Mathematica Pannonica
14/2 (2003), 283–291

KOEBE DOMAIN FOR CERTAIN ANALYTIC FUNCTIONS IN THE UNIT DISC UNDER THE MONTEL NORMALIZATION

Yaşar **Polatoğlu**

Department of Mathematics, Faculty of Science and Arts, Kültür University, Istanbul, Turkey

Metin **Bolcal**

Department of Mathematics, Faculty of Science and Arts, Kültür University, Istanbul, Turkey

Received: May 2002

MSC 2000: 30 C 45

Keywords: Starlike functions, starlike functions of complex order, Carathéodory functions, Koebe domain with Montel normalization.

Abstract: A function $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ which is regular in the open unit disc $D = \{z \,|\, |z| < 1\}$ belongs to the class $S^*(A, B, b)$ if

$$1 + \frac{1}{b}\Big(z\frac{f'(z)}{f(z)} - 1\Big) = \frac{1 + A\omega(z)}{1 + B\omega(z)} \qquad (0 \neq b \in \mathbb{C}),$$

where $\omega(z)$ is regular in D and satisfies the conditions $\omega(0) = 0$ $|\omega(z)| < 1$, and A, B are real arbitrary fixed numbers such that $-1 < A \leq 1$, $-1 \leq B <$ $< A$. The aim of this paper is to give the Koebe domain under the Montel type normalization of the class $S^*(A, B, b)$.

E-mail address: y.polatoglu@iku.edu.tr

1. Introduction

Let Ω be the family of the functions $\omega(z)$ regular in the unit disc D and satisfying $\omega(0) = 0, |\omega(z)| < 1$ for $z \in D$.

Next, for the arbitrary fixed numbers $A, B, -1 < A \leq 1, -1 \leq \leq B < A$, denote by $P(A, B)$ the family of functions

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots ,$$

regular in D such that

$$p(z) \in P(A, B) \text{ if and only if } p(z) = \frac{1 + A\omega(z)}{1 + B\omega(z)}$$

for some function $\omega(z) \in \Omega$ and every $z \in D$.

Moreover, let $S^*(A, B, b)$ denote the family of functions

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots ,$$

regular in D such that

$$f(z) \in S^*(A, B, b) \text{ if and only if } 1 + \frac{1}{b}\Big(z\frac{f'(z)}{f(z)} - 1\Big) = p(z)$$

for some $p(z)$in $P(A, B)$ and all z in D.

We note that the class $P(A, B)$ was introduced by W. Janowski [10]. $p(1, -1)$ is the class of Carathéodory functions $p(z)$ for which $p(0) = 1$, $\operatorname{Re} p(z) > 0$ in D. Therefore the set $S^*(A, B, b)$ contains the following classes:

1. $S^*(1, -1, 1)$ – the well-known class of starlike functions [1].
2. $S^*(1, -1, b)$ – the class of starlike functions of complex order, introduced by P. Wiatrowski [7].
3. $S^*(1, -1, 1 - \beta), 0 \leq \beta < 1$ – the class of starlike functions of order β, introduced by M. S. Robertson [6].
4. $S^*(1, -1, e^{-i\lambda} \cos \lambda)$, $|\lambda| < \frac{\pi}{2}$ – the class of λ-spirallike functions, introduced by L. Spacek [5].
5. $S^*\big(1, -1, (1-\beta)e^{-i\lambda} \cos \lambda\big)$, $0 \leq \beta < 1$, $|\lambda| < \frac{\pi}{2}$ – the class of λ-spirallike functions of order β, introduced by R. J. Libera [8].
6. $S^*(1, 0, b)$ – the set defined by $|ST(b) - 1| < 1$, where $ST(b) = 1 + + \frac{1}{b}\big(z\frac{f'(z)}{f(z)} - 1\big)$.
7. $S^*(\beta, 0, b), 0 \leq \beta < 1$ – the set defined by$|ST(b) - 1| < \beta$.
8. $S^*(\beta, -\beta, b), 0 \leq \beta < 1$ – the set defined by$\left|\frac{ST(b)-1}{ST(b)+1}\right| < \beta$.
9. $S^*(1, (-1 + \frac{1}{M}), b), M > 1$ – the set defined by$|ST(b) - M| < M$.
10. $S^*(1 - 2\beta, -1, b), 0 \leq \beta < 1$ – the set defined by $\operatorname{Re} ST(b) > \beta$.

2. Auxiliary lemmas

From the definition of the classes $P(A,B)$ and $S^*(A,B,b)$ we easily obtain the following lemmas.

Lemma 2.1. *$f(z) = z + a_2z^2 + a_3z^3 + \cdots$ belongs to $S^*(A,B,b)$ if and only if*

$$f(z) = \begin{cases} z(1+B\omega(z))^{\frac{b(A-B)}{B}} & B \neq 0 \\ ze^{bA\omega(z)} & B = 0 \end{cases}$$

where $\omega(z) \in \Omega$.

Proof. We prove first the necessity of the condition.

Let $B \neq 0$ and

$$f(z) = z(1+B\omega(z))^{\frac{b(A-B)}{B}}.$$

If we take logarithmic derivative, from this we obtain

$$\frac{1}{b}\Big(z\frac{f'(z)}{f(z)} - 1\Big) = (A-B)\frac{z\omega'(z)}{1+B\omega(z)}.$$

Using Jack's lemma [2] in this equality we obtain

$$\frac{1}{b}\Big(z\frac{f'(z)}{f(z)} - 1\Big) = \frac{(A-B)\omega(z)}{1+B\omega(z)}.$$

It follows that

$$1 + \frac{1}{b}\Big(z\frac{f'(z)}{f(z)} - 1\Big) = \frac{1+A\omega(z)}{1+B\omega(z)}.$$

This means that $f(z) \in S^*(A,B,b)$.

Let $B = 0$ and

$$f(z) = ze^{bA\omega(z)}.$$

If we take logarithmic derivative we obtain

$$\frac{1}{b}\Big(z\frac{f'(z)}{f(z)} - 1\Big) = Az\omega'(z).$$

Using Jack's lemma we deduce

$$\frac{1}{b}\Big(z\frac{f'(z)}{f(z)} - 1\Big) = A\omega(z).$$

This equality can be written in the form

$$1 + \frac{1}{b}\Big(z\frac{f'(z)}{f(z)} - 1\Big) = 1 + A\omega(z) = \frac{1+A\omega(z)}{1+B\omega(z)}.$$

This shows that $f(z) \in S^*(A,B,b)$.

The condition is also sufficient.

Let $f(z) \in S^*(A,B,b)$ and $B \neq 0$. Then

$$1+\frac{1}{b}\left(z\frac{f'(z)}{f(z)}-1\right)=p(z)$$

for some $p(z) \in P(A,B)$. On the other hand the boundary function $p_0(z)$of $P(A,B)$ with respect to this equality has the form

$$p_0(z)=\frac{1+Az}{1+Bz}.$$

Therefore we have the equality

$$1+\frac{1}{b}\left(z\frac{f'(z)}{f(z)}-1\right)=\frac{1+Az}{1+Bz}$$

for every boundary function. After simple calculations we deduce

$$f(z)=z(1+Bz)^{\frac{b(A-B)}{B}} \Longleftrightarrow \frac{f(z)}{z}=(1+Bz)^{\frac{b(A-B)}{B}}.$$

If we use the subordination principle [1] to this equality we obtain

$$f(z)=z(1+B\omega(z))^{\frac{b(A-B)}{B}}.$$

Let $f(z) \in S^*(A,B,b)$ and $B=0$. Then

$$1+\frac{1}{b}\left(z\frac{f'(z)}{f(z)}-1\right)=p(z)$$

for some $p(z) \in P(A,B)$ and so we obtain

$$f(z)=ze^{bA\omega(z)}.$$

The assertion is also proved. ◊

Lemma 2.2. *If $f(z) \in S^*(A,B,b)$, then the function*

$$f_*(z)=\begin{cases} \dfrac{azf\left(\frac{z+a}{1+\bar{a}z}\right)}{f(a)(z+a)(1+\bar{a}z)^{\frac{-b(A-B)}{B}-1}} & B\neq 0, a\in D \\[2ex] \dfrac{azf\left(\frac{z+a}{1+\bar{a}z}\right)}{f(a)(z+a)(1+\bar{a}z)^{2bA-1}} & B=0, a\in D \end{cases}$$

is in $S^*(A,B,b)$.

Proof. Let $B\neq 0$, $a\in D$. We consider for real $g, 0<g<1$,

$$f_g(z)=\frac{azf\left(\frac{z+a}{1+\bar{a}z}\right)}{f(a)(z+a)(1+\bar{a}z)^{\frac{-b(A-B)}{B}-1}}, \quad z\in D.$$

Then

$$1+\frac{1}{b}\Big(z\frac{f_g'(z)}{f_g(z)}-1\Big)=$$

$$=\frac{z}{(z+a)(1+\bar{a}z)}(1-|a|^2)\left[1+\frac{1}{b}\left(g\Big(\frac{z+a}{1+\bar{a}z}\Big)\frac{f'\big(g(\frac{z+a}{1+\bar{a}z})\big)}{f\big(g(\frac{z+a}{1+\bar{a}z})\big)}-1\right)\right]+$$
$$+\Big(\frac{1}{b}-1\Big)\frac{z}{(z+a)(1+\bar{a}z)}(1-|a|^2)+\left[1-\frac{1}{b}\frac{z}{(z+a)(1+\bar{a}z)}(1+\bar{a}z)-\right.$$
$$\left.-\left(\frac{-(A-B)}{B}-\frac{1}{b}\right)\frac{z}{(z+a)(1+\bar{a}z)}\bar{a}(z+a)\right].$$

Letting $z=e^{i\theta}$ and $\omega=g(\frac{z+a}{1+\bar{a}z})$ we obtain

$$1+\frac{1}{b}\Big(z\frac{f_g'(z)}{f_g(z)}-1\Big)=$$
$$=\frac{(1-|a|^2)}{|1+ae^{-i\theta}|^2}\left[1+\frac{1}{b}\left(\omega\frac{f'(\omega)}{f(\omega)}-1\right)\right]+\Big(\frac{1}{b}-1\Big)\frac{(1-|a|^2)}{|1+ae^{-i\theta}|^2}+$$
$$+\left[1-\frac{1}{b}\frac{1+\bar{a}e^{i\theta}}{|1+ae^{-i\theta}|^2}-\Big(\frac{-(A-B)}{B}-\frac{1}{b}\Big)\frac{\bar{a}(a+e^{i\theta})}{|(1+ae^{-i\theta}|^2}\right].$$

On the other hand, from Lemma 2.1 we have

$$\frac{(1-|a|^2)}{|1+ae^{-i\theta}|^2}\left[1+\frac{1}{b}\left(\omega\frac{f'(\omega)}{f(\omega)}-1\right)\right]+\Big(\frac{1}{b}-1\Big)\frac{(1-|a|^2)}{|1+ae^{-i\theta}|^2}+$$
$$+\left[1-\frac{1}{b}\frac{1+\bar{a}e^{i\theta}}{|1+ae^{-i\theta}|^2}-\left(\frac{-(A-B)}{B}-\frac{1}{b}\right)\frac{\bar{a}(a+e^{i\theta})}{|1+ae^{-i\theta}|^2}\right]=\frac{1+A\omega(z)}{1+B\omega(z)}.$$

It results from the last two equalities that

$$1+\frac{1}{b}\Big(z\frac{f_g'(z)}{f_g(z)}-1\Big)=\frac{1+A\omega(z)}{1+B\omega(z)},$$

consequently $f_g(z)$ is in $S^*(A,B,b)$ for admissible g. Using also the property of compactness of $S^*(A,B,b)$ we conclude that

$$f_*(z)=\lim_{g\to 1}f_g(z)$$

is in $S^*(A,B,b)$.

Let $B=0$, $a\in D$. We obtain similarly

$$1+\frac{1}{b}\Big(z\frac{f_g'(z)}{f_g(z)}-1\Big)=\frac{(1-|a|^2)}{|1+ae^{-i\theta}|^2}\Big[1+\frac{1}{b}\Big(\omega\frac{f'(\omega)}{f(\omega)}-1\Big)\Big]+$$
$$+\Big(\frac{1}{b}-1\Big)\frac{(1-|a|^2)}{|1+ae^{-i\theta}|^2}+\Bigg[1-\frac{1}{b}\frac{1+\bar{a}e^{i\theta}}{|1+ae^{-i\theta}|^2}-$$
$$-\Big(2A-\frac{1}{b}\Big)\frac{a+e^{i\theta}}{|1+ae^{-i\theta}|^2}\Bigg]=\frac{1+A\omega(z)}{1+B\omega(z)}$$

and

$$f_*(z)=\lim_{g\to 1} f_g(z)$$

which ends the proof. ◊

Lemma 2.3. *If $f(z)=z+a_2z^2+a_3z^3+\cdots$ belongs to $S^*(A,B,b)$ then the set of the values $\Big(z\frac{f'(z)}{f(z)}\Big)$ is the closed disc with the centre $c(r)$ and the radius $g(r)$ where*

$$c(r)=\frac{1-[B-b(AB-B^2)]r^2}{1-B^2r^2},\quad g(r)=\frac{|b|(A-B)r^2}{1-B^2r^2}.$$

Proof. The images of the closed disc $|z|\le r$ under the transformation $p(z)=\frac{1+A\omega(z)}{1+B\omega(z)}$ are contained in the closed disc with the centre $c_1(r)$ and the radius $g_1(r)$ [10], where

$$c_1(r)=\frac{1-ABr^2}{1-B^2r^2},\quad g_1(r)=\frac{(A-B)r}{1-B^2r^2}.$$

Therefore by Lemma 2.1 we have

$$\left|\Big[1+\frac{1}{b}\Big(z\frac{f'(z)}{f(z)}-1\Big)\Big]-\frac{1-ABr^2}{1-B^2r^2}\right|\le\frac{(A-B)}{1-B^2r^2}.$$

After simple calculations we obtain the desired result of the Lemma 2.3. ◊

Theorem 2.1. *If $f(z)\in S^*(A,B,b)$, then*

$$G(r,-A,-B,|b|)\le|f(z)|\le G(r,A,B,|b|),$$

where

$$G(r,A,B,|b|)=\begin{cases} r(1+Br)^{\frac{b(A-B)}{B}} & B\neq 0,\\ re^{|b|Ar} & B=0.\end{cases}$$

Remark. This bound is sharp, because the extremal function is

$$f_*(z) = \begin{cases} z(1+Bz)^{\frac{b(A-B)}{B}} & B \neq 0 \\ ze^{bAz} & B = 0. \end{cases} \quad \text{and} \quad w = \frac{r\left(r - \sqrt{\frac{\bar{b}}{b}}\right)}{1 - r\sqrt{\frac{\bar{b}}{b}}}.$$

Proof of Theorem 2.1. Since $f(z) \in S^*(A, B, b)$, we have

$$1 + \frac{1}{b}\left(z\frac{f'(z)}{f(z)} - 1\right) = p(z), p(z) \in P(A, B)$$

and after simple calculations we obtain

$$f(z) = z\text{Exp}\left(\int_0^z \frac{b(p(\xi) - 1)}{\xi} d\xi\right).$$

Therefore

$$|f(z)| = |z|\text{Exp}\left(\text{Re}\left(\int_0^z \frac{b(p(\xi) - 1)}{\xi} d\xi\right)\right).$$

Substituting $\xi = zt$ we obtain

$$|f(z)| = |z|\text{Exp}\left(\text{Re}\left(\int_0^1 \frac{b(p(zt) - 1)}{t} dt\right)\right).$$

On the other hand it follows from Lemma 2.3 that

$$\max_{|zt|=rt}\left(\frac{b(p(zt) - 1)}{t}\right) = \frac{|b|(A - B)r}{1 + Brt}.$$

After integration we obtain the upper bounds for $|f(z)|$. Similarly we obtain the lower bounds for $|f(z)|$, which shows that the proof of Th. 2.1 is complete. ◊

3. Koebe domain with Montel normalization for the class $S^*(A, B, b)$

In this section we shall give the sharp bound of the Koebe domain with Montel normalization for the class $S^*(A, B, b)$. Therefore we shall need the following definition.

Definition. The *Koebe domain* $K(F)$ for a family F of regular functions $f(z)$ in D is the set of all points ω contained in $f(D)$ for every function $f(z)$ in F. In symbols

$$K(F) = \bigcap_{f(z) \in F} f(D).$$

Supposing the set F is invariant under the rotation, so that $e^{i\alpha} f(e^{-i\alpha} z)$ is in F whenever $f(z)$ is in F. Then the Koebe domain will be either the single point $w = 0$ or an open disc $|w| < R$. In the second case R is often easy to find. Indeed supposing that we have sharp lower bound $M(r)$ for $|f(re^{i\theta})|$ for all functions in F and F contains only univalent functions, then

$$R = \lim_{R\to 1^-} M(r)$$

gives the disc $|w| < R$ as the Koebe domain for the set F [see 1].

We can also impose a Montel type normalization. This means that for some fixed $r_0, 0 < r_0 < 1$, we consider the family of normalized functions $f(z)$ regular and univalent in D with $f(0) = 0$, $f'(0) = 1$, $f(r_0) = r_0$ [3].

Theorem 3.1. *If $f(z) \in S^*_M(A,B,b)$, then the Koebe domain of $S^*(A,B,b)$ is*

$$R = \begin{cases} \dfrac{(1-B)^{\frac{|b|(A-B)}{B}}(1-r_0^2)^{\frac{-b(A-B)}{B}+1}}{(1-2r_0\cos\theta + r_0^2)^{\frac{-b(A-B)}{B}+\frac{1}{2}}} & B \neq 0 \\ \dfrac{e^{-|b|A}(1-r_0^2)^{2bA}}{(1-2r_0\cos\theta + r_0^2)^{2bA-\frac{1}{2}}} & B = 0. \end{cases}$$

Proof. Let $B \neq 0$. If we take $a = v = r_0, f(r_0) = r_0$,

$$u = \frac{a+z}{1+\bar{a}z} = \frac{v+z}{1+\bar{v}z} \Longleftrightarrow z = \frac{u-v}{1-\bar{v}u},$$

then Lemma 2.2 gives

$$f_*(z) = \frac{v(u-v)(1-\bar{v}u)^{\frac{-b(A-B)}{B}} f(u)}{uf(v)(1-|v|^2)^{\frac{-b(A-B)}{B}+1}}.$$

Using Lemma 2.2, Th. 2.1 and the definition of the Koebe domain under the Montel type normalization, after simple calculations we obtain

$$R = \frac{(1-B)^{\frac{|b|(A-B)}{B}}(1-r_0^2)^{\frac{-b(A-B)}{B}+1}}{(1-2r_0\cos\theta + r_0^2)^{\frac{-b(A-B)}{B}+\frac{1}{2}}}.$$

Similarly for $B = 0$ we obtain

$$R = \frac{e^{-|b|A}(1-r_0^2)}{(1-2r_0\cos\theta + r_0^2)^{2bA-\frac{1}{2}}}. \ \Diamond$$

Corollary. *If we take $A = 1$, $B = -1$, $b = 1$, $r_0 = 0$ we obtain $R = \frac{1}{4}$. This is the well-known Koebe domain for the starlike function. Therefore the Koebe domains under the Montel normalization for the* *classes* $S^*(1,-1,1)$, $S^*(1,-1,b)$, $S^*(1,-1,1-\beta)$, $S^*(1,-1,e^{-i\lambda}\cos\lambda)$, $S^*\big(1,-1,(1-\beta)e^{-i\lambda}\cos\lambda\big)$, $S^*(1,0,b)$, $S^*(\beta,0,b)$, $S^*(\beta,-\beta,b)$, $S^*(1,(-1+\frac{1}{M}),b)$, $S^*(1-2\beta,-1,b)$ *are obtained.*

References

[1] GOODMAN, A. W.: Univalent functions, Volume I and Volume II, Mariner Publ. Co. Inc., Tampa Florida, 1983.

[2] JACK, I. S.: Functions Starlike and Convex of order α, *J. London. Math. Soc.* (2), **3** (1971), 469–474.

[3] KRZYZ, J. K.: On a problem of Montel, *Ann. Polon. Math.* **12** (1962), 55–60.

[4] FAN, Kay: Distortion of univalent functions, *J. Math. Anal. Appl.* **66** (1979), 626–631.

[5] SPACEK, L.: Prispeek k teorii funkci prostych, *Casopis Pěst. Mat. Fys.* **62** (1933), 12–19.

[6] ROBERTSON, M. S.: On the theory of univalent functions, *Ann. Math.* **37** (1936), 374–408.

[7] WIATROWSKI, P.: The coefficients of a certain family of holomorphic functions, *Zeszyty Nauk. Univ. Todzk. Nauki. Mat. przyrod. ser II.* **39** (1971), 57–85.

[8] LIBERA, R. J.: Univalent α-spiral functions, *Canad. J. Math.* **19** (1967), 449–456.

[9] LIBERA, R. J. and ZIEGLER, M. R.: Regular functions f(z) for which $zf'(z)$ is α-spiral, *Trans. Amer. Math. Soc.* **166** (1972), 361–370.

[10] JANOWSKI, W.: Some extremal problems for certain families of analytic functions. I, *Ann. Pol. Math.* **28** (1973), 297–326.