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ON RIEMANNIAN MANIFOLDS WITH CERTAIN CURVATURE CONDITIONS

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Abstract: The object of the present paper is to study a type of non-flat Riemannian manifolds called generalized projective 2-recurrent Riemannian manifolds. At first we give a concrete example of generalized projective 2-recurrent Riemannian manifold. In section 3, we obtain necessary and sufficient condition for a generalized projective 2-recurrent Riemannian manifold to be a generalized 2-recurrent Riemannian manifold. We also prove that a generalized projective 2-recurrent Riemannian manifold is a generalized conformally 2-recurrent Riemannian manifold. In section 4, we prove that an Einstein generalized projective 2-recurrent Riemannian manifold is of constant curvature. Finally it is shown that an Einstein generalized projective 2-recurrent Riemannian manifold admitting a unit parallel vector field is a generalized 2-recurrent Riemannian manifold.

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1. Introduction

As is well known, symmetric spaces play an important role in differential geometry. The study of Riemannian symmetric spaces was initiated in the late twenties by E. Cartan, who, in particular, obtained a classification of those spaces.

Let (M^n, g) be a Riemannian manifold of dimension n and let ∇ be the Levi-Civita connection of (M^n, g). A Riemannian manifold is called locally symmetric if $\nabla R = 0$, where R is the Riemannian curvature tensor of (M^n, g). This notion has been generalized in many ways by various authors, such as recurrent manifolds introduced by A. G. Walker [1], weakly symmetric manifolds by L. Tamássy and T. Q. Binh [2], pseudo symmetric manifolds introduced by M. C. Chaki [3], conformally symmetric manifolds by Chaki and Gupta [4], projective symmetric manifolds by G. Soós [5] etc. These notions have been further generalized by E. M. Patterson [6] by introducing Ricci-recurrent manifolds, weakly Ricci-symmetric manifolds by L. Tamássy and T. Q. Binh [7], 2-recurrent manifolds by A. Lichnerowicz [8], projective 2-recurrent manifolds by D. Ghosh [9] and others.

In 1972, A. K. Roy [10] generalized the notion of 2-recurrent Riemannian manifold. A non-flat Riemannian manifold (M^n, g) is called generalized 2-recurrent if the Riemannian curvature tensor R satisfies the condition

$$(1)\quad (\nabla_V \nabla_U R)(X,Y)Z = A(V)(\nabla_U R)(X,Y)Z + B(U,V)R(X,Y)Z,$$

where A is a non-zero 1-form and B is a non-zero (0,2) tensor. Such a manifold is denoted by $G\{^2K_n\}$. In the same paper A. K. Roy introduced the notion of generalized projective 2-recurrent Riemannian manifold and generalized conformally 2-recurrent Riemannian manifold. In a Riemannian manifold the projective curvature tensor is defined by

$$(2)\qquad P(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}[S(Y,Z)X - S(X,Z)Y],$$

where S is the Ricci tensor. A Riemannian manifold is called a generalized projective 2-recurrent Riemannian manifold if the projective curvature tensor defined by (2) satisfies the condition

$$(3)\quad (\nabla_V \nabla_U P)(X,Y)Z = A(V)(\nabla_U P)(X,Y)Z + B(U,V)P(X,Y)Z,$$

where A and B are as stated earlier, and such a manifold is denoted by $G\{P(^2K_n)\}$.

In a Riemannian manifold the conformal curvature tensor is defined by

$$
\begin{aligned}
(4)\qquad C(X,Y)Z &= R(X,Y)Z - \frac{1}{n-2}[g(Y,Z)QX - g(X,Z)QY \\
&+S(Y,Z)X - S(X,Z)Y] + \frac{r}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y],
\end{aligned}
$$

where Q is the Ricci operator defined by $g(QX,Y) = S(X,Y)$ and r is the scalar curvature.

A Riemannian manifold is called a generalized conformally 2-recurrent Riemannian manifold if the conformal curvature tensor defined by (4) satisfies the condition

$$(5)\quad (\nabla_V \nabla_U C)(X,Y)Z = A(V)(\nabla_U C)(X,Y)Z + B(U,V)C(X,Y)Z,$$

where A and B are as stated earlier, and such a manifold is denoted by $G\{C({}^2K_n)\}$.

In a recent paper [11] U. C. De and S. Bondyopadhyay introduced and studied generalized Ricci 2-recurrent Riemannian manifolds which are defined as follows: A non-flat Riemannian manifold is called a generalized Ricci 2-recurrent Riemannian manifold if the Ricci tensor S is non-zero and satisfies the condition

$$(6)\quad (\nabla_V \nabla_U S)(X,Y) = A(V)(\nabla_U S)(X,Y) + B(U,V)S(X,Y).$$

Such a manifold is denoted by $G({}^2R_n)$.

In the present paper we study some properties of generalized projective 2-recurrent Riemannian manifolds. At first we give a concrete example of a $G\{P({}^2K_n)\}$. From (1) and (3) it follows that a $G\{{}^2K_n\}$ is a $G\{P({}^2K_n)\}$, but the converse is not true in general. In section 3, we obtain a necessary and sufficient condition for a $G\{P({}^2K_n)\}$ to be a $G\{{}^2K_n\}$. Also we prove that a $G\{P({}^2K_n)\}$ is a $G\{C({}^2K_n)\}$. It is known [12] that a manifold of constant curvature is an Einstein manifold, but the converse is not true, in general. The converse will be true if the dimension of $M^n = 3$. In section 4, it is shown that an Einstein $G\{P({}^2K_n)\}$ is a manifold of constant curvature, i.e. it is locally isometric to a unit sphere. Finally we study an Einstein $G\{P({}^2K_n)\}$ admitting a unit parallel vector field.

2. Example of a $G\{P(^2K_n)\}$

In this section we give a concrete example of a $G\{P(^2K_n)\}$.

Let each Latin index run over $1, 2, \ldots, n$, and each Greek index over $2, 3, \ldots, n-1$. We define the metric g on M^n ($n \geq 4$) by the formula

$$ds^2 = \Phi(dx^1)^2 + K_{\alpha\beta}dx^\alpha dx^\beta + 2dx^1 dx^n \tag{2.1}$$

where $\Phi(x^1, x^2, \ldots, x^n) = K_{\alpha\beta}dx^\alpha dx^\beta$ and

$$[K_{\alpha\beta}] = \begin{bmatrix} 1 & 0 & \ldots. & 0 \\ 0 & 1 & \ldots. & 0 \\ . & . & \ldots.. & . \\ 0 & 0 & \ldots. & 1 \end{bmatrix}$$

So $\Phi_{.\alpha\beta} = 2K_{\alpha\beta}e^{x^1}$, where (.) denotes partial differentiation. Since $K_{\alpha\beta} = 0$ for $\alpha \neq \beta$, the non-zero components of $\Phi_{.\alpha\beta}$ are $\Phi_{.\alpha\alpha} = 2e^{x^1}$.

In the metric considered, the only non-vanishing components of the curvature tensor R_{hijk} and of the Ricci-tensor R_{ij} are (see [13])

$$\begin{aligned} R_{1\alpha\alpha 1} &= \frac{1}{2}Q_{.\alpha\alpha} = e^{x^1} \\ R_{11} &= \frac{1}{2}K^{\alpha\alpha}Q_{.\alpha\alpha} = (n-2)e^{x^1}. \end{aligned} \tag{2.2}$$

So the only non-zero components of $\nabla_1 R_{hijk}$, $\nabla_m \nabla_l R_{hijk}$, $\nabla_l R_{ij}$, $\nabla_m \nabla_l R_{ij}$ and the relation between them are given by

$$\begin{aligned} \nabla_l R_{1\alpha\alpha 1} &= \nabla_l \nabla_l R_{1\alpha\alpha 1} = R_{1\alpha\alpha 1} = e^{x^1} \\ \nabla_1 R_{11} &= \nabla_1 \nabla_1 R_{11} = R_{11} = (n-2)e^{x^1}. \end{aligned} \tag{2.3}$$

Here the expression for the projective curvature tensor in local coordinates is given by

$$P_{hijk} = R_{hijk} - \frac{1}{n-1}[g_{hk}R_{ij} - g_{hj}R_{ik}].$$

Since the non-zero components of g_{ij} from (2.1) are $g_{11} (= \Phi)$, g_{1n}, g_{n1}, $g_{\alpha\alpha}$, we get that, by virtue of (2.2) the only non-zero components of P_{hijk} are

$$P_{1\alpha\alpha 1} = R_{1\alpha\alpha 1},\ P_{\alpha 1\alpha 1} = -R_{1\alpha\alpha 1} + \frac{1}{n-1}R_{11},\ P_{111n} = -\frac{1}{n-1}R_{11}. \tag{2.4}$$

Now, on account of (2.3), the non-zero components of the covariant derivatives of P_{abcd} satisfy the relation

$$\nabla_1\nabla_1 P_{abcd} = \nabla_1 P_{abcd} = P_{abcd}.$$

Therefore we can write

$$\nabla_1\nabla_1 P_{abcd} = \frac{1}{2}\nabla_1 P_{abcd} + \frac{1}{2}P_{abcd}.$$

The above relation holds trivially for the components of P_{hijk} which is zero. Hence the relation

$$\nabla_m\nabla_l P_{hijk} = \beta_m\nabla_l P_{hijk} + a_{lm}P_{hijk}$$

holds for $\beta_m(=\frac{1}{2})\neq 0$ and $a_{lm}(=\frac{1}{2})\neq 0$.

Therefore M^n with the metric g defined by (2.1) is a $G\{P({}^2K_n)\}$.

3. Generalized projective 2-recurrent Riemannian manifolds

In this section we deduce a necessary and sufficient condition for a $G\{P({}^2K_n)\}$ to be a $G\{{}^2K_n\}$. We prove the following

Theorem 1. *A necessary and sufficient condition for a generalized projective 2-recurrent Riemannian manifold to be a generalized 2-recurrent Riemannian manifold is that the manifold is a generalized Ricci 2-recurrent Riemannian manifold.*

Proof. First we assume that a $G\{P({}^2K_n)\}$ is a $G\{{}^2K_n\}$. Then from (3) by virtue of (2) we get

$$(\nabla_V\nabla_U R)(X,Y)Z - \frac{1}{n-1}[(\nabla_V\nabla_U S)(Y,Z)X - (\nabla_V\nabla_U S)(X,Z)Y]$$
$$= A(V)[(\nabla_U R)(X,Y)Z - \frac{1}{n-1}\{(\nabla_U S)(Y,Z)X - (\nabla_U S)(X,Z)Y\}]$$
$$+ B(U,V)[R(X,Y)Z - \frac{1}{n-1}\{S(Y,Z)X - S(X,Z)Y\}]$$

or

(3.1) $$(\nabla_V\nabla_U R)(X,Y)Z - A(V)(\nabla_U R)(X,Y)Z - B(U,V)R(X,Y)Z$$
$$= \frac{1}{n-1}[(\nabla_V\nabla_U S)(Y,Z)X - A(V)(\nabla_U S)(Y,Z)X - B(U,V)S(Y,Z)X]$$
$$- \frac{1}{n-1}[(\nabla_V\nabla_U S)(X,Z)Y - A(V)(\nabla_U S)(X,Z)Y - B(U,V)S(X,Z)Y]$$

and consequently, using (1), we obtain

$$(3.2)\quad \frac{1}{n-1}\{(\nabla_V\nabla_U S)(Y,Z)g(X,W) - A(V)(\nabla_U S)(Y,Z)g(X,W) -B(U,V)S(Y,Z)g(X,W) - (\nabla_V\nabla_U S)(X,Z)g(Y,W) +A(V)(\nabla_U S)(X,Z)g(Y,W) + B(U,V)S(X,Z)g(Y,W)\} = 0.$$

Putting $X = W = e_i$ in (3.2), where $\{e_i\}$, $i = 1, 2, \ldots, n$ is an orthonormal basis of the tangent space at any point of the manifold and taking summation over i, $1\leq i\leq n$, we get

$$(3.3)\quad (\nabla_V\nabla_U S)(Y,Z) - A(V)(\nabla_U S)(Y,Z) - B(U,V)S(Y,Z) = 0.$$

That is, the manifold is a $G\{^2R_n\}$. Conversely, suppose that the condition (3.3) holds. Then from (3.1) it follows that the manifold is a $G\{^2K_n\}$. This completes the proof. ◊

Next we prove

Theorem 2. *Every generalized projective 2-recurrent Riemannian manifold $G\{P(^2K_n)\}$ $(n > 3)$ is a generalized conformally 2-recurrent Riemannian manifold $G\{C(^2K_n)\}$ $(n > 3)$.*

Proof. Now we consider a generalized projective 2-recurrent Riemannian manifold. Then we have (3.3). Putting in (3.3) again $Y = Z = e_i$, where $\{e_i\}$, $i = 1, 2, ..., n$ is an orthonormal basis of the tangent space at any point of the manifold and taking summation over i, $1\leq i\leq n$, we get

$$(3.4)\qquad \nabla_V\nabla_U r - A(V)\nabla_U r - B(U,V)r = 0.$$

Also, we have from (3.3)

$$(3.5)\qquad (\nabla_V\nabla_U Q)(Y) - A(V)(\nabla_U Q)(Y) - B(U,V)Q(Y) = 0,$$

where Q is the Ricci operator defined by $S(X,Y) = g(QX,Y)$.

Using (4) in (5) we obtain

$$(\nabla_V\nabla_U C)(X,Y)Z = (\nabla_V\nabla_U R)(X,Y)Z - \frac{1}{n-2}[g(Y,Z)(\nabla_V\nabla_U Q)X - g(X,Z)(\nabla_V\nabla_U Q)Y + (\nabla_V\nabla_U S)(Y,Z)X - (\nabla_V\nabla_U S)(X,Z)Y] + \frac{(\nabla_V\nabla_U r)}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y].$$

With the help of (1), (4), (3.4) and (3.5) we get from the above equation

$$(\nabla_V\nabla_U C)(X,Y)Z = A(V)(\nabla_U C)(X,Y)Z + B(U,V)C(X,Y)Z.$$

That is, the manifold is a $G\{C(^2K_n)\}$. ◊

4. Einstein generalized projective 2-recurrent Riemannian manifolds

In this section we obtain

Theorem 3. *An Einstein $G\{P(^2K_n)\}$ is of constant curvature.*

Proof. Let us suppose that a $G\{P(^2K_n)\}$ is an Einstein manifold. Then the projective curvature tensor takes the form

$$P(X,Y)Z = R(X,Y)Z - \frac{r}{n(n-1)}[g(Y,Z)X - g(X,Z)Y] \tag{4.1}$$

where r is the scalar curvature of the manifold.

Now from (4.1) it follows that

$$(\nabla_U P)(X,Y)Z = (\nabla_U R)(X,Y)Z, \quad \text{since } r \text{ is constant.} \tag{4.2}$$

Using Bianchi's 2nd identity we get

$$(\nabla_U P)(X,Y)Z + (\nabla_X)(Y,U)Z + (\nabla_Y P)(U,X)Z = 0. \tag{4.3}$$

Now covariant differentiation of (4.3) yields

$$(\nabla_V \nabla_U P)(X,Y)Z + (\nabla_V \nabla_X P)(Y,U)Z + (\nabla_V \nabla_Y P)(U,X)Z = 0. \tag{4.4}$$

Using (4.1) and (4.3) we get from (4.4)

$$B(U,V)P(X,Y)Z + B(X,V)P(Y,U)Z + B(Y,V)P(U,X)Z = 0. \tag{4.5}$$

Contracting (4.5) we get

$$B(P(X,Y)Z,V) = 0$$

that is

$$g(P(X,Y)Z,LV) = 0 \tag{4.6}$$

where the associated $(0,2)$ tensor B is induced by a linear endomorphism

$$L : T_P \to T_P(M)$$

given by

$$B(X,Y) = g(X,LY). \tag{4.7}$$

From (4.1) we find that

$$\begin{aligned} g(P(X,Y)Z,W) &= -g(P(X,Y)W,Z) \\ &= -g(P(Y,X)Z,W) = -g(P(Z,W)X,Y). \end{aligned} \tag{4.8}$$

Now using (4.7) and putting $U = LV$, expression (4.5) takes the form

(4.9)
$$g(LV,LV)P(X,Y)Z+g(X,LV)P(Y,LV)Z+g(Y,LV)P(LV,X)Z=0.$$

Using (4.6) and (4.8) it follows from (4.9) that

$$g(LV,LV)P(X,Y)Z=0,$$

since $g(LV,LV)\neq 0$, $P(X,Y)Z=0$.

But it is known [11] that a projectively flat manifold is a manifold of constant curvature and this completes the proof of the above theorem. ◊

Next we obtain a necessary and sufficient condition for an Einstein $G\{P({}^2K_n)\}$ to be a $G\{{}^2K_n\}$

Theorem 4. *The necessary and sufficient condition for an Einstein $G\{P({}^2K_n)\}$ to be a $G\{{}^2K_n\}$ is that the scalar curvature r vanishes.*

Proof. We suppose that an Einstein $G\{P({}^2K_n)\}$ is a $G\{{}^2K_n\}$. Then we have (4.1), (4.2) and also

$$(\nabla_V\nabla_U P)(X,Y)Z=(\nabla_V\nabla_U R)(X,Y)Z. \tag{4.10}$$

Now applying (4.1), (4.2) and (4.10) in (3) we get

$$\begin{aligned}&(\nabla_V\nabla_U R)(X,Y)Z-A(V)(\nabla_U R)(X,Y)Z-B(U,V)R(X,Y)Z\\&+B(U,V)\frac{r}{n(n-1)}[g(Y,Z)X-g(X,Z)Y]=0.\end{aligned} \tag{4.11}$$

Since the manifold is a $G\{{}^2K_n\}$, we get from (4.11) by applying (1) that

$$B(U,V)\frac{r}{n(n-1)}[g(Y,Z)X-g(X,Z)Y]=0.$$

But $B(U,V)\neq 0$, and thus, for appropriately chosen X,Y,Z $[g(Y,Z)X-$ $-g(X,Z)Y]\neq 0$, $r=0$.

Next suppose that $r=0$. Then from (4.1) we get

$$P(X,Y)Z=R(X,Y)Z,$$

which means that $G\{P({}^2K_n)\}$ is a $G\{{}^2K_n\}$. ◊

Finally we consider an Einstein $G\{P({}^2K_n)\}$ which admits a unit parallel vector field, then we obtain the following theorem.

Theorem 5. *If an Einstein $G\{P({}^2K_n)\}$ admits a unit parallel vector field, then manifold reduces to a $G\{{}^2K_n\}$.*

We suppose that an Einstein $G\{P({}^2K_n)\}$ admits a unit parallel vector field V [14], [15]. Then

$$\nabla_X V = 0. \tag{4.12}$$

Applying Ricci identity in (4.12) we get

$$R(X,Y)V = 0. \tag{4.13}$$

Contracting (4.13) we obtain $S(Y,V) = 0$. Therefore

$$\frac{r}{n} g(Y,V) = 0. \tag{4.14}$$

Putting $Y = V$ in (4.14) we get $r = 0$, since V is a unit vector field. Thus Th. 5 follows from Th. 4. ◊

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