-1)^n u^{-n}v^{-n}a^{-n(n+1)/2}\prod_{i=1}^n(a^i-1)^{-1}}.$$

In [4] we found some Tasoev continued fractions with period 3. Namely,

$$[0;\overline{ua^{2k-1}-1,1,va^{2k}-1}]_{k=1}^\infty=$$
$$=\frac{\sum_{n=0}^\infty u^{-n-1}v^{-n}a^{-(n+1)^2}\prod_{i=1}^n(a^{2i}-(-1)^i)^{-1}}{\sum_{n=0}^\infty(-1)^n u^{-n}v^{-n}a^{-n^2}\prod_{i=1}^n(a^{2i}-(-1)^i)^{-1}},$$

$$[0;\overline{ua^k-1,1,va^k-1}]_{k=1}^\infty=$$
$$=\frac{\sum_{n=0}^\infty u^{-n-1}v^{-n}a^{-(n+1)(n+2)/2}\prod_{i=1}^n(a^i-(-1)^i)^{-1}}{\sum_{n=0}^\infty(-1)^n u^{-n}v^{-n}a^{-n(n+1)/2}\prod_{i=1}^n(a^i-(-1)^i)^{-1}}.$$

In [5] we got some other Tasoev continued fractions with period 3 by obtaining the following

$$[0;\overline{ua^k-1,1,v-1}]_{k=1}^{\infty}=$$

$$=\frac{\sum_{n=0}^{\infty} u^{-2n-1}v^{-2n}a^{-(n+1)^2}\prod_{i=1}^{n}(a^{2i}-1)^{-1}}{\sum_{n=0}^{\infty}\left((uv)^{-2n}a^{-n^2}-(uv)^{-2n-1}a^{-(n+1)^2}\right)\prod_{i=1}^{n}(a^{2i}-1)^{-1}},$$

$$[0;\overline{v-1,1,ua^k-1}]_{k=1}^{\infty}=$$

$$=\frac{\sum_{n=0}^{\infty}\left(u^{-2n}v^{-2n-1}a^{-n^2}+u^{-2n-1}v^{-2n-2}a^{-(n+1)^2}\right)\prod_{i=1}^{n}(a^{2i}-1)^{-1}}{\sum_{n=0}^{\infty}(uv)^{-2n}a^{-n^2}\prod_{i=1}^{n}(a^{2i}-1)^{-1}}.$$

Rational approximations to the various Hurwitz continued fractions have been studied by many authors. Especially, Tasoev [10] obtained a general result. Let

$$\alpha=[a_0;a_1,\dots,a_s,\overline{c_1+kd_1,\dots,c_m+kd_m}]_{k=1}^{\infty},$$

where a_0 is an integer and other partial quotients take positive integral values. Let $\Omega>0$ be the number of nonzero numbers d_i and $d=$ $=\max_{1\le i\le m} d_i$. Then for $C=\Omega/d$ and any $\epsilon>0$

$$\left|\alpha-\frac{p}{q}\right|<(C+\epsilon)\frac{\log\log q}{q^2\log q}$$

for infinitely many integers p, q, while there is a positive constant q_0 such that

$$\left|\alpha-\frac{p}{q}\right|>(C-\epsilon)\frac{\log\log q}{q^2\log q}$$

for all integers p, $q(\ge q_0)$.

Tasoev gave a result in the exponential case, too. Let $\alpha=$ $=[a_0;\overline{\underbrace{a^k,\dots,a^k}_{m}}]_{k=1}^{\infty}$ with integers a_0, $a>1$ and $m>1$. Then for $C=1/\sqrt{a}$ and any $\epsilon>0$

$$\left|\alpha-\frac{p}{q}\right|<(C+\epsilon)q^{-2-\sqrt{2\log a/(m\log q)}}$$

for infinitely many integers p, q, while there is a positive constant q_0 depending on a, m and ϵ such that

$$\left|\alpha - \frac{p}{q}\right| > (C-\epsilon)q^{-2-\sqrt{2\log a/(m\log q)}}$$

for all integers $p, q(\geq q_0)$. Some other minor results can be found in [9].

As seen in [4], some of the Tasoev continued fractions coincides with some of the Rogers–Ramanujan continued fractions. From this point of view, Shiokawa [7] proved the following. Let $f(\alpha, x)$ be the Rogers–Ramanujan continued fraction defined by $f(\alpha, x) = 1 + + \frac{\alpha x}{1} + \frac{\alpha x^2}{1} + \frac{\alpha x^3}{1} + \cdots$. Let a, b and d be positive integers such that $\gcd(b,d) = 1$, $a \geq 2$ and d divides a and let $C = \sqrt{b/d}$ if $(d/b)^2 > > a$; $\sqrt{d/(ab)}$ otherwise. Then, for any $\epsilon > 0$

$$\left|f\left(\frac{d}{b}, \frac{1}{a}\right) - \frac{p}{q}\right| < (C+\epsilon)q^{-2-\sqrt{\log a/\log q}}$$

for infinitely many integers p, q, while there is a positive constant $q_0 = = q_0(a, b, d, \epsilon)$ such that

$$\left|f\left(\frac{d}{b}, \frac{1}{a}\right) - \frac{p}{q}\right| > (C-\epsilon)q^{-2-\sqrt{\log a/\log q}}$$

for all integers p, $q(\geq q_0)$. Note that $f(d/b, 1/a) = [1; ba^k/d, a^k]_{k=1}^\infty$.

In this paper we shall give the rational approximation to a general Tasoev continued fraction of the type

$$\alpha = [b_0; b_1, \ldots, b_s, \overline{u_1 a_1^k + v_1, \ldots, u_m a_m^k + v_m}]_{k=1}^\infty ,$$

where b_0 is an integer, b_1, ..., b_s are positive integers, $u_j a_j^k + v_j$ ($j = = 1, 2, \ldots, m$) takes a positive integral value for $k = 1, 2, \ldots$ and at least one of u's is not zero.

2. Main result

It is sufficient to consider the case where

$$\alpha = [0; \overline{u_1 a_1^k + v_1, \ldots, u_r a_r^k + v_r, v_{r+1}, \ldots, v_{r+l}}]_{k=1}^\infty ,$$

where $u_j > 0$ $(1 \leq j \leq r)$ and $r + l = m$.

Theorem 1. *Let* $A = a_1 \ldots a_r$, $U = \prod_{j=1}^r u_j$ *and* $V = \prod_{\nu=1}^l v_{r+\nu}$. *Then for any* $\epsilon > 0$,

$$\left|\alpha - \frac{p}{q}\right| < (1+\epsilon)q^{-2-C^*}$$

for infinitely many integers p, q, *while there is a positive constant* q_0

depending on a_j, u_j $(1 \le j \le r)$, v_j $(1 \le j \le m)$ *and* ϵ *such that*

$$\left|\alpha - \frac{p}{q}\right| > (1-\epsilon)q^{-2-C^*}$$

for all integers p, $q(\ge q_0)$, *where*

$$C^* = \max_{1\le j\le r}\left(\left(\log(u_j\sqrt{a_j}) - \frac{\log a_j \log(a_1\dots a_{j-1}UV)}{\log A}\right)\frac{1}{\log q} + \right.$$
$$\left. + \frac{\sqrt{2}\log a_j}{\sqrt{\log A\cdot\log q}}\right).$$

Remark. (1) If $r = m$, $u_j = 1$, $a_j = a$ and $v_j = 0$ $(1 \le j \le m)$, then Th. 1 coincides with the Tasoev's approximation above.

(2) If $r = m = 2$, $u_1 = b/d$, $a_1 = a_2 = a$ and $v_1 = v_2 = 0$, then Th. 1 coincides with the Shiokawa's approximation above.

As some applications of Th. 1 we can obtain the following new results.

Example 1. Let

$$\alpha = \frac{\sum\limits_{n=0}^{\infty} u^{-n-1}v^{-n}a^{-(n+1)^2}\prod\limits_{i=1}^{n}(a^{2i}-(-1)^i)^{-1}}{\sum\limits_{n=0}^{\infty}(-1)^n u^{-n}v^{-n}a^{-n^2}\prod\limits_{i=1}^{n}(a^{2i}-(-1)^i)^{-1}} =$$
$$= [0; \overline{ua^{2k-1}-1, 1, va^{2k}-1}]_{k=1}^{\infty}.$$

Set $r = 2$, $l = 1$, $a_1 = a_2 = a^2$, $u_1 = u/a$, $u_2 = v$, $v_1 = v_2 = -1$ and $v_3 = 1$ in Th. 1. Let

$$C = \begin{cases} \sqrt{v/(ua)} & \text{if } u \ge v; \\ \sqrt{u/(va)} & \text{if } u < v. \end{cases}$$

Then, for any $\epsilon > 0$

$$\left|\alpha - \frac{p}{q}\right| < (C+\epsilon)q^{-2-\sqrt{2\log a/\log q}}$$

for infinitely many integers p, q, while there is a positive constant $q_0 = = q_0(a, u, v, \epsilon)$ such that

$$\left|\alpha - \frac{p}{q}\right| > (C-\epsilon)q^{-2-\sqrt{2\log a/\log q}}$$

for all integers p, $q(\ge q_0)$.

Example 2. Let

$$\beta = \frac{\sum_{n=0}^{\infty} u^{-2n-1} v^{-2n} a^{-(n+1)^2} \prod_{i=1}^{n} (a^{2i}-1)^{-1}}{\sum_{n=0}^{\infty} \left((uv)^{-2n} a^{-n^2} - (uv)^{-2n-1} a^{-(n+1)^2}\right) \prod_{i=1}^{n} (a^{2i}-1)^{-1}} =$$

$$= [0; \overline{ua^k - 1, 1, v-1}]_{k=1}^{\infty} .$$

Set $r = 1$, $l = 2$, $a_1 = a$, $u_1 = u$, $v_1 = -1$, $v_2 = 1$ and $v_3 = v - 1$ in Th. 1. Then, for any $\epsilon > 0$

$$\left|\beta - \frac{p}{q}\right| < \left(\frac{v-1}{\sqrt{a}} + \epsilon\right) q^{-2-\sqrt{2\log a/\log q}}$$

for infinitely many integers p, q, while there is a positive constant $q_0 = = q_0(a, u, v, \epsilon)$ such that

$$\left|\beta - \frac{p}{q}\right| > \left(\frac{v-1}{\sqrt{a}} - \epsilon\right) q^{-2-\sqrt{2\log a/\log q}}$$

for all integers p, $q(\geq q_0)$.

3. Proof of Theorem

For the proof we need the following.

Lemma 1. *Let $[a_0; a_1, a_2, \dots]$ be a continued fraction with its convergents $p_n/q_n = [a_0; a_1, \dots, a_n]$ $(n = 0, 1, \dots)$. If $\sum_{n=1}^{\infty} (a_n a_{n+1})^{-1} < \infty$, then $q_n/(a_1 a_2 \cdots a_n)$ converges to a finite non-zero limit as $n \to \infty$.*

Proof ([6], Lemma 1). By the definition,

$$q_1 = a_1, \quad q_2 = a_1 a_2 \left(1 + \frac{1}{a_1 a_2}\right)$$

and

$$q_3 = a_1 a_2 a_3 \left(1 + \frac{1}{a_1 a_2}\right) \left(1 + \frac{1}{a_2 a_3} \left(1 + \frac{1}{a_1 a_2}\right)^{-1}\right) =$$

$$= a_1 a_2 a_3 \left(1 + \frac{1}{a_1 a_2}\right) \left(1 + \frac{t_2}{a_2 a_3}\right)$$

for some t_2 with $1/2 \leq t_2 < 1$. Similarly, we get

$$q_n = a_1 a_2 \cdots a_n \prod_{j=1}^{n-1} \left(1 + \frac{t_j}{a_j a_{j+1}}\right)$$

for some t_j with $1/2 < t_j < 1$ $(3 \le j \le n-1)$. Hence, if $\sum_{n=1}^{\infty} (a_n a_{n+1})^{-1} < \infty$, then $q_n/(a_1 a_2 \cdots a_n)$ converges to a finite non-zero limit as $n \to \infty$. ◊

Proof of Th. 1. When $n = (k-1)m$, we have $a_{n+1} = u_1 a_1^k + v_1$ and

$$a_1 a_2 \cdots a_n = \prod_{i=1}^{k-1} \prod_{j=1}^{r} (u_j a_j^i + v_j) \cdot \prod_{\nu=1}^{l} v_{r+\nu}^{k-1}.$$

By Lemma 1

$$\begin{aligned} \log q_n &= \sum_{i=1}^{k-1} \sum_{j=1}^{r} \log(u_j a_j^i + v_j) + \sum_{\nu=1}^{l} \log v_{r+\nu}^{k-1} + O(1) = \\ &= \frac{k(k-1)}{2} \log A + (k-1) \log(UV) + O(1) = \\ &= \left(k - \frac{1}{2} + \frac{\log(UV)}{\log A}\right)^2 \frac{\log A}{2} + O(1). \end{aligned}$$

It follows that

$$k \sim \frac{1}{2} - \frac{\log(UV)}{\log A} + \sqrt{\frac{2 \log q_n}{\log A}}.$$

As well-known,

$$\frac{1}{(a_{n+1}+2)q_n^2} < \left|\alpha - \frac{p_n}{q_n}\right| = \frac{1}{(\alpha_{n+1} q_n + q_{n+1})} < \frac{1}{a_{n+1} q_n^2},$$

where $\alpha_{n+1} = [a_{n+1}; a_{n+2}, \ldots]$. Therefore,

$$\begin{aligned} \left|\alpha - \frac{p_n}{q_n}\right| &\sim \frac{1}{u_1 q_n^{k \log a_1 / \log q_n} q_n^2} \sim \\ &\sim q_n^{-2 - \left(\log(u_1 \sqrt{a_1}) - \frac{\log a_1 \log(UV)}{\log A}\right) \frac{1}{\log q_n} + \frac{\sqrt{2} \log a_1}{\sqrt{\log A \cdot \log q_n}}}. \end{aligned}$$

When $n = (k-1)m + 1$, we have $a_{n+1} = u_2 a_2^k + v_2$ and

$$a_1 a_2 \cdots a_n = (u_1 a_1^k + v_1) \prod_{i=1}^{k-1} \prod_{j=1}^{r} (u_j a_j^i + v_j) \cdot \prod_{\nu=1}^{l} v_{r+\nu}^{k-1}.$$

Hence,

$$\log q_n =$$
$$= \log(u_1 a_1^k + v_1) + \sum_{i=1}^{k-1}\sum_{j=1}^{r} \log(u_j a_j^i + v_j) + \sum_{\nu=1}^{l} \log v_{r+\nu}^{k-1} + O(1) =$$
$$= \frac{k(k-1)}{2}\log A + (k-1)\log(UV) + k\log a_1 + \log u_1 + O(1) =$$
$$= \left(k - \frac{1}{2} + \frac{\log(a_1 UV)}{\log A}\right)^2 \frac{\log A}{2} + O(1)\,.$$

It follows that

$$k \sim \frac{1}{2} - \frac{\log(a_1 UV)}{\log A} + \sqrt{\frac{2\log q_n}{\log A}}\,.$$

Therefore, we obtain

$$\left|\alpha - \frac{p_n}{q_n}\right| \sim \frac{1}{u_2 q_n^{k\log a_2/\log q_n} q_n^2} \sim$$
$$\sim q_n^{-2-\left(\log(u_2\sqrt{a_2}) - \frac{\log a_2 \log(a_1 UV)}{\log A}\right)\frac{1}{\log q_n} + \frac{\sqrt{2}\log a_2}{\sqrt{\log A\cdot\log q_n}}}\,.$$

Similarly, when $n = (k-1)m + (j-1)$ $(j = 1, 2, \dots, r)$, we have

$$\left|\alpha - \frac{p_n}{q_n}\right| \sim q_n^{-2-\left(\log(u_j\sqrt{a_j}) - \frac{\log a_j \log(a_1\dots a_{j-1} UV)}{\log A}\right)\frac{1}{\log q_n} + \frac{\sqrt{2}\log a_j}{\sqrt{\log A\cdot\log q_n}}}\,.$$

On the other hand, when $n = (k-1)m + (r-1+\nu)$ $(\nu = 1, 2, \dots, l)$, we have

$$\frac{1}{(v_{r+\nu}+2)q_n^2} < \left|\alpha - \frac{p_n}{q_n}\right| < \frac{1}{v_{r+\nu} q_n^2}\,.\ \Diamond$$

References

[1] JONES, W. B. and THRON, W. J.: Continued Fractions: Analytic theory and applications, Encyclopedia of mathematics and its applications; vol. 11, Addison-Wesley Reading, 1980.

[2] KOMATSU, T.: On Tasoev's continued fractions, *Math. Proc. Cambridge Philos. Soc.* **134** (2003), 1–12.

[3] KOMATSU, T.: On Hurwitzian and Tasoev's continued fractions, *Acta Arith.* **107** (2003), 161–177.

[4] KOMATSU, T.: On Tasoev's continued fractions and Rogers–Ramanujan continued fractions, *J. Number Theory*, to appear.

[5] KOMATSU, T.: Hurwitz and Tasoev continued fractions, *Monatsh. Math.*, to appear.

[6] SHIOKAWA, I.: Rational approximations to the values of certain hypergeometric functions, in: Number Theory and Combinatorics, Japan 1984 (eds. J. Akiyama et al.), World Sci. Publ. Co., Singapore, 1985, pp. 353–367.

[7] SHIOKAWA, I.: Rational approximations to the Rogers–Ramanujan continued fraction, *Acta Arith.* **50** (1988), 23–30.

[8] TASOEV, B. G.: Certain problems in the theory of continued fractions, *Trudy Tbiliss. Univ. Mat. Mekh. Astronom.* **16/17** (1984), 53–83 (in Russian).

[9] TASOEV, B. G.: Rational approximations to some infinite continued fractions, *Trudy Tbiliss. Univ. Mat. Mekh. Astronom.* **24** (1988), 104–138 (in Russian).

[10] TASOEV, B. G.: Rational approximations to certain numbers, *Mat. Zametki* **67** (2000), 931–937. English transl. in *Math. Notes* **67** (2000), 786–791.