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DIVISIBILITY ORDERS ON SEMI-GROUPS II

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Abstract: In continuation of [10], Green's $\mathcal{L}$-preorder: $a \leq_{\mathcal{L}} b \Leftrightarrow a = xb$ for some $x \in S^1$, and a particular case of it: $a \leq_E b \Leftrightarrow a = eb$ for some $e \in E_S^1$, are studied with respect to the question when these relations are downwards directed partial orders on a semigroup S. The special classes of: (cancellative) monoids or finite or (E-unitary) E-inversive semigroups are considered. In particular, the case when $\leq_{\mathcal{L}}$ is a total order (with greatest element) is studied. Characterizations of such semigroups in different classes, as left or right simple, respectively right archimedean and right cancellative (or not), are given.

1. Introduction

Let $(S, \cdot)$ be a semigroup. Then Green's $\mathcal{L}$-preorder on S is defined by:

$$a \leq_{\mathcal{L}} b \Leftrightarrow a = xb \text{ for some } x \in S^1 \text{ (see [4]).}$$

The right dual of $\leq_{\mathcal{L}}$ is Green's $\mathcal{R}$-preorder $\leq_{\mathcal{R}}$. As a particular case, the relation

$$a \leq_E b \Leftrightarrow a = eb \text{ for some } e \in E_S^1$$

(E_S the set of idempotents of S) appears as a generalization of the natural partial order of inverse semigroups (see [2]). These two, in

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general not antisymmetric divisibility relations were studied in [10]: $\leq_{\mathcal{L}}$ is a partial order iff S is $\mathcal{L}$-trivial (see [9]); if E_S is a band then $\leq_E$ is a partial order iff E_S is a rightregular band. In the following we will go a step further investigating those semigroups S which have the property that $\leq_{\mathcal{L}}$ resp. $\leq_E$ are directed downwards, that is,

for any $a, b \in S$ there exists $c \in S$ such that $c \leq_{\mathcal{L}} a, b$ resp. $c \leq_E a, b$.

In Section 2 this problem is dealt with concerning $\leq_{\mathcal{L}}$. Characterizations are given for monoids, left-right duo semigroups, and cancellative monoids, which are directed downwards. For semigroups is which every left ideal is left principal, under these conditions the particular case of a lattice order is obtained. In Section 3, the relation $\leq_E$ is studied on (necessarily) E-inversive semigroups S whose idempotents commute (see [11]). If the idempotents of S are central then a necessary and sufficient condition is given in order that $\leq_E$ is directed downwards. If S is E-unitary than this occurs iff S is a semilattice. Also, the particular cases that S is inverse or finite is dealt with. In Section 4, we investigate semigroups S, for which $\leq_{\mathcal{L}}$ is a total order. If S is left- or right simple a description is provided. The main results concern semigroups S for which $\leq_{\mathcal{L}}$ is a total order with greatest element. The two cases, when S is right cancellative or not, are investigated under the hypothesis that S is right archimedean. In the first case, S is isomorphic with $(\mathbb{N}, +, \leq_d)$, where $\leq_d$ is the dual of the usual total order on $\mathbb{N}$. In the second, S is either a finite cylic nil-semigroup or an infinite nil-semigroup with right 0-cancellation, which is dense except possibly at 0. In the finite case this yields a characterization.

Throughout the paper, only non-trivial semigroup S are considered. The natural partial order on S is defined by

$$a \leq_S b \Leftrightarrow a = xb = by, xa = a (= ay), \text{ for some } x, y \in S^1 \text{ (see [7])}.$$

In general, $\leq_S$ is not right compatible with multiplication ([8]), whereas both $\leq_{\mathcal{L}}$ and $\leq_E$ above satisfy: $a \leq_{\mathcal{L}} b \quad (a \leq_E b) \Rightarrow ac \leq_{\mathcal{L}} bc \quad (ac \leq_E bc)$ for any $c \in S$. Generally, a semigroup S, which is partially ordered by a relation $\leq$, is called right partially ordered if $\leq$ is compatible with multiplication on the right. If $\leq$ is also compatible on the left then $(S, \cdot, \leq)$ is called a partially ordered semigroup. In case that S has an identity, which is the greatest or least element of $(S, \leq)$, then S is called integrally (right) partially ordered.

2. Green's $\mathcal{L}$-preorder $\leq_{\mathcal{L}}$

In this Section we deal with the problem when for a semigroup $(S,\cdot)$ the relation:

$$a \leq_{\mathcal{L}} b \Leftrightarrow a = xb \text{ for some } x \in S^1,$$

is a downwards directed partial order on S, that is, when for any $a, b \in$ $\in S$ there exists $c \in S$ such that $c \leq_{\mathcal{L}} a, b$. Evidently, this is the case if S contains a left zero z (then $z \leq_{\mathcal{L}} x$ for any $x \in S$). For the general case we have the following characterization. Recall that a semigroup S is right reversible (see [2]) if for any $a, b \in S$, $Sa \cap Sb \neq \varnothing$; furthermore S is $\mathcal{L}$-trivial if Green's equivalence $\mathcal{L}$ is the identity relation on S. Using [10], Th. 2.7, we immediately obtain

Lemma 2.1. *Let S be a semigroup. Then the relation $\leq_{\mathcal{L}}$ is a downwards directed partial order on S iff S is $\mathcal{L}$-trivial and right reversible.*

In case that S is a left-right duo semigroup, that is, every principal left ideal of S is twosided, we have

Corollary 2.2. *Let S be a semigroup such that $aS \subseteq Sa$ for any $a \in$ $\in S$. Then S is a downwards directed partially ordered semigroup with respect to $\leq_{\mathcal{L}}$ iff S is $\mathcal{L}$-trivial.*

Proof. *Necessity* holds by [10], Th. 2.7.

Sufficiency. By [10], Th. 2.7, the relation $\leq_{\mathcal{L}}$ is a right compatible partial oder on S. It is also left compatible by the proof of [10], Cor. 2.10. S is right reversible since for any $a, b \in S, ab = xa$ for some $x \in S$. It follows by Lemma 2.1 that $(S, \leq_{\mathcal{L}})$ is directed downwards. $\Diamond$

If S is a monoid then Cor. 2.2 and [10], Cor. 2.10, yield

Corollary 2.3. *Let S be a monoid. Then with respect to the relation $\leq_{\mathcal{L}}$, S is a downwards directed partially ordered semigroup (with 1_S as greatest element) iff S is $\mathcal{L}$-trivial and $aS \subseteq Sa$ for any $a \in S$.*

For cancellative monoids we have the following characterization. Recall that the negative cone of a (right) partially ordered group $(G, \cdot, \preccurlyeq)$ is the set $\{x \in G | x \preccurlyeq 1_G\}$.

Theorem 2.4. *Let S be a semigroup. Then S is a cancellative monoid, for which $\leq_{\mathcal{L}}$ is a downwards directed partial order, iff S is isomorphic with the negative cone of a downwards directed right partially ordered group.*

Proof. *Necessity.* This holds by Lemma 2.1 and Satz 8 of [6].

Sufficiency. Let S be (isomorphic with) the negative cone of the right partially ordered group $(G, \cdot, \preccurlyeq)$. Then by Satz 8 in [6], S is a cancellative right reversible monoid. Since $(S, \cdot, \preccurlyeq)$ is right partially ordered with $1_G = 1_S$ as greatest element, S is $\mathcal{L}$-trivial (see [9], Section 2). Hence by Lemma 2.1, $(S, \leq_{\mathcal{L}})$ is a downwards directed partially ordered set. $\Diamond$

Examples. Right reversible semigroups:

1. Every commutative semigroup S. Note that $ab \leq_{\mathcal{L}} a, b$ for all $a, b \in S$.

2. Any semigroup S with left-zeros. If $\leq_{\mathcal{L}}$ is a partial order on S then there exists only one left zero and this is the zero of S($z \leq_{\mathcal{L}} x$ for any $x \in S$ and $az \leq_{\mathcal{L}} z$ for any $a \in S$ imply that $az = z$).

3. Every E-dense semigroup S, that is, an E-inversive semigroup in which the idempotents commute (see [11]). Indeed, let $a, b \in S$; then by [11], $xa, yb \in E_S$ for some $x, y \in S$, whence $xa \cdot yb = yb \cdot xa \in Sa \cap Sb$. Note that $ef \leq_{\mathcal{L}} a, b$ for $e = xa, f = yb \in E_S$.

4. Any right archimedean semigroup S, that is, for all $a, b \in S$ there exist $n > 0$, $x \in S^1$, such that $a^n = xb$. Note that for any $a, b \in S$, there exist $m, n > 0$ such that $a^m \leq_{\mathcal{L}} a, b$ and $b^n \leq_{\mathcal{L}} a, b$.

5. Every archimedean semigroup S, that is, S is commutative and right archimedean (see **4**). If $E_S = \varnothing$ then by [10], Cor. 2.5(3), $\leq_{\mathcal{L}}$ is a partial order on S.

6. Any periodic (in particular, finite) semigroup S such that E_S forms a left zero semigroup. Indeed, S is right archimedean (see **4**): let $a, b \in S$; then $a'a = e \in E_S$ for some $a' \in S$ ($a^m \in E_S$ for some $m > 0$) and $b^n = f \in E_S$ for some $n > 0$; hence $f \cdot a'a = fe = f = b^n$, i.e., $b^n = xa$ for $x = fa' \in S$. The relation $\leq_{\mathcal{L}}$ is a partial order on S iff S is a nil-semigroup (by [10], Prop. 2.16).

Remarks. 1. It follows from Lemma 2.1, that for every $\mathcal{L}$-trivial semigroup S in the above list, the relation $\leq_{\mathcal{L}}$ is a downwards directed partial order. If in particular, S is cancellative then S is embeddable in a group (by [2], Th. 1.23). For the case that S has also an identity, see Th. 2.4, above. Note also that every right cancellative semigroup without right identity and every right cancellative monoid without proper units is $\mathcal{L}$-trivial (see [9], Prop. 2.1 and 2.2).

2. Every partially ordered monoid $(S, \cdot, \leq)$ with 1_S as greatest element is directed downwards. Indeed, if $a, b \in S$ then $a, b \leq 1_S$ implies $ab \leq a, b$. If S is an integrally right partially ordered monoid then by [10], Th. 2.9, S is $\mathcal{L}$-trivial. Supposing that E_S is finite we have the following

Proposition 2.5. *Let S be an $\mathcal{L}$-trivial semigroup such that E_S is* *finite and commutative. Then $(S, \cdot)$ has a zero (hence $(S, \leq_{\mathcal{L}})$ is a downwards directed partially ordered set) iff S is E-inversive (i.e., for any $a \in S$ there exists $x \in S$ such that $ax \in E_S$).*

Proof. *Necessity* is evident since $a0 = 0 \in E_S$ for any $a \in S$.

Sufficiency. Let $E_S = \{e_1, \cdots, e_n\}$ and put $e = e_1 \cdots e_n$. Then with respect to the usual (natural) partial oder of idempotents, $e \in S$ is the least idempotent of S: $e = (e_1 \cdots e_{i-1} e_{i+1} \cdots e_n) e_i$ implies $e \leq_S e_i$ and $e \in E_S$. Since S is E-inversive it follows by [9], Cor. 4.10, that $e \in E_S$ is the zero of $(S, \cdot)$. Because of $|S| > 1$ it follows by [10], Th. 2.7, that $\leq_{\mathcal{L}}$ is a nontrivial partial order on S with 0 as least element. Thus $(S, \leq_{\mathcal{L}})$ is directed downwards. $\lozenge$

Remark. In particular, every finite integrally right partially ordered monoid with commuting idempotents has a zero. Note that by [10], Th. 2.9, every $\mathcal{L}$-trivial monoid is an integrally right partially ordered semigroup.

Examples. 1. Let $S = (\mathbb{N}_0, \cdot, \leq)$ be the multiplicative monoid of natural numbers including zero, where $\leq$ denotes the dual of the divisibility order: $a \leq b \Leftrightarrow a = xb$ for some $x \in \mathbb{N}_0$ (note that $\leq$ is equal to $\leq_{\mathcal{L}}$). Then $n \leq 1$ for every $n \in \mathbb{N}_0, E_S = \{0, 1\}$ is finite and commutative, and $0 \in S$ is the zero of $(S, \cdot)$. Note that $0 \in S$ is the least element of $(S, \leq)$, hence $(S, \leq)$ is directed downwards ($\leq$ is even a lattice order).

2. Let $S = (\mathbb{N}, +, \leq)$ be the additive semigroup of natural numbers (without 0), where $\leq$ denotes the dual of the usual (total) order: $a \leq b \Leftrightarrow a = b$ or $a = x + b$ for some $x \in \mathbb{N}$ (note that $\leq$ is equal to $\leq_{\mathcal{L}}$). Then $n \leq 1$ for every $n \in \mathbb{N}$, $E_S = \varnothing$ and S has no zero element. Nevertheless, $(S, \leq)$ is directed downwards since $(\mathbb{N}, \leq)$ is a chain. This case will be investigated in Section 4.

3. Let $(Y, \cdot)$ be a finite semilattice and let $S = \bigcup_{e \in Y} T_e$ be an (infinite) inflation of Y (see [2]). Since Y is $\mathcal{L}$-trivial, so is S (see [9]). Since $E_S = Y$, the set of idempotents of S is finite and commutative. Furthermore, S is E-inversive: $ab \in Y = E_S$ for any $a, b \in S$. Finally, S has a zero–namely the least element of the finite semilattice. Hence $(S, \leq_{\mathcal{L}})$ is directed downwards, even an inf-semilattice: $\inf\{a, b\} = ab$ for all $a, b \in S$ (see Ex. 3 at the end of Section 3).

We conclude this Section with a particular case of a downwards directed partially ordered set $(X, \leq)$, a lattice, i.e., for all $a, b \in X$, $\inf\{a, b\}$ and $\sup\{a, b\}$ exists in $(X, \leq)$. Denoting by $(a)_L$ the principal left ideal of S generated by $a \in S$ we have:

Proposition 3.8. *Let S be a finite monoid with commuting idempotents. Then the following are equivalent:*

(i) $(S, \leq_E)$ *is a downwards directed partially ordered set.*
(ii) $(S, \leq_E)$ *has a least element.*
(iii) $(S, \cdot)$ *admits a (unique) left zero element.*

Proof. (i) $\Rightarrow$ (ii): This is evident.

(ii) $\Rightarrow$ (iii): Let $m \in S$ denote the least element of $(S, \leq_E)$. Then $m \leq_E 1_S$ implies that $m = e1_S = e$ for some $e \in E_S$. Let $f \in E_S$; since $e \leq_E f$ we have that $e = gf$ for some $g \in E_S$, thus $ef = gf \cdot f = e$. Let $a \in S$; then $e \leq_E a$ implies that $e = fa$ for some $f \in E_S$. Hence $e = e \cdot e = e \cdot fa = ea$, that is, $e \in S$ is a left zero of S. If $z \in S$ is an arbitrary left zero of $(S, \cdot)$ then $z = za \leq_E a$ for every $a \in S$. Thus $z \in S$ is the least element of $(S, \leq_E)$, hence $z = m = e$.

(iii) $\Rightarrow$ (i): If $z \in S$ is the left zero of $(S, \cdot)$ then $z = za \leq_E a$ for every $a \in S$. Therefore, $(S, \leq_E)$ is directed downwards.

If the idempotents of a semigroup S are central then a left zero of S is the zero of all of S. If S is also finite then by [7]:

$$a \leq_E b \Leftrightarrow a = eb \ (e \in E_S) \Leftrightarrow a = eb = be \Leftrightarrow a \leq_S b,$$

i.e., $\leq_E$ is the natural partial order of S. Thus Prop. 3.8 yields

Corollary 3.9. *Let S be a finite monoid with central idempotents. Then the following are equivalent with respect to the natural partial order of S:*

(i) $(S, \leq_S)$ *is a downwards directed partially ordered set.*
(ii) $(S, \leq_S)$ *has a least element.*
(iii) $(S, \cdot)$ *admits a zero element.*

Example. Every residue class semigroup $S = (\mathbb{Z}_n, \cdot)$ modulo $n \geq 2$ satisfies all the conditions in Cor. 3.9. Note that for $\bar{a}, \bar{b} \in S$ there are possibly lower bounds in $(S, \leq_S)$ different from $\bar{0} \in S$. For instance, in $S = (\mathbb{Z}_6, \cdot)$: $\bar{3} \leq_S \bar{1}, \bar{5}$. In this case, $(S, \leq_S)$ is even an inf-semilattice, but not a semilattice (note that S is not E-unitary: compare with Th. 3.7).

The existence of a (left) zero in a semigroup S evidently forces $(S, \leq_E)$ to be directed downwards. Besides the cases encountered above this happens also in the following situation (compare with Prop. 2.5):

Proposition 3.10. *Let S be an $\mathcal{L}$-trivial semigroup such that E_S is finite and commutative. Then the following are equivalent:*

(i) $(S, \leq_E)$ *is a downwards directed partially ordered set.*
(ii) *S is E-inversive.*

(iii) $(S, \cdot)$ *has a zero element.*

Proof. (i) $\Rightarrow$ (ii): This was proved at the beginning of this Section.

(ii) $\Rightarrow$ (iii) $\Rightarrow$ (i): This is shown as in the proof of sufficiency in Prop. 2.5. $\Diamond$

Examples. 1. Let $S = (\mathbb{N}_0, \cdot, \leq)$ be the semigroup given in Ex. 1 at the end of Section 2. Then S is an $\mathcal{L}$-trivial monoid (see [9]) with commuting idempotents: $0, 1$. Since S has a zero, $(S, \leq_E)$ is directed downwards (such that all elements of $S \backslash 0$ are incomparable). Note that $\leq_E$ is equal to $\leq_S$.

2. Let $S = (\mathbb{N}, +, \leq)$ be the semigroup given in Ex. 2 at the end of Section 2. Then S is an $\mathcal{L}$-trivial semigroup (see [9]). Since $E_S = \varnothing$, $\leq_E$ is the identity relation (and so is $\leq_S$),hence $(S, \leq_E)$ is not directed downwards.

3. Let S be the semigroup given in Ex. 3 at the end of Section 2. Then S is an $\mathcal{L}$-trivial (E-inversive) semigroup such that E_S is finite and commutative. Since S has a zero, $(S, \leq_E)$ is directed downwards–even an inf-semilattice: for $a \in T_e$, $b \in T_f$, say, $\inf\{a, b\} = {}$ $= ab = ef$ ($ef \leq_E a, b$ since $ef = eb$, $ef = fe = fa$; if $c \leq_E a, b$ then $c = ga = ge$ and $c = hb = hf$ for some $g, h \in E_S^1$, hence $c = hf = {}$ $= hf \cdot f = cf = gef$, i.e., $c \leq_E ef$). Note that the relations $\leq_E$, $\leq_{\mathcal{L}}$ and $\leq_S$ coincide on S.

4. Let $(X, \leq)$ be an infinite well ordered set and let $S = \{f : X \to$ $\to X | x < y$ implies $f(x) < f(y)\}$, that is, the set of all strictly monotone maps of X into itself. With respect to composition of functions " $\circ$ " and pointwise ordering "$\preccurlyeq$", S is a partially (lattice) ordered monoid. Every $f \in S$ satisfies $f(x) \geq x$ for any $x \in X$: assume that $Y = \{x \in$ $\in X | f(x) < x\} \neq \varnothing$ and let $y \in Y$ be the least element of Y; then $f(y) < y$, whence $f[f(y)] < f(y)$, so that $f(y) \in Y$ contradicting the minimality of $y \in Y$. Let $f \in E_S$ and assume that there exists $x_0 \in X$ such that $f(x_0) > x_0$; then $f(x_0) = f[f(x_0)] > f(x_0)$, a contradiction. Therefore, f is the identity function and $E_S = \{\text{id}\}$. Thus $\leq_E$ is the identity relation and $(S, \leq_E)$ is not directed downwards. Since id $\in S$ is the least element of S with respect to $\preccurlyeq$, S is integrally partially ordered. It follows by [10], Th. 2.9, that S is $\mathcal{L}$-trivial (see also [9], Section 3). Observe that $(S, \circ)$ has no zero since $E_S = \{\text{id}\}$ and $|S| >$ > 1. Note that also $\leq_S$ is the identity relation on S: if $f <_S g$ then $f = g \circ k = f \circ k$ for some $k \in S \backslash \text{id}$; thus there exists $x_0 \in X$ such that $k(x_0) > x_0$, whence $f(x_0) = f[k(x_0)] > f(x_0)$, a contradiction. But $\leq_{\mathcal{L}}$

is a non-trivial right partial order on S with id as greatest element (by [10], Th. 2.7).

Omitting the finiteness condition on the idempotents, Prop. 3.10 can be generalized in the following way:

Proposition 3.11. *Let S be an $\mathcal{L}$-trivial semigroup with commuting idempotents. Then $(S, \leq_E)$ is a downwards directed partially ordered set iff S is E-inversive.*

Proof. *Necessity* was shown at the beginning of this Section.

Sufficiency. First, by [10], Th. 3.1, $\leq_E$ is a partial order on S. Since S is $\mathcal{L}$-trivial and E-inversive, for any $a, b \in S$ there exist $e, f \in \in E_S$ such that $e = ea, f = fb$ (by [9], Cor. 4.9). Therefore $e \leq_E a$ and $f \leq_E b$. Since $ef = fe \leq_E e, f$ it follows that $ef \leq_E a, b$; hence $(S, \leq_E)$ is directed downwards. $\Diamond$

Example. Generalizing Ex. 3 above we obtain: Every inflation S of an arbitrary semilattice Y is $\mathcal{L}$-trivial and E-inversive with commuting idempotents, thus $(S, \leq_E)$ is a downwards directed partially ordered set, in fact, it is an inf-semilattice ($\inf\{a, b\} = ab$ for all $a, b \in S$). Note that S not necessarily has a zero.

4. The totally ordered case

Evidently, any totally ordered set is directed downwards (and upwards). We will consider this particular case for semigroups S with respect to the relation: $a \leq_{\mathcal{L}} b \Leftrightarrow a = xb$ for some $x \in S^1$ (see Section 2).

Lemma 4.1. *Let S be a semigroup. Then $(S, \leq_{\mathcal{L}})$ is a totally ordered set iff S is $\mathcal{L}$-trivial and the set of principal left ideals of S forms a (non-trivial) chain with respect to inclusion.*

Proof. *Necessity.* Since $\leq_{\mathcal{L}}$ is a partial order on S, S is $\mathcal{L}$-trivial (by [10], Th. 2.7). Let $(a)_L$, $(b)_L$ be arbitrary principal left ideals. Since $\leq_{\mathcal{L}}$ is a total order we have either $a \leq_{\mathcal{L}} b$ or $b \leq_{\mathcal{L}} a$. Hence $a = xb (x \in S^1)$ or $b = ya (y \in S^1)$; therefore $(a)_L \subseteq (b)_L$ or $(b)_L \subseteq (a)_L$. If $(a)_L = = (b)_L$ for all $a, b \in S$ then $a = b$ (since S is $\mathcal{L}$-trivial), hence $|S| = 1$: contradiction.

Sufficiency. Since S is $\mathcal{L}$-trivial, $\leq_{\mathcal{L}}$ is a partial order on S (by [10], Th. 2.7). Let $a, b \in S$ be such that $a \neq b$. Since $(a)_L \subseteq (b)_L$ or $(b)_L \subseteq (a)_L$ we have $a = xb (x \in S)$ or $b = ya (y \in S)$. Thus $a <_{\mathcal{L}} b$ or $b <_{\mathcal{L}} a$, i.e., $\leq_{\mathcal{L}}$ is a total order on S. This also shows that $\leq_{\mathcal{L}}$ is not the identity relation. $\Diamond$

Remark. In [3], right simple semigroups S whose principal left ideals are totally ordered by inclusion were studied. If $E_S = \varnothing$ then by [10], Cor. 2.5(2), $\leq_{\mathcal{L}}$ is a partial, hence a total order on S (by Lemma 4.1, proof of sufficiency). If $E_S \neq \varnothing$ then $\leq_{\mathcal{L}}$ can not be an order relation as the following result shows:

Theorem 4.2. *Let S be a semigroup, for which $\leq_{\mathcal{L}}$ is a total order.*

(1) *If S is left simple then $|S| = 1$.*

(2) *If S is right simple then either $|S| = 1$ (if $E_S \neq \varnothing$) or S is right cancellative without idempotents, hence embeddable into a Baer–Levi semigroup.*

Proof. (1) This holds by [10], Remark following Th. 2.7.

(2) First we show that S is right cancellative (following the proof in [3]). Let $ac = bc$ for some $a, b, c \in S$. Assume that $a \neq b$; then $a <_{\mathcal{L}} b$, say. Thus $a = xb$ for some $x \in S$, so that $xbc = bc$. Since S is right simple there exists $y \in S$ such that $b = bc \cdot y$. It follows that $a = xb = x \cdot bcy = xbc \cdot y = bc \cdot y = b$: contradiction.

Next suppose that $E_S \neq \varnothing$. Then for $a \in S, e \in E_S$, the equation $ae = aee$ implies that $ae = e$. Hence $e \in E_S$ is a right identity of S. Further, $aa' = e$ for some $a' \in S$; hence S is a group. Let $a, b \in S$; then $a = ab^{-1} \cdot b$, $b = ba^{-1} \cdot a$ imply that $a \leq_{\mathcal{L}} b$, $b \leq_{\mathcal{L}} a$. By the antisymmetry of $\leq_{\mathcal{L}}$, it follows that $a = b$, i.e., $|S| = 1$. If $E_S = \varnothing$ then by [2], Th. 8.5, S is embeddable into a Baer–Levi semigroup of type (p, p) where $p = |S|$.

Let us suppose now that S is a semigroup for which $\leq_{\mathcal{L}}$ is a total order with greatest element. We will consider the class of those S which are right archimedean, whence right reversible (see Ex. 4 in Section 2). If S is cancellative then S is embeddable in a group (by [2], Th. 1.23). Here we will distinguish two cases: S is right cancellative or not. In the first case we have the following generalization of [5] on cancellative, (commutative) archimedean, naturally totally ordered semigroups without identity and with least element.

Theorem 4.3. *Let S be a right archimedean, right cancellative semigroup, for which $\leq_{\mathcal{L}}$ is a total order with greatest element. Then $(S, \cdot, \leq_{\mathcal{L}})$ is semigroup- and order isomorphic with the additive semigroup $(\mathbb{N}, +, \leq_d)$ of natural numbers (without zero), where $\leq_d$ denotes the dual of the usual total order of $\mathbb{N}$.*

Proof. We first show that $E_S = \varnothing$. Let $a \in S, e \in E_S \cdot$, then $ae = aee$ implies that $a = ae \leq_{\mathcal{L}} e$. Since there exist $k \in \mathbb{N}$, $x \in S^1$ such that $e = e^k = xa \leq_{\mathcal{L}} a$, it follows that $a = e$, i.e., $|S| = 1$: contradiction.

If $m \in S$ denotes the greatest element of $(S, \leq_{\mathcal{L}})$, it follows that $m^2 <_{\mathcal{L}} m$. Therefore by right compatibility of $\leq_{\mathcal{L}}$, $m^{i+1} \leq_{\mathcal{L}} m^i$ for every $i \in \mathbb{N}$. If $m^{i+1} = m^i$ for some $i > 1$ then $m^2 \cdot m^{i-1} = m \cdot m^{i-1}$ implies that $m^2 = m$. Hence $m >_{\mathcal{L}} m^2 >_{\mathcal{L}} m^3 >_{\mathcal{L}} \cdots$.

Let $a \in S$, $a \neq m$; we will show that $a = m^k$ for some $k \in \mathbb{N}$. Since $m^n = xa$ for some $n \in \mathbb{N}$, $x \in S^1$, we have $m^n \leq_{\mathcal{L}} a$. Note that $n \neq 1$, since $m \leq_{\mathcal{L}} a$ implies that $a = m$. Therefore, the set $M = \{n \in \mathbb{N} \backslash 1 | \; m^n \leq_{\mathcal{L}} a\}$ is not empty and has a least element k, say (with respect to the usual total order of $\mathbb{N}$). Thus $k \neq 1$ and $m^k \leq_{\mathcal{L}} a < _{\mathcal{L}} m^{k-1}$. Hence $a = ym^{k-1}$ for some $y \in S$. Since $y \leq_{\mathcal{L}} m$ it follows that $a = ym^{k-1} \leq_{\mathcal{L}} mm^{k-1} = m^k$, whence $a = m^k$.

Thus we obtain that $S = \{m^k | k \in \mathbb{N}\}$, i.e., an infinite cyclic semigroup. It is well-known that the mapping $\varphi : (S, \cdot) \to (\mathbb{N}, +)$, $\varphi(m^k) = k$, is a semigroup isomorphism. It is also orderpreserving: Let $m^i <_{\mathcal{L}} m^k$ in S and assume that $k <_d i$ in $\mathbb{N}$ (note that $i \neq k$); then $k = p + i$ for some $p \in \mathbb{N}$ and $m^k = m^{p+i} = m^p \cdot m^i \leq_{\mathcal{L}} m^i$: contradiction. Since $(S, \leq_{\mathcal{L}})$ is a chain, it follows that also φ^{-1} is orderpreserving. $\Diamond$

Remarks. 1. In Th. 4.3, the condition "S is right archimedean" can be replaced by "there is no $c \in S$ such that $c <_{\mathcal{L}} m^j$ for any $j \in \mathbb{N}$":

First we show that $m^2 <_{\mathcal{L}} m$. If $m^2 = m$ then $m^j = m$ for any $j \in \mathbb{N}$; since $|S| > 1$ there exists $c \in S$ such that $c \neq m$, that is, $c < m = m^j$ for every $j \in \mathbb{N}$: contradiction. Therefore we obtain again: $m >_{\mathcal{L}} m^2 >_{\mathcal{L}} m^3 >_{\mathcal{L}} \cdots$. The third paragraph in the proof of Th. 4.3 has to be replaced by the following:

Let $a \in S$, $a \neq m$; then $a <_{\mathcal{L}} m$ and $a = xm$ for some $x \in S$. Since $x \leq_{\mathcal{L}} m$ we have $a = xm \leq_{\mathcal{L}} mm = m^2$. If $a = m^2$ we are done. If $a <_{\mathcal{L}} m^2$ then as before, $a \leq_{\mathcal{L}} m^3$ and so on. Since there is no $c \in S$ such that $c <_{\mathcal{L}} m^j$ for any $j \in \mathbb{N}$, we obtain that $a = m^j$ for some $j \in \mathbb{N}$.

2. With respect to Th. 2.4 above we make the following observation. Let S be a semigroup satisfying the conditions given in Th. 4.3. Then $(S, \cdot, \leq_{\mathcal{L}})$ is isomorphic with $(\mathbb{N}, +, \leq_d)$. Adjoining an identity (that is, 0) then $(\mathbb{N}^0, +, \leq_d)$ is the negative cone of the totally ordered group of integers $(\mathbb{Z}, +, \leq_d)$.

If the semigroup S in Th. 4.3 is not right cancellative then we have the following version of [1] on (commutative) archimedean, naturally totally ordered semigroups without cancellation. The first alternative occurs if S that a greatest element, the second if there is no such element in $(S, \leq_{\mathcal{L}})$. Recall that a semigroup with zero is nil if for every $a \in S$ there exists $n > 0$ such that $a^n = 0$. S is right 0-cancellative if

$ac = bc \neq 0$ $(a, b, c \in S)$ implies that $a = b$. A partially ordered set $(X, \leq)$ is dense if for any $a < b$ in X there is some $c \in X$ such that $a < c < b$.

Theorem 4.4. *Let S be a right archimedean, not right cancellative semigroup, for which $\leq_{\mathcal{L}}$ is a total order. Then S is either a finite cyclic nil-semigroup or an infinite nil-semigroup with right 0-cancellation, which is dense except possibly at 0.*

Proof. By hypothesis, there exist $a, b, c \in S$ such that $ac = bc$ and $a <_{\mathcal{L}} b$, say. Then $a = xb$ for some $x \in S$; put $p = bc$. Then $xp = xbc = ac = bc = p$, thus $x^n p = p$ for every $n \in \mathbb{N}$. Since S is right-archimedean there exist $k > 0$, $y \in S^1$, such that $x^k = yp \leq_{\mathcal{L}} p$. Therefore, $p = x^k \cdot p \leq_{\mathcal{L}} p \cdot p \leq_{\mathcal{L}} p$, hence $p^2 = p$. Let $a \in S$; then $p = p^i = za \leq_{\mathcal{L}} a$ for some $(i > 0) z \in S^1$. Thus p is the least element of $(S, \leq_{\mathcal{L}})$. Furthermore, $a^j = up \leq_{\mathcal{L}} p$ for some $j > 0$, $u \in S^1$; hence $a^j = p$. It follows that $p \cdot a = a^j \cdot a = a \cdot a^j = ap \leq_{\mathcal{L}} p$, so that $ap = pa = p$. Hence $p \in S$ is the zero of $(S, \cdot)$: $p = 0$, and S is a nil-semigroup. Note that $E_S = \{0\}$: if $e \in E_S$ then $e = e^k = xp = 0$ for some $(k > 0) x \in S^1$.

Next we show that S is right 0-cancellative: let $x, y, z \in S$ be such that $xz = yz \neq 0$ and assume that $x \neq y$, $x \underset{\mathcal{L}}{<} y$ say; then as above, $q = yz \in S$ is the zero element of $(S, \cdot)$: contradiction.

We have to distinguish three cases:

Case 1. $(S, \leq_{\mathcal{L}})$ has a greatest element m and $m^2 = m$. Then since $E_S = \{0\}$, it follows that $m = 0 = p$, hence $|S| = 1$: contradiction.

Case 2. $(S, \leq_{\mathcal{L}})$ has a greatest element m and $m^2 \neq m$.

Then by the proof of Th. 4.3 (third paragraph), S is the cyclic nil-semigroup generated by $m \in S$. In particular, for $m \in S$ there is a (least) $k \in \mathbb{N}$ such that $m^k = 0$ (above). Hence $m^{k+i} = 0$ for any $i \in \mathbb{N}$, i.e., there are only finitely many distinct powers of m : $S = \{m, m^2, \cdots, m^{k-1}, m^k = 0\}$.

Case 3. $(S, \leq_{\mathcal{L}})$ has no greatest element.

Then by the above, S is an infinite nil-semigroup. We show that $(S, \leq_{\mathcal{L}})$ is dense except possibly at 0. Let $a, b \in S$ be such that $a <_{\mathcal{L}} b$; then $a = xb$ for some $x \in S$. By hypothesis on $(S, \leq_{\mathcal{L}})$ there exists $y \in S$ with $x <_{\mathcal{L}} y$; therefore $a = xb \leq_{\mathcal{L}} yb \leq_{\mathcal{L}} b$. We have $yb \neq b$, since $yb = b$ implies that $y^n b = b$ for any $n > 0$; but $y^j = 0$ for some $j > 0$, whence $b = 0$: contradiction. It follows that $a = xb \leq_{\mathcal{L}} yb <_{\mathcal{L}} b$. If $a \neq 0$ then $a = xb \neq yb$ - otherwise $x = y$, by right 0-cancellation. Thus for $0 \neq a <_{\mathcal{L}} b$ there exists $c \in S$ such that $a <_{\mathcal{L}} c <_{\mathcal{L}} b$. ◊

[8] MITSCH, H.: Semigroups and their natural order, *Math. Slovaca* **44** (1994), 445–462.
[9] MITSCH, H.: $\mathcal{L}$-trivial semigroups, *Pure Math. Appl.* **12** (2002), 209–218.
[10] MITSCH, H.: Divisibility orders on semigroups I, *Contrib. Gen. Alg.* **15** (2003).
[11] MITSCH, H., PETRICH, M.: Basic properties of E-inversive semigroups, *Comm. Algebra* **28** (2000), 5169–5182.