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WEIGHTED (0,1,3)-INTERPOLATION ON THE ROOTS OF THE CLASSICAL ORTHOGONAL POLYNOMIALS

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Abstract: The aim of this paper is to give the existence, uniqueness and representations of the weighted $(0,1,3)$-interpolation polynomials on the roots of all classical orthogonal polynomials.

1. Introduction and preliminaries

1.1. Recently the weighted Lagrange and Hermite–Fejér interpolation are researched intensively (see e.g. [5], [12], [18], [16], [17] and the references therein). In this paper we shall investigate a weighted lacunary (or weighted Birkhoff type) interpolation process. As G. G. Lorentz [6] has remarked the Birkhoff interpolation problem differs from the more familiar Lagrange and Hermite interpolation in both its problems and its methods. The Lagrange and Hermite interpolation problems always uniquely solvable for every choice of nodes, but a given Birkhoff interpolation problem may not give a (unique) solution. For example the $(0,2)$-interpolation problem can be unsolvable or can have infinitely many solutions for a suitable choice of the nodal points. Another

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difficulty is that they have no simple explicit form and therefore convergence theorems on these polynomials are rather complicated (see [13, Chapter VII]). In order to avoid these difficulties, in 1961 J. Balázs [1] introduced the investigation of the weighted $(0,2)$-interpolation and he showed that using a suitable weight function this problem has a unique solution when the nodal points are the roots of the ultraspherical polynomials. He also proved a convergence theorem. Further investigations showed that similar results hold on the roots of the Hermite polynomials (cf. [14], [2]), Jacobi polynomials (cf. [3]) and Laguerre polynomials (cf. [4]).

The analogue problem with respect to the weighted $(0,1,3)$-interpolation on the roots of the Hermite polynomials was studied by K. K. Mathur and R. B. Saxena in [8].

The aim of this paper is to show that similarly to the weighted $(0,2)$-interpolation (cf. [15]) the weighted $(0,1,3)$-interpolation problem can be treated in a unified way on the roots of all classical orthogonal polynomials with respect to its existence, uniqueness and representation. Convergence theorems will be proved in forthcoming papers.

1.2. Classical orthogonal polynomials. First we recall that classical orthogonal polynomials can be derived in a unified way (see [9, Part I]).

Consider the following second order linear homogeneous differential equation

$$\sigma y'' + \tau y' + \lambda y = 0, \tag{1.1}$$

where σ (resp. τ) is a polynomial of degree not greater than 2 (resp. 1) and λ is a real parameter. Let ϱ be a solution of the differential equation

$$(\sigma\varrho)' = \tau\varrho. \tag{1.2}$$

The equation (1.1) can be written in the following self adjoint form

$$(\sigma\varrho y')' + \lambda\varrho y = 0. \tag{1.3}$$

It can be shown (see [9]) that the equation (1.3) has a polynomial solution if and only if

$$\lambda = \lambda_n = -n\tau' - \frac{n(n-1)}{2}\sigma'' \qquad (n \in \mathbb{N}),$$

and

$$\mu_k = \lambda + k\tau' + \frac{k(k-1)}{2}\sigma'' \neq 0 \qquad (k = 0, 1, \dots, n-1).$$

These polynomials have the following explicit forms

$$(1.4)\qquad y_n(x) = \frac{B_n}{\varrho(x)}\,[\sigma^n(x)\varrho(x)]^{(n)} \qquad (n \in \mathbb{N}),$$

where B_n are arbitrary real numbers. (This is the so called Rodrigues formula.)

We want to find the solutions of (1.2). Denote $I = (a,b) \subset \mathbb{R}$ the maximal interval in which the solution ϱ of (1.2) can be defined, and suppose that the function ϱ is positive and integrable on I. It can be shown (see [11, II.1]) that under these condition the equation (1.2) has the following three different solutions only (disregarding a linear variable transformation)

$$\varrho(x) = e^{-x^2} \qquad (x \in \mathbb{R}),$$
$$\varrho(x) = x^\alpha e^{-x} \qquad (x \in (0,+\infty),\ \alpha > -1),$$
$$\varrho(x) = (1-x)^\alpha(1+x)^\beta \quad (x \in (-1,1),\ \alpha,\beta > -1).$$

From these it follows that the polynomial solutions of (1.2) under the given conditions above are exactly the well known classical orthogonal polynomials, that is the Jacobi polynomials $P_n^{(\alpha,\beta)}$ $(\alpha,\beta > -1)$, the Laguerre polynomials L_n^α $(\alpha > -1)$ and Hermite polynomials H_n. The common explicit representation of these polynomials are given by (1.4) (see also Table 1).

$p_n(x)$	$P_n^{(\alpha,\beta)}(x)$ $(\alpha>-1,\beta>-1)$	$L_n^{(\alpha)}(x)$ $(\alpha>-1)$	$H_n(x)$
(a,b)	$(-1,1)$	$(0,+\infty)$	$(-\infty,+\infty)$
$\varrho(x)$	$(1-x)^\alpha(1+x)^\beta$	$x^\alpha e^{-x}$	e^{-x^2}
$\sigma(x)$	$1-x^2$	x	1
$\tau(x)$	$\beta-\alpha-(\alpha+\beta+2)x$	$1+\alpha-x$	$-2x$

Table 1

1.3. Weighted $(0,1,3)$-interpolation. Let a system of the nodal points $\{x_{k,n} \mid k = 1,2,\dots,n\}$ $(n \in \mathbb{N})$ be given in the finite or infinite interval (a,b) and let $w \in C^3(a,b)$ be a weight function. Determine a polynomial R_n of lowest possible degree satisfying the conditions

$$(1.5)\qquad R_n(x_{i,n}) = y_{i,n}, \quad R_n'(x_{i,n}) = y_{i,n}', \quad \left(w^2R_n\right)'''(x_{i,n}) = y_{i,n}''' \qquad (i = 1,2,\dots,n),$$

where $y_{i,n}, y_{i,n}'$ and $y_{i,n}'''$ $(i = 1,2,\dots,n)$ are arbitrarily given real numbers.

K. K. Mathur and R. B. Saxena [8] investigated this problem when $x_{k,n}$'s ($k = 1, 2, \dots, n$) are the roots of the n-th Hermite polynomial H_n and $w(x) := e^{-x^2/2}$ ($x \in \mathbb{R}$). They proved that there does not exist – in general – a polynomial R_n of degree $\leq 3n - 1$ satisfying the conditions (1.5). If n is even then under some additional condition for $R_n(0)$ there exists a unique polynomial of degree $\leq 3n$. (If n is odd then the uniqueness is not true.) They gave the explicit form of these polynomials and also proved a convergence theorem.

In this note we shall study the above problem in those cases when the nodal points are the roots of the classical orthogonal polynomials.

2. Results

In the sequel (p_n, $n \in \mathbb{N}$) denotes a system of the classical orthogonal polynomials on the interval (a, b), and

$$(2.1) \qquad -\infty \leq a < x_{n,n} < x_{n-1,n} < \dots < x_{1,n} < b \leq +\infty$$

the roots of p_n ($n \in \mathbb{N}$). Let $\ell_{k,n}$ represent the Lagrange-fundamental polynomials corresponding to the nodal points $x_{k,n}$, i.e.

$$(2.2) \qquad \ell_{k,n}(x) = \frac{p_n(x)}{p_n'(x_{k,n})(x - x_{k,n})} \qquad (k = 1, 2, \dots, n; \ n \in \mathbb{N}).$$

Choose the weight of the weighted (0, 1, 3)-interpolation by

$$(2.3) \qquad w(x) := \sqrt{\sigma(x)\varrho(x)} \qquad (x \in (a, b)).$$

Now we formulate a negative result with respect to the existence.

Theorem 2.1. *In general there is no polynomial R_n of degree $\leq 3n - 1$ satisfying conditions* (1.5).

Fortunately we can construct such polynomials of degree $\leq 3n$ in relatively simple form.

Lemma 2.2. *The polynomials*

$$(2.4) \qquad C_{k,n}(x) = \frac{p_n^2(x)}{6w^2(x_{k,n})\left[p_n'(x_{k,n})\right]^2} \int_0^x \ell_{k,n}(t)dt \qquad (k = 1, 2, \dots, n; \ n \in \mathbb{N})$$

are of degree $3n$ satisfying the requirements

$$C_{k,n}(x_{j,n})=0,\quad C'_{k,n}(x_{j,n})=0,\quad \big(w^2C_{k,n}\big)'''(x_{j,n})=\delta_{k,j} \qquad (j,k=1,2,\dots,n), \tag{2.5}$$

where $\delta_{k,j}$ *denotes the Kronecker symbol.*

Lemma 2.3. *The polynomials*

$$B_{k,n}(x)=(x-x_{k,n})\ell^3_{k,n}(x)+ + \frac{p_n^2(x)}{\big[p'_n(x_{k,n})\big]^2}\int_0^x \frac{\big[\gamma_{k,n}(t-x_{k,n})+\ell'_{k,n}(x_{k,n})\big]\ell_{k,n}(t)-\ell'_{k,n}(t)}{t-x_{k,n}}dt \qquad (k=1,2,\dots,n;\ n\in\mathbb{N}) \tag{2.6}$$

are of degree $3n$, *where*

$$\gamma_{k,n}=-\frac{w''(x_{k,n})}{w(x_{k,n})}+\big[\ell'_{k,n}(x_{k,n})\big]^2-\frac{1}{2}\ell''_{k,n}(x_{k,n})$$

satisfy the requirements

$$B_{k,n}(x_{j,n})=0,\quad B'_{k,n}(x_{j,n})=\delta_{k,j},\quad \big(w^2B_{k,n}\big)'''(x_{j,n})=0 \qquad (j,k=1,2,\dots,n). \tag{2.7}$$

Lemma 2.4. *The polynomials*

$$A_{k,n}(x)=\ell^3_{k,n}(x)-3\ell'_{k,n}(x_{k,n})B_{k,n}(x)+\frac{p_n^2(x)}{\big[p'_n(x_{k,n})\big]^2}\times \times\int_0^x \frac{\big[\alpha_{k,n}(t-x_{k,n})^2+\beta_{k,n}(t-x_{k,n})+\ell'_{k,n}(x_{k,n})\big]\ell_{k,n}(t)-\ell'_{k,n}(t)}{(t-x_{k,n})^2}dt \qquad (k=1,2,\dots,n;\ n\in\mathbb{N}) \tag{2.8}$$

are of degree $3n$, *where*

$$\beta_{k,n}=\ell''_{k,n}(x_{k,n})-\big[\ell'_{k,n}(x_{k,n})\big]^2,$$

$$\alpha_{k,n}=-\frac{1}{3}\frac{w'''(x_{k,n})}{w(x_{k,n})}-2\ell'_{k,n}(x_{k,n})\frac{w''(x_{k,n})}{w(x_{k,n})}+ +3\big[\ell'_{k,n}(x_{k,n})\big]^3-\frac{3}{2}\ell'_{k,n}(x_{k,n})\ell''_{k,n}(x_{k,n})$$

and they satisfy the requirements

$$A_{k,n}(x_{j,n})=\delta_{k,j},\quad A'_{k,n}(x_{j,n})=0,\quad \big(w^2A_{k,n}\big)'''(x_{j,n})=0 \qquad (j,k=1,2,\dots,n). \tag{2.9}$$

In the following statement we give the explicit form of all polynomials of degree $\leq 3n+1+s$ ($s \geq -1$ is a fixed integer) satisfying the requirements (1.5).

Theorem 2.5. *Let $s \geq -1$ be a fixed integer and $A_{k,n}, B_{k,n}, C_{k,n}$ ($k = 1, 2, \ldots, n$) are given by Lemma 2.4, Lemma 2.3 and Lemma 2.2. Then for every $h \in \mathcal{P}_s$ and $c \in \mathbb{R}$ the polynomial*

$$(2.10)\qquad \begin{aligned} R_n(x) &= \sum_{k=1}^{n} y_{k,n} A_{k,n} + \sum_{k=1}^{n} y'_{k,n} B_{k,n}(x) + \sum_{k=1}^{n} y'''_{k,n} C_{k,n}(x) + \\ &+ p_n^2(x) \left\{ \int_0^x p_n(t) h(t) dt + c \right\} \qquad (n \in \mathbb{N}) \end{aligned}$$

has of degree $\leq 3n+1+s$ satisfying (1.5). Conversely, if $R_n \in \mathcal{P}_{3n+1+s}$ obeys (1.5) then R_n has the form (2.10) with suitable $h \in \mathcal{P}_s$ and $c \in \mathbb{R}$. For $s = -1$, $h \in \mathcal{P}_s$ denotes $h(x) \equiv 0$ for all $x \in \mathbb{R}$.

From this result it follows that for the uniqueness of the weighted (0,1,3)-interpolation polynomials $R_n \in \mathcal{P}_{3n}$ we have to make an additional condition beside of (1.5). In the following statement we choose a Balázs type additional condition.

Theorem 2.6. *If $p_n(0) \neq 0$ then there exists exactly one polynomial $\overline{R}_n$ of degree $\leq 3n$ satisfying*

$$(2.11)\qquad \begin{aligned} &\overline{R}_n(x_{i,n}) = y_{i,n}, \quad \overline{R}'_n(x_{i,n}) = y'_{i,n}, \quad \left(w^2 \overline{R}_n\right)'''(x_{i,n}) = y'''_{i,n} \\ &\qquad\qquad (i = 1, 2, \ldots, n), \\ &\overline{R}_n(0) = \sum_{k=1}^{n} \left[y_{k,n} + 3 y_{k,n} x_{k,n} \ell'_{k,n}(x_{k,n}) - y'_{k,n} x_{k,n} \right] \ell_{k,n}^3(0), \end{aligned}$$

where $y_{i,n}, y'_{i,n}$ and $y'''_{i,n}$ ($i = 1, 2, \ldots, n$) are arbitrarily given real numbers. The explicit form of $\overline{R}_n$ is

$$(2.12)\qquad \overline{R}_n(x) = \sum_{k=1}^{n} y_{k,n} A_{k,n} + \sum_{k=1}^{n} y'_{k,n} B_{k,n}(x) + \sum_{k=1}^{n} y'''_{k,n} C_{k,n}(x),$$

where $A_{k,n}$, $B_{k,n}$ and $C_{k,n}$ are given by (2.8), (2.6) and (2.4).

Remarks. 1. If n is a such number for which $p_n(0) = 0$ then there are infinitely many polynomials of degree $\leq 3n$ satisfying (2.11). Indeed, in these cases for every $c \in \mathbb{R}$ the polynomials $\overline{R}_n(x) + c p_n^2(x)$ satisfy (2.11) (see (3.5)).

2. If $(p_n, n \in \mathbb{N})$ is a system of the classical orthogonal polynomials

then there is a subsequence $(p_{n_k}, k \in \mathbb{N})$ for which $p_{n_k}(0) \neq 0$ for all $k \in \mathbb{N}$. For the Laguerre polynomials $L_n^{(\alpha)}$ it is true for all $n \in \mathbb{N}$ and $\alpha > -1$. It is known that $H_n(0) \neq 0$ for all even $n \in \mathbb{N}$. Finally for the Jacobi polynomials $P_n^{(\alpha,\beta)}$ we have $P_n^{(\alpha,\beta)}(0) \neq 0$ if n is an odd number and $\alpha - \beta = 4l + 2$ $(l \in \mathbb{Z})$, or n is an even number and $\alpha - \beta = 4l$ $(l \in \mathbb{Z})$, or $n \in \mathbb{N}$ is an arbitrary number for other $\alpha, \beta > -1$ (see [3, p. 45]).

Corollary 2.7. *Let n be a natural number satisfying the condition $p_n(0) \neq 0$. If S is an arbitrary polynomial of degree $\leq 3n$ then for all $x \in \mathbb{R}$ we have*

$$S(x) = \sum_{k=1}^{n} S(x_{k,n}) A_{k,n}(x) + \sum_{k=1}^{n} S'(x_{k,n}) B_{k,n}(x) + \\ + \sum_{k=1}^{n} (w^2 S)'''(x_{k,n}) C_{k,n}(x) + c_n p_n^2(x),$$

where

$$c_n = \frac{1}{p_n^2(0)} \Big(S(0) - \sum_{k=1}^{n} \big[S(x_{k,n}) + 3S(x_{k,n}) x_{k,n} \ell'_{k,n}(x_{k,n}) - \\ - S'(x_{k,n}) x_{k,n} \big] \ell_{k,n}^3(0) \Big).$$

3. Proofs

3.1. First we mention some relations which will be used later. For the fundamental polynomials of Lagrange interpolation $\ell_{k,n}(x)$ (see (2.2)) we have

$$\ell'_{k,n}(x_{j,n}) = \frac{p'_n(x_{j,n})}{p'_n(x_{k,n})(x_{j,n} - x_{k,n})} \quad (j = 1, 2, \dots, n;\ j \neq k), \tag{3.1}$$

$$\ell'_{k,n}(x_{k,n}) = \frac{p''_n(x_{k,n})}{2p'_n(x_{k,n})}, \tag{3.2}$$

$$\ell''_{k,n}(x_{k,n}) = \frac{p'''_n(x_{k,n})}{3p'_n(x_{k,n})}. \tag{3.3}$$

If p_n denotes a classical orthogonal polynomial and w is given by (2.3) then using (1.1) and (1.2) we get

$$
(3.4)\qquad
\begin{aligned}
(wp_n)(x_{k,n}) &= 0,\\
(wp_n)'(x_{k,n}) &= w(x_{k,n})p_n'(x_{k,n}),\\
(wp_n)''(x_{k,n}) &= 0,\\
(wp_n)'''(x_{k,n}) &= 3w''(x_{k,n})\left[p_n'(x_{k,n})\right]^2+\\
&\qquad + 3w'(x_{k,n})p_n''(x_{k,n}) + w(x_{k,n})p_n'''(x_{k,n})
\end{aligned}
$$

and

$$
(3.5)\qquad
\begin{aligned}
(w^2p_n^2)(x_{k,n}) &= 0,\\
(w^2p_n^2)'(x_{k,n}) &= 0,\\
(w^2p_n^2)''(x_{k,n}) &= 2w^2(x_{k,n})\left[p_n'(x_{k,n})\right]^2,\\
(w^2p_n^2)'''(x_{k,n}) &= 0
\end{aligned}
$$

for all $k = 1, 2, \dots, n$ and $n \in \mathbb{N}$.

For the product of the fundamental polynomial of Lagrange interpolation $\ell_{k,n}(x)$ and the weight w (see (2.3)) we have the following relations: If $j, k = 1, 2, \dots, n$ then

$$
(3.6)\qquad
\begin{aligned}
(w\ell_{k,n}^2)(x_{j,n}) &= \delta_{k,j}w(x_{j,n}),\\
(w\ell_{k,n}^2)''(x_{j,n}) &= 2w(x_{j,n})\left[\ell_{k,n}'(x_{j,n})\right]^2 \quad k \neq j;
\end{aligned}
$$

$$
(3.7)\qquad
\begin{aligned}
(w\ell_{k,n}^2)''(x_{k,n}) &= w''(x_{k,n}) + 4w'(x_{k,n})\ell_{k,n}'(x_{k,n})+\\
&+2w(x_{k,n})\left[\ell_{k,n}'(x_{k,n})\right]^2 + 2w(x_{k,n})\ell_{k,n}''(x_{k,n}) =\\
&= w(x_{k,n})\left(\frac{w''(x_{k,n})}{w(x_{k,n})} - 2\left[\ell_{k,n}'(x_{k,n})\right]^2 + 2\ell_{k,n}''(x_{k,n})\right);
\end{aligned}
$$

$$
(3.8)\qquad \left(w^2\ell_{k,n}^3\right)'''(x_{j,n}) = 6w^2(x_{j,n})\left[\ell_{k,n}'(x_{j,n})\right]^3 \quad \text{if} \quad k \neq j;
$$

$$
(3.9)\qquad
\begin{aligned}
\left(w^2\ell_{k,n}^3\right)'''(x_{k,n}) &= 6w^2(x_{k,n})\Big(\frac{w'''(x_{k,n})}{3w(x_{k,n})}+\\
&+2\frac{w''(x_{k,n})}{w(x_{k,n})}\ell_{k,n}'(x_{k,n}) - 2\left[\ell_{k,n}'(x_{k,n})\right]^3 + \frac{1}{2}\ell_{k,n}'''(x_{k,n})\Big).
\end{aligned}
$$

3.2. Proof of Theorem 2.1. Let $n \in \mathbb{N}$ and $j \in \{1, 2, \dots, n\}$ be fixed numbers and choose $y_{k,n}', y_{k,n}', y_{k,n}'''$ $(k = 1, 2, \dots, n)$ such that

(3.10) $$y_{k,n}=0, \quad y'_{k,n}=0, \quad y'''_{k,n}=\delta_{k,j}.$$

Assume that there exists a polynomial R_n of degree $\le 3n-1$ satisfying the requirements (1.5). Then R_n has the following form

$$R_n(x)=p_n^2(x)Q_{n-1}(x),$$

where $Q_{n-1}(x)$ is a polynomial of degree $\le n-1$. Using (3.5) and (3.10) we obtain that

(3.11) $$\begin{aligned} y'''_{k,n} &= (w^2R_n)'''(x_{k,n})=3(w^2p_n^2Q_{n-1})'''(x_{k,n})= \\ &= (w^2p_n^2)''(x_{k,n})Q'_{n-1}(x_{k,n})= \\ &= 6w^2(x_{k,n})\left[p'_n(x_{k,n})\right]^2Q'_{n-1}(x_{k,n})=\delta_{k,j}. \end{aligned}$$

Since $w(x_{k,n})\neq 0$ and $p'_n(x_{k,n})\neq 0$ for all $k=1,2,\dots,n$ thus

$$Q'_{n-1}(x_{k,n})=0 \qquad (k=1,2,\dots,j-1,j+1,\dots,n);$$

from which it follows that $Q'_{n-1}(x)=0$ for all $x\in\mathbb{R}$, contradicting (3.11). This completes the proof of Th. 2.1. ◊

3.3. Proof of Lemma 2.2. Fix the index $k=1,2,\dots,n$. The polynomials $C_{k,n}$ are of degree $3n$, indeed, and it is clear that they satisfy the first and second condition of (2.5). Let

$$Q_{k,n}(x):=\int_0^x \ell_{k,n}(t)dt.$$

By (3.5) we get

$$(w^2C_{k,n})'''(x_{j,n})=\frac{1}{6w^2(x_{k,n})\left[p'_n(x_{k,n})\right]^2}(w^2p_n^2Q_{k,n})'''(x_{j,n})=$$

$$=\frac{3}{6w^2(x_{k,n})\left[p'_n(x_{k,n})\right]^2}(w^2p_n^2)''(x_{j,n})Q'_{k,n}(x_{j,n})=\ell_{k,n}(x_{j,n})=\delta_{k,j}.$$

Thus the third conditions of (2.5) also hold and this completes the proof of Lemma 2.2. ◊

3.4. Proof of Lemma 2.3. Fix the number $k=1,2,\dots,n$ and let

$$Q_{k,n}(x):=\int_0^x \frac{\left[\gamma_{k,n}(t-x_{k,n})+\ell'_{k,n}(x_{k,n})\right]\ell_{k,n}(t)-\ell'_{k,n}(t)}{t-x_{k,n}}dt.$$

It is clear that $Q_{k,n}$ is a polynomial and $\deg Q_{k,n}=n$. Therefore the polynomial $B_{k,n}$ is of degree $3n$, indeed. It is easy to see that $B_{k,n}$ satisfies the first and second condition of (2.7). Applying (3.4), (3.5) we get

$$(w^2B_{k,n})'''(x_{j,n}) =$$

$$= \frac{1}{p_n'(x_{k,n})}(w^2p_n\ell_{k,n}^2)'''(x_{j,n}) + \frac{1}{[p_n'(x_{k,n})]^2}(w^2p_n^2Q_{k,n})'''(x_{j,n}) =$$

$$= \frac{1}{p_n'(x_{k,n})}\Big((wp_n)'''(x_{j,n})(w\ell_{k,n}^2)(x_{j,n}) + 3(wp_n)'(x_{j,n})(w\ell_{k,n}^2)''(x_{j,n})\Big) +$$

$$+ \frac{3}{[p_n'(x_{k,n})]^2}(w^2p_n^2)''(x_{j,n})Q_{k,n}'(x_{j,n}).$$

For the proof of the third condition of (2.7) first we suppose that $k \neq j$ ($j = 1, 2, \ldots, n$). Then using

$$Q_{k,n}'(x_{j,n}) = -\frac{\ell_{k,n}'(x_{j,n})}{x_{j,n} - x_{k,n}},$$

(3.1), (3.4) and (3.6) we get

$$(w^2B_{k,n})'''(x_{j,n}) = 6w^2(x_{j,n})\frac{p_n'(x_{j,n})}{p_n'(x_{k,n})}\left[\ell_{k,n}'(x_{j,n})\right]^2 -$$

$$-6w^2(x_{j,n})\frac{p_n'(x_{j,n})}{p_n'(x_{k,n})} \cdot \frac{p_n'(x_{j,n})}{p_n'(x_{k,n})(x_{j,n} - x_{k,n})}\ell_{k,n}'(x_{j,n}) = 0.$$

If $k = j$ then

$$Q_{k,n}'(x_{k,n}) = \gamma_{k,n} + \left[\ell_{k,n}'(x_{k,n})\right]^2 - \ell_{k,n}''(x_{k,n})$$

so from (3.5) we obtain that

$$\frac{1}{\left[p_n'(x_{k,n})\right]^2}(w^2p_n^2Q_{k,n})'''(x_{k,n}) =$$

$$= \frac{3}{[p_n'(x_{k,n})]^2}(w^2p_n^2)''(x_{k,n})Q_{k,n}'(x_{k,n}) =$$

$$= 6w^2(x_{k,n})\left(-\frac{w''(x_{k,n})}{w(x_{k,n})} + 2\left[\ell_{k,n}'(x_{k,n})\right]^2 - \frac{3}{2}\ell_{k,n}''(x_{k,n})\right).$$

Moreover by (3.2), (3.3), (3.4) and (3.7)

$$\frac{1}{p_n'(x_{k,n})}(w^2p_n\ell_{k,n}^2)'''(x_{k,n}) =$$

$$= \frac{1}{p_n'(x_{k,n})}\left((wp_n)'''(x_{k,n})(w\ell_{k,n}^2)(x_{k,n}) + \right.$$

$$\left. + 3(wp_n)'(x_{k,n})(w\ell_{k,n}^2)''(x_{k,n})\right) =$$

$$= 3w^2(x_{k,n})\left(\frac{w''(x_{k,n})}{w(x_{k,n})} - 2\left[\ell_{k,n}'(x_{k,n})\right]^2 + \ell_{k,n}''(x_{k,n})\right) +$$

$$+3w^2(x_{k,n})\left(\frac{w''(x_{k,n})}{w(x_{k,n})}-2\big[\ell'_{k,n}(x_{k,n})\big]^2+2\ell''_{k,n}(x_{k,n})\right)=$$

$$=6w^2(x_{k,n})\left(\frac{w''(x_{k,n})}{w(x_{k,n})}-2\big[\ell'_{k,n}(x_{k,n})\big]^2+\frac{3}{2}\ell''_{k,n}(x_{k,n})\right).$$

From the above relations it follows that $(w^2B_{k,n})'''(x_{k,n})=0$. Thus the third condition of (2.7) holds and this completes the proof of Lemma 2.3. ◊

3.5. Proof of Lemma 2.4. Fix the number $k=1,2,\dots,n$ and let

$q_{k,n}(t):=\big[\alpha_{k,n}(t-x_{k,n})^2+\beta_{k,n}(t-x_{k,n})+\ell'_{k,n}(x_{k,n})\big]\ell_{k,n}(t)-\ell'_{k,n}(t)$,

where $\alpha_{k,n},\beta_{k,n}$ given in Lemma 2.4. Since

$$q_{k,n}(x_{k,n})=0,\quad q'_{k,n}(x_{k,n})=0,$$

thus $q_{k,n}(t)/(t-x_{k,n})^2$ is a polynomial of degree $n-1$. Consequently $A_{k,n}$ is a polynomial and $\deg A_{k,n}=3n$.

It is easy to see that $A_{k,n}$ satisfies the first and second condition of (2.9).

Now we prove that

$$(w^2A_{k,n})'''(x_{j,n})=0\qquad(j=1,2,\dots,n).\tag{3.12}$$

If

$$Q_{k,n}(x):=\int_0^x\frac{q_{k,n}(t)}{(t-x_{k,n})^2}dt$$

then for $j=1,2,\dots,n$ and $k\neq j$ we get

$$Q'_{k,n}(x_{j,n})=-\frac{\ell'_{k,n}(x_{j,n})}{(x_{j,n}-x_{k,n})^2}$$

and

$$Q'_{k,n}(x_{k,n})=\frac{1}{2}q''_{k,n}(x_{k,n})=$$

$$=\alpha_{k,n}+\frac{3}{2}\ell'_{k,n}(x_{k,n})\ell''_{k,n}(x_{k,n})-\big[\ell'_{k,n}(x_{k,n})\big]^3-\frac{1}{2}\ell'''_{k,n}(x_{k,n}).$$

From (2.7) and (2.8) it follows that

$$(w^2A_{k,n})'''(x_{j,n})=(w^2\ell^3_{k,n})'''(x_{j,n})+\frac{(w^2p_n^2Q_{k,n})'''(x_{j,n})}{\big[p'_n(x_{k,n})\big]^2}$$

for all $j=1,2,\dots,n$.

If $j=1,2,\dots,n$ and $k\neq j$ then using (3.1) and (3.5) we get

$$\frac{(w^2p_n^2Q_{k,n})'''(x_{j,n})}{\big[p_n'(x_{k,n})\big]^2} = 6w^2(x_{j,n})\frac{\big[p_n'(x_{j,n})\big]^2}{\big[p_n'(x_{k,n})\big]^2}Q_{k,n}'(x_{j,n}) =$$
$$= 6w^2(x_{j,n})\frac{\big[p_n'(x_{j,n})\big]^2}{\big[p_n'(x_{k,n})\big]^2}\cdot\frac{-\ell_{k,n}'(x_{j,n})}{(x_{j,n}-x_{k,n})^2} = -6w^2(x_{j,n})\big[\ell_{k,n}'(x_{j,n})\big]^3.$$

Combining this with (3.8) we obtain (3.12) for $j \neq k$.

If $k = j$ then

$$\frac{(w^2p_n^2Q_{k,n})'''(x_{k,n})}{\big[p_n'(x_{k,n})\big]^2} = 6w^2(x_{k,n})Q_{k,n}'(x_{k,n}) =$$
$$= 6w^2(x_{k,n})\Big(\alpha_{k,n}+\frac{3}{2}\ell_{k,n}'(x_{k,n})\ell_{k,n}''(x_{k,n})-\big[\ell_{k,n}'(x_{k,n})\big]^3-\frac{1}{2}\ell_{k,n}'''(x_{k,n})\Big).$$

Using (3.9) and the definition of $\alpha_{k,n}$ we get $(w^2A_{k,n})'''(x_{k,n}) = 0$. Thus the third conditions of (2.9) hold and this completes the proof of Lemma 2.4. ◊

3.6. Proof of Theorem 2.5. From Lemmas 2.2, 2.3 and 2.4 it follows that the polynomial

$$\sum_{k=1}^{n} y_{k,n}A_{k,n}(x) + \sum_{k=1}^{n} y_{k,n}'B_{k,n}(x) + \sum_{k=1}^{n} y_{k,n}'''C_{k,n}(x)$$

has of degree $\leq 3n$ and it satisfies conditions (1.5). Let us denote by D_n the remainder part of R_n:

$$D_n(x) := p_n^2(x)\Big\{\int_0^x p_n(t)h(t)dt + c\Big\} \qquad (n \in \mathbb{N}), \tag{3.13}$$

where $h \in \mathcal{P}_s$ is an arbitrary polynomial and c is an arbitrary real number. Using (3.5) we have

$$D_n(x_{j,n}) = 0, \quad D_n'(x_{j,n}) = 0, \quad \big(w^2D_n\big)'''(x_{j,n}) = 0 \qquad (j,k = 1,2,\dots,n). \tag{3.14}$$

which means that the polynomial R_n (see (2.10)) satisfies (1.5).

Conversely, if $R_n \in \mathcal{P}_{3n+1+s}$ obeys (1.5) then the polynomial

$$D_n(x) := R_n(x) - \sum_{k=1}^{n} y_{k,n}A_{k,n}(x) - \sum_{k=1}^{n} y_{k,n}'B_{k,n}(x) - \sum_{k=1}^{n} y_{k,n}'''C_{k,n}(x)$$

has of degree $\leq 3n+1+s$ and satisfies (3.14). Thus D_n has the following form

$$D_n(x) = p_n^2(x)Q_n(x),$$

where $Q_n(x)$ is a polynomial of degree $\leq n+1+s$. Since by (3.5)

$$(w^2D_n)'''(x_{k,n}) = (w^2p_n^2Q_n)'''(x_{k,n}) = (w^2p_n^2)''(x_{k,n})Q_n'(x_{k,n}) =$$
$$= 6w^2(x_{k,n})\big[p_n'(x_{k,n})\big]^2Q_n'(x_{k,n}) = 0,$$

and $w(x_{k,n}) \neq 0$ moreover $p_n'(x_{k,n}) \neq 0$ thus we get

$$Q_n'(x_{k,n}) = 0 \qquad (k = 1, 2, \ldots, n).$$

Consequently $Q_n'(x) = p_n(x)h(x)$ and

$$Q_n(x) = \int_0^x p_n(t)h(t)dt + c$$

with suitable $h \in \mathcal{P}_s$ and $c \in R$. This completes the proof of Th. 2.5. ◊

3.7. Proof of Theorem 2.6. Applying Th. 2.5 with $h \equiv 0$ and $c = 0$ moreover the explicit forms of $A_{k,n}$, $B_{k,n}$ and $C_{k,n}$ (see (2.8), (2.6) and (2.4)) we obtain that the polynomial $\overline{R}_n$ obeys (2.11).

For the proof of the uniqueness suppose that there exists another polynomial R_n^* of degree at most $3n$ satisfying (2.11). Then for $k = 1, 2, \ldots, n$ $(n \in \mathbb{N})$ we have

$$\big(\overline{R}_n - R_n^*\big)(x_{k,n}) = 0, \qquad \big(\overline{R}_n - R_n^*\big)'(x_{k,n}) = 0,$$
$$\big(w^2(\overline{R}_n - R_n^*)\big)'''(x_{k,n}) = 0, \qquad \overline{R}_n(0) - R_n^*(0) = 0.$$

Hence it follows that

$$\overline{R}_n(x) - R_n^*(x) = p_n^2(x)Q_n(x),$$

where the polynomial Q_n is of degree at most n. By our condition $p_n(0) \neq 0$ so $Q_n(0) = 0$.

For the third derivative we get (see (3.5))

$$\big(w^2(\overline{R}_n - R_n^*)\big)'''(x_{k,n}) = 2w^2(x_{k,n})\big[p_n'(x_{k,n})\big]^2Q_n'(x_{k,n}) = 0$$
$$(k = 1, 2, \ldots, n),$$

i.e. $Q_n'(x_{k,n}) = 0$ $(k = 1, 2, \ldots, n)$ and $\deg Q_n \leq n$. This means that Q_n is a constant. Since $Q_n(0) = 0$ thus $\overline{R}_n(x) = R_n^*(x)$ for all $x \in \mathbb{R}$ and this completes the proof of Th. 2.6. ◊

3.8. Proof of Corollary 2.7. Let S be an arbitrary polynomial of degree $\leq 3n$ and consider the polynomial

$$T(x) := S(x) - \sum_{k=1}^{n} S(x_{k,n})A_{k,n}(x) - \sum_{k=1}^{n} S'(x_{k,n})B_{k,n}(x) -$$
$$- \sum_{k=1}^{n} \left(w^2 S\right)'''(x_{k,n})C_{k,n}(x).$$

By Lemmas 2.2, 2.3 and 2.4 we have

$$T(x_{k,n}) = 0 \quad \text{and} \quad T'(x_{k,n}) = 0 \quad (k = 1, 2, \dots, n),$$

i.e. the polynomial T has of the form

$$T(x) = p_n^2(x)Q_n(x),$$

where the polynomial Q_n has of degree at most n.

Using (3.5) we obtain that

$$\left(w^2 T\right)'''(x_{k,n}) = 0 = 2w^2(x_{k,n})\left[p_n'(x_{k,n})\right]^2 Q_n'(x_{k,n})$$

for all $k = 1, 2, \dots, n$. Therefore $Q_n \equiv c_n$ is a constant polynomial. Hence

$$c_n p_n^2(x) = S(x) - \sum_{k=1}^{n} S(x_{k,n})A_{k,n}(x) - \sum_{k=1}^{n} S'(x_{k,n})B_{k,n}(x) -$$
$$- \sum_{k=1}^{n} \left(w^2 S\right)'''(x_{k,n})C_{k,n}(x).$$

The value of c_n follows from the above relations. ◊

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