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ON A THEOREM OF DABOUSSI RELATED TO THE SET OF GAUSSIAN INTEGERS

Nader L. **Bassily**

Ain Shams University, Abbassia, Cairo, Egypt

Jean-Marie **De Koninck**[1]

Département de mathématiques, Université Laval, Québec G1K 7P4, Canada

Imre **Kátai**[2]

Computer Algebra Department, Eötvös Loránd University, Budapest H–1117, Pázmány Péter Sétány 1/C, Hungary

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Abstract: It has been established by Daboussi that if f is a complex valued multiplicative function such that $|f(n)| \leq 1$ and α is an arbitrary irrational number, then $\lim\limits_{x\to\infty} \frac{1}{x}\sum\limits_{n\leq x} f(n)e^{2\pi i n\alpha} = 0$. If W stands for the union of finitely many convex bounded domains in $\mathbb{C}$ and if $\mathcal{A}$ is the set of those additive characters χ such that $\chi(1) = e^{2\pi i A}$ and $\chi(i) = e^{2\pi i B}$, where at least one of A and B is irrational, we prove that, given $\chi \in \mathcal{A}$, then for every multiplicative function $g : \mathbb{Z}[i] \setminus \{0\} \to \mathbb{C}$ such that $|g(\alpha)| \leq 1$, $\lim\limits_{x\to\infty} \frac{1}{|xW|}\sum\limits_{\beta\in xW} g(\beta)\chi(\beta) = 0$, where the convergence is uniform in g.

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1. Introduction

Let $G = \mathbb{Z}[i]$ be the set of Gaussian integers, $U = \{1, -1, i, -i\}$ the set of units of G and set $G^* = G \setminus \{0\}$. We say that a function $g : G^* \to \mathbb{C}$ is multiplicative if $g(\varepsilon) = 1$ for each $\varepsilon \in U$ and if $g(\alpha_1 \alpha_2) = g(\alpha_1)g(\alpha_2)$ for each coprime pair $\alpha_1, \alpha_2 \in G^*$. We say that the integers α_1 and α_2 are associates if $\alpha_1 = \varepsilon\alpha_2$ for some unit ε. Let $\mathcal{M}$ be the set of multiplicative functions defined on G^* and let $\mathcal{M}^*$ be the subset of $\mathcal{M}$ made of those $g \in \mathcal{M}$ satisfying $|g(\alpha)| \le 1$ for all $\alpha \in G^*$.

Let χ be an arbitrary additive character, that is a function $\chi : G \to \{z : |z| = 1\}$ for which $\chi(0) = 1$ and $\chi(\alpha_1 + \alpha_2) = \chi(\alpha_1)\chi(\alpha_2)$ for all $\alpha_1, \alpha_2 \in G$. We shall say that χ is periodic if there is some $\gamma \in G$, $\gamma \ne 0$, for which $\chi(\gamma) = 1$. Let $\mathcal{N}$ be the set of nonperiodic characters.

Using the standard notation $e(u) = e^{2\pi i u}$, we set $\chi(1) = e(A)$ and $\chi(i) = e(B)$, and denote by $\mathcal{A}$ the set of those χ's for which at least one of A and B is irrational. Clearly $\mathcal{N} \subset \mathcal{A}$.

Let W be the union of finitely many convex bounded domains in $\mathbb{C}$. Given $x > 0$, we denote by xW the set $\{xz : z \in W\}$. With the Lebesgue measure $|\cdot|$, we have

$$|xW| = x^2\,|W|\,.$$

It is known from Gauss that the number of Gaussian integers located in xW is equal to $\pi x^2|W| + O(x)$ as $x \to \infty$.

Daboussi (see Daboussi and Delange [1]) proved that if $f : \mathbb{N} \to \mathbb{C}$ is a multiplicative function such that $|f(n)| \le 1$ and α an arbitrary irrational number, then

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n \le x} f(n)e(n\alpha) = 0.$$

Further generalisations have been obtained by Daboussi and Delange [2], Dupain, Hall and Tenenbaum [3], Kátai [6], Indlekofer and Kátai [5], as well as Goubain [4].

In this paper, we prove the following analogous result.

Theorem 1. *Let W and $\mathcal{A}$ be as above, and let $\chi \in \mathcal{A}$. Then, for every $g \in \mathcal{M}^*$,*

$$\lim_{x\to\infty} \frac{1}{|xW|} \sum_{\beta \in xW} g(\beta)\chi(\beta) = 0,$$

where the convergence is uniform in g.

The proof is based on a simple version of a Turán–Kubilius type inequality which we express as Lemma 1.

2. The key lemmas

Lemma 1. *Let $\wp = \{\rho_1, \rho_2, \dots, \rho_r\}$ be a finite set of Gaussian primes, with $|\rho_1| \le |\rho_2| \le \dots \le |\rho_r|$, such that no two of them are associates. Set*

$$A_\wp := \sum_{j=1}^{r} \frac{1}{|\rho_j|^2} \qquad \text{and} \qquad \omega_\wp(\alpha) := \sum_{\substack{\rho \mid \alpha \\ \rho \in \wp}} 1.$$

Then, for all $x \ge 1$,

$$\sum_{\alpha \in xW} (\omega_\wp(\alpha) - A_\wp)^2 \le c_1 |W| A_\wp x^2 + c_2 \left(\sum_{j=1}^{r} \frac{1}{|\rho_j|} \right) x, \tag{2.1}$$

where c_1 is an absolute constant and where c_2 is a constant which may depend on W.

Proof. One can proceed as Turán did (see for instance Kubilius [7], Chapter 10, Lemmas 10.1 and 10.2). We omit the details. ◊

Let V be a convex domain in the complex plane.

Lemma 2. *Let $\chi \in \mathcal{A}$. Then*

$$\lim_{x \to \infty} \frac{1}{x^2} \sum_{\gamma \in xV} \chi(\gamma) = 0. \tag{2.2}$$

Proof. Let $\varepsilon > 0$ and ε_1 be arbitrary small positive numbers. We can approximate V by a finite family of squares I_i, $i = 1, 2, \dots, h$, each of area ε^2 such that $I_1 \cup I_2 \cup \dots \cup I_h \subset V$, interior$(I_i \cap I_j) = \emptyset$ for $i \ne j$, and $\left| V \setminus \left(\cup_{i=1}^{h} I_i \right) \right| < \varepsilon_1$. Such a family of I_j's clearly exists.

Let $I_r = [u, u+\varepsilon) \times [v, v+\varepsilon)$ be one of these squares. Since

$$L_r(x) := \sum_{\gamma \in xI_r} \chi(\gamma) = \left\{ \sum_{xu \le m < x(u+\varepsilon)} \chi(1)^m \right\} \left\{ \sum_{xv \le n < x(v+\varepsilon)} \chi(i)^n \right\},$$

and either A or B is irrational, it follows that $L_r(x) = O(x)$, where the constant in the $O(\dots)$ may depend on A or B. This proves (2.2). ◊

3. Proof of Theorem 1

Let $\wp$ be as in Lemma 1 and set

$$T(x) := \sum_{\beta \in xW} g(\beta)\chi(\beta), \tag{3.1}$$

$$T_1(x) := \sum_{\substack{\rho\gamma \in xW \\ \rho \in \wp}} g(\rho\gamma)\chi(\rho\gamma), \tag{3.2}$$

$$T_2(x) := \sum_{\substack{\rho\gamma \in xW \\ \rho \in \wp}} g(\rho)g(\gamma)\chi(\rho\gamma). \tag{3.3}$$

Since $g(\rho\gamma) = g(\rho)g(\gamma)$, if ρ does not divide γ, it follows that for each positive constant c,

$$|T_1(x) - T_2(x)| \le \sum_{\rho \in \wp} \#\{\rho^2\beta \in xW\} \le cx^2 \sum_{j=1}^{r} \frac{1}{|\rho_j|^4} \le \frac{cx^2}{|\rho_1|^2} A_{\wp}. \tag{3.4}$$

Furthermore, since

$$T_1(x) = \sum_{\beta \in xW} g(\beta)\chi(\beta)\omega_{\wp}(\beta),$$

it follows from Lemma 1 that for some positive constant c_3,

$$|A_{\wp} T(x) - T_1(x)| \le \sum_{\beta \in xW} |\omega_{\wp}(\beta) - A_{\wp}| \le$$

$$\le \left(c_1 |W| A_{\wp} x^2 + c_2 \left(\sum_{j=1}^{r} \frac{1}{|\rho_j|} \right) x \right)^{1/2} \left(c_3 x^2 |W|\right)^{1/2}. \tag{3.5}$$

We now proceed to estimate (3.3). Let

$$a(\gamma) := g(\gamma), \qquad b(\gamma) := \sum_{\substack{\rho \in \wp \\ \rho \in \frac{x}{\gamma} W}} g(\rho)\chi(\rho\gamma).$$

Thus

$$T_2(x) = \sum_{\gamma} a(\gamma) b(\gamma),$$

where the γ's run over the set of Gaussian integers $\bigcup_{\rho} \left(\frac{x}{\rho} W\right)$.

Thus, by using the Cauchy–Schwarz inequality, we have $|T_2(x)| \leq \leq \sum_1^{1/2} \cdot \sum_2^{1/2}$, where $\Sigma_1 = \sum |a(\gamma)|^2$ and $\Sigma_2 = \sum |b(\gamma)|^2$.

It is immediate that

$$\Sigma_1 \ll x^2. \tag{3.6}$$

Indeed, if W is covered by the disk of radius s centered at the origin, then all the γ's occurring in Σ_1 satisfy $|\gamma| \leq \frac{xs}{|\rho_1|}$, which proves (3.6).

On the other hand,

$$\Sigma_2 = \sum_{\gamma} \sum_{\substack{\rho\in\wp \\ \rho\gamma\in xW}} 1 + \sum_{\nu\neq j} \sum_{\gamma\in S_{\nu,j}} g(\rho_\nu)\overline{g(\rho_j)}\chi(\rho_\nu\gamma)\overline{\chi(\rho_j\gamma)}, \tag{3.7}$$

where $S_{\nu,j} = \dfrac{xW}{\rho_\nu} \bigcap \dfrac{xW}{\rho_j}$. Let

$$\mathcal{B}_{\nu,j} := \sum_{\gamma\in S_{\nu,j}} \chi((\rho_\nu - \rho_j)\gamma). \tag{3.8}$$

Observe that $\chi^*(\gamma) = \chi((\rho_\nu - \rho_j)\gamma)$ is an additive character which, furthermore, belongs to $\mathcal{A}$. Indeed, if we write

$$\rho_\nu - \rho_j = k + \ell i \neq 0, \qquad \gamma = m + ni,$$

then

$$\chi^*(m+ni) = \chi(((mk - \ell n) + (nk + \ell m)i) =$$
$$= e\left(A(mk - \ell n) + B(nk + \ell m)\right) = e\left((Ak + B\ell)m + (Bk - A\ell)n\right).$$

If $\chi^* \notin \mathcal{A}$, then both $Ak + B\ell$ and $Bk - A\ell$ are rationals, which implies that A and B are rationals, a contradiction.

Furthermore, observe that $\dfrac{W}{\rho_\nu} \bigcap \dfrac{W}{\rho_j}$ is a collection of finitely many convex domains. Whence, by Lemma 2, we have that

$$\mathcal{B}_{\nu,j} = o(x^2) \tag{3.9}$$

Thus it follows from (3.7) that

$$\Sigma_2 \leq cx^2\sqrt{A_\wp} + o(x^2). \tag{3.10}$$

Consequently

$$\limsup_{x\to\infty} \frac{|T_2(x)|}{x^2} \leq c\sqrt{A_\wp}.$$

Thus, from (3.4),

$$\limsup_{x\to\infty} \frac{|T_1(x)|}{x^2} \le \frac{c}{|\rho_1|^2} A_\wp + c\sqrt{A_\wp},$$

and so, by (3.5),

$$\limsup_{x\to\infty} \frac{|T(x)|}{x^2} \le \frac{c}{|\rho_1|^2} + \frac{c}{\sqrt{A_\wp}}. \tag{3.11}$$

Let $0 < \varepsilon < 1$ be arbitrary. Then we can choose a finite set of Gaussian primes $\rho_1, \rho_2, \dots, \rho_r$ such that

$$\frac{1}{\varepsilon} \le |\rho_1| \le |\rho_2| \le \dots \le |\rho_r| \qquad \text{and} \qquad \sum_{j=1}^{r} \frac{1}{|\rho_j|^2} > \frac{1}{\varepsilon^2}.$$

Thus the right-hand side of (3.11) is less than $2\varepsilon c$. We therefore obtain that the left-hand side of (3.11) equals zero, which concludes the proof of Th. 1. $\Diamond$

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