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ON QUASI-CONTINUOUS ITERATION GROUPS ON THE UNIT CIRCLE

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Abstract: The aim of this paper is to give a characterization of iteration groups defined on the unit circle S^1, continuous with respect to the iterative parameter. Such groups are named quasi-continuous. The problem of the embeddability of a given function $T : S^1 \longrightarrow S^1$ into quasi-continuous iteration groups is also considered.

Let $S^1 = \{x \in \mathbb{C} : |x| = 1\}$ be the unit circle. A set $L \subset S^1$ is said to be an open arc if

$$L = \overrightarrow{(x_1, x_2)} := \{e^{2\pi i t} : t \in (t_1, t_2)\},$$

where $t_1, t_2 \in \mathbb{R}$ are such that $t_1 < t_2 \leq t_1 + 1$ and $x_1 = e^{2\pi i t_1}$, $x_2 = e^{2\pi i t_2}$. Similarly we define $\overrightarrow{(x_1, x_2]}$, $\overrightarrow{[x_1, x_2)}$, $\overrightarrow{[x_1, x_2]}$, but with one different detail: $t_1 < t_2 < t_1 + 1$. Each of these four will be called an arc in this paper.

Let L be an arc or $L = S^1$ or L be a singleton. Let us introduce the following:

Definitions (see [3], also [6]). A family $\{T^t, t \in \mathbb{R}\}$ of functions $T^t : L \longrightarrow L$ is said to be an iteration group on L if

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$$T^t \circ T^s = T^{t+s} \qquad \text{for } t, s \in \mathbb{R}.$$

If for every $x \in L$ the mapping $h_x : \mathbb{R} \longrightarrow L$ given by $h_x(t) := T^t(x), t \in \mathbb{R}$ is continuous then the iteration group is said to be quasi-continuous.

If, moreover, all functions T^t are continuous then the quasi-continuous iteration group will be called a continuous iteration group.

The general construction of quasi-continuous iteration groups of real functions is given in [6]. On the base of these results we give a construction of quasi-continuous iteration groups on the unit circle.

Given an iteration group $\{T^t, t \in \mathbb{R}\}$ on S^1 and $x \in S^1$ put

$$C(x) := \{T^t(x), t \in \mathbb{R}\},$$

$$B(x) := \{t \in \mathbb{R} : T^t(x) = x\},$$

and

$$p(x) := \inf\{t > 0 : T^t(x) = x\}, \quad (\inf \emptyset := \infty).$$

$p(x)$ is called the period of the point x.

For any mapping $T : S^1 \longrightarrow S^1$ we also put

$$A_T := \{x \in S^1 : T(x) = x\}.$$

We begin with some elementary properties of iteration groups on the circle.

Proposition 1 (see also [2]). *Let $\{T^t, t \in \mathbb{R}\}$ be an iteration group on S^1, then*

(i) *for $x, y \in S^1$ we have $C(x) = C(y)$ or $C(x) \cap C(y) = \emptyset$,*
(ii) *$T^0[S^1] = T^t[S^1]$ for $t \in \mathbb{R}$,*
(iii) *$T^0|_{T^0[S^1]} = Id|_{T^0[S^1]}$.*

Proof. To prove (i) fix $t \in \mathbb{R}$, $x, y \in S^1$ such that $x \neq y$. Suppose that $C(x) \cap C(y) \neq \emptyset$, i.e. there exist $t_1, t_2 \in \mathbb{R}$ such that $T^{t_1}(x) = T^{t_2}(y)$. Then

$$T^t(x) = T^{t-t_1+t_1}(x) = T^{t-t_1}(T^{t_1}(x)) = $$
$$= T^{t-t_1}(T^{t_2}(y)) = T^{t-t_1+t_2}(y) \in C(y).$$

Thus, $C(x) \subset C(y)$. In the same way we can show that $C(y) \subset C(x)$.

In order to prove (ii), fix a $t \in \mathbb{R}$. First, take an $x \in T^0[S^1]$ and let $y \in S^1$ be such that $x = T^0(y)$. Then

$$x = T^0(y) = T^{t-t}(y) = T^t(T^{-t}(y)) \in T^t[S^1].$$

If $x \in T^t[S^1]$, then there exists a $y \in S^1$ such that $x = T^t(y)$, and consequently

$$x = T^t(y) = T^{0+t}(y) = T^0(T^t(y)) \in T^0[S^1].$$

The proof of (iii) is trivial. ◊

Remark 1. *Let $\{T^t, t \in \mathbb{R}\}$ be a quasi-continuous iteration group on S^1. Then for every $x \in S^1$, $C(x)$ is either a singleton or the circle or an arc.*

Proof. Since for every $x \in S^1$, $C(x) = h_x[\mathbb{R}]$ and the function $h_x : \mathbb{R} \longrightarrow S^1$ is continuous, the set $C(x)$ is connected, and our assertion follows. ◊

The following lemmas are similar to Th. 1.13 in [2] but these ones give more facts and work with another assumptions.

Lemma 1. *Let $\{T^t, t \in \mathbb{R}\}$ be a quasi-continuous iteration group on S^1 and $x \in S^1$. Then*

(i) *the following three conditions are equivalent*

  (a) $p(x) = 0$,

  (b) $C(x) = \{x\}$,

  (c) $B(x) = \mathbb{R}$;

(ii) *the following three conditions are equivalent*

  (a) $0 < p(x) < \infty$,

  (b) $C(x) = S^1$,

  (c) $B(x)$ *is a nontrivial cyclic subgroup of* $\mathbb{R}$.

Proof. First, note that if $B(x) \neq \emptyset$ then $B(x)$ is a closed additive subgroup of $\mathbb{R}$, so it is either $\mathbb{R}$ or a cyclic subgroup of $\mathbb{R}$. This together with the definitions of $C(x)$ and $p(x)$ gives (i).

Next, we prove (ii). To do this, let us first assume that $B(x) = \{nt_0, n \in \mathbb{Z}\}$ for a positive t_0. Then $h_x|_{\langle 0,t_0)}$ is one-to-one. Indeed, assuming $h_x(s) = h_x(p)$ for some $p, s \in \langle 0, t_0)$ we get

$$x = T^0(x) = T^{s-s}(x) = T^{-s}(T^s(x)) = T^{-s}(T^p(x)) = T^{p-s}(x),$$

since $0 \in B(x)$. Then $p - s \in B(x)$, so $p = s$. Next, note that $h_x(0) = h_x(t_0) = x$. From this, Remark 1 and the fact that $h_x|_{\langle 0,t_0)}$ is a continuous injection we have

$$C(x) = h_x[\mathbb{R}] = h_x[\langle 0, t_0)] = S^1.$$

Conversely, assume that $C(x) = S^1$. Then $B(x) \neq \mathbb{R}$, and there exists a $t \in \mathbb{R}$ such that $x = T^t(x)$. Therefore $B(x) \neq \emptyset$. If $B(x) = \{0\}$, then h_x is easily seen to be one-to-one, which contradicts the known fact that there does not exist a continuous injection from $\mathbb{R}$ onto S^1 (see for instance [2]). Consequently, $B(x)$ is a nontrivial cyclic subgroup of $\mathbb{R}$. The rest of the proof is immediate. ◊

Lemma 2. *Let $\{T^t, t \in \mathbb{R}\}$ be a quasi-continuous iteration group on S^1. If $x \in S^1$ and $p(x) = \infty$ then one of the following conditions occurs:*
(H1) $x \neq T^0(x)$ *and* $B(T^0(x)) = \mathbb{R}$,
(H2) $x \neq T^0(x)$ *and* $B(T^0(x)) = \{0\}$,
(H3) $x = T^0(x)$ *and* $B(x) = \{0\}$.

Proof. Let us assume that $x \in S^1$ and $p(x) = \infty$. Then $B(x) \cap \mathbb{R}^+ = \emptyset$. Thus, $B(x) = \emptyset$ or $B(x) = \{0\}$. Assuming $B(x) = \emptyset$ we have $T^t(x) \neq x$ for every $t \in \mathbb{R}$, so $x \notin C(x)$. Note that for $y = T^0(x)$, $C(x) = C(y)$. Obviously, $T^0(y) = y$, so $0 \in B(y)$. If $B(y)$ is a cyclic nontrivial subgroup of $\mathbb{R}$, then by Lemma 1, $S^1 = C(y) = C(x)$, contrary to $x \notin C(x)$. Hence $B(y) = \mathbb{R}$ or $B(y) = \{0\}$. If $B(x) = \{0\}$, then $x = T^0(x)$. $\diamondsuit$

Lemma 3. *Let $\{T^t, t \in \mathbb{R}\}$ be a quasi-continuous iteration group on S^1. Let $x \in S^1$, then*
(i) *if* (H1) *then* $C(x) = \{T^0(x)\}$, $p(x) = \infty$ *and the function* h_x *is constant,*
(ii) *if* (H2) *then* $C(x)$ *is an arc such that* $x \notin C(x)$ *and* $p(x) = \infty$,
(iii) *if* (H3) *then* $C(x)$ *is an arc such that* $x \in C(x)$ *and* $p(x) = \infty$.
Moreover, the following three conditions are equivalent
(a) (H2) *or* (H3) *occurs,*
(b) h_x *is a homeomorphism,*
(c) $C(x)$ *is an arc.*

Proof. Fix an $x \in S^1$ and put $y := T^0(x)$. Let us first assume that $x \neq T^0(x)$. Then $B(x) = \emptyset$ and, by Lemma 1, $p(x) = \infty$. If $B(y) = \mathbb{R}$ then, by Lemma 1, $\{T^0(x)\} = C(y) = C(x)$, and consequently h_x is also constant. If $B(y) = \{0\}$, then $h_x(s) = h_x(p)$ implies $h_y(s) = h_y(p)$, and consequently $s = p$. Therefore h_x is one-to-one, and Remark 1 now shows that $C(x)$ is an arc with $x \notin C(x)$. Next, assume that (H3) holds true. Then h_x is an injection, and consequently $C(x)$ is an arc with $x \in C(x)$. Moreover, Lemma 1 now shows that $p(x) = \infty$.

From Lemmas 1, 2 and the proved part of Lemma 3 it follows that conditions (a) i (c) are equivalent. Moreover, it is obvious that (b) implies (c). To complete the proof let us assume that (c) holds true. Then there is an open arc L such that $C(x) \subset L$. Let g be a homeomorphism from L onto $\mathbb{R}$. Since h_x is one-to-one, the mapping $f : \mathbb{R} \longrightarrow \mathbb{R}$ given by $f := g \circ h_x$ is a continuous injection, and consequently f is a homeomorphism. Therefore so is h_x. $\diamondsuit$

Corollary 1. *Let $\{T^t, t \in \mathbb{R}\}$ be a quasi-continuous iteration group on S^1 and $x \in S^1$. If $C(x)$ is an arc, then it is an open arc.*

From Lemma 1, Prop. 1 and Remark 1 we have

Remark 2. *Let $\{T^t, t \in \mathbb{R}\}$ be a quasi-continuous iteration group on S^1. If there exists an $x_0 \in S^1$ such that $0 < p(x_0) < \infty$, then $0 < p(x) < \infty$ for every $x \in S^1$.*

We can now prove the following

Theorem 1 (see also [2]). *If $\{T^t, t \in \mathbb{R}\}$ is a quasi-continuous iteration group on S^1, then for every $x \in S^1$, $\{T^t|_{C(x)}, t \in \mathbb{R}\}$ is a continuous iteration group on $C(x)$.*

Proof. Fix $x \in S^1$ and $t \in \mathbb{R}$. First, suppose that $0 < p(x) < \infty$. By Lemma 1 we see that $C(x) = S^1$. Moreover, from Prop. 1(iii) we conclude that $T^0 = Id_{S^1}$. Therefore, by Th. 1.19 in [2], we deduce that $\{T^t, t \in \mathbb{R}\}$ is a continuous iteration group on S^1.

Now, assume that $p(x) = 0$ or $p(x) = \infty$. The proof is completed by showing that $T^t|_{C(x)}$ is continuous. If the orbit contains only one point our assertion follows. By Lemma 1 we only need to show the continuity of $T^t|_{C(x)}$ in the case when the orbit is an arc.

Since $C(x)$ is a metric space, it is sufficient to show that for every sequence $(y_n)_{n \in \mathbb{N}}$ of elements of $C(x)$ such that $y_n \to y \in C(x)$, we have $T^t(y_n) \to T^t(y)$. Fix such a sequence and an $n \in \mathbb{N}$. Since $C(x) = C(y)$, we can find $s, s_n \in \mathbb{R}$ such that $y_n = T^{s_n}(y)$ and $y = T^s(x)$. Thus

$$y_n = T^{s_n}(y) = T^{s_n}(T^s(x)) = T^{s_n+s}(x).$$

Since $y_n \to y$, we have $T^{s_n+s}(x) \to T^s(x)$, i.e. $h_x(s_n + s) \to h_x(s)$. By Lemma 3 we see that h_x is a homeomorphism, so $s_n \to 0$. Hence and from the fact that h_y is continuous we obtain

$$T^t(y_n) = T^t(T^{s_n}(y)) = T^{t+s_n}(y) \to T^t(y). \ \diamond$$

The general form of continuous iteration groups on the unit circle is well known (see for instance [5]), but we will remind it. We first need to prove

Theorem 2. *Let $\{T^t, t \in \mathbb{R}\}$ be a quasi-continuous iteration group on S^1. If there exists an $x_0 \in S^1$ such that $0 < p(x_0) < \infty$ then*

(i) *for every $t \in \mathbb{R}$, either $T^t \equiv Id_{S^1}$ or $T^t(x) \neq x$ for $x \in S^1$,*

(ii) *$T^0 \equiv Id_{S^1}$,*

(iii) *T^t is a homeomorphism for $t \in \mathbb{R}$.*

Proof. Assume that $A_{T^a} \neq \emptyset$ for an $a \neq 0$. Fix an $x' \in A_{T^a}$. Then $T^a(x') = x'$. We claim that $C(x') \subset A_{T^a}$. Indeed, let $y \in C(x')$. Then there exists a $u \in \mathbb{R}$ such that $y = T^u(x')$. Thus

$$T^a(y) = T^a(T^u(x')) = T^u(T^a(x')) = T^u(x') = y,$$

and consequently $y \in A_{T^a}$. By Remark 2 we see that $0 < p(x') < \infty$ and Lemma 1(ii) now shows that $S^1 = C(x') \subset A_{T^a}$. Consequently, $A_{T^a} = S^1$.

Next, by Lemma 1(ii), $T^0[S^1] = S^1$, since $S^1 = C(x_0) \subset T^0[S^1]$. By Prop. 1(ii) and (iii), $T^0 \equiv Id_{S^1}$ and $T^t[S^1] = S^1$ for every $t \in \mathbb{R}$. Hence $T^{-t} \circ T^t = Id_{S^1}$, so T^t is invertible. Consequently, by Th. 1, every T^t is a homeomorphism from S^1 onto S^1. $\Diamond$

Th. 2 lets us to use Th. 2 in [5]. Thus, the general form of quasi-continuous iteration groups $\{T^t, t \in \mathbb{R}\}$ on S^1 such that $0 < p(x_0) < \infty$ for an $x_0 \in S^1$ is given by

$$T^t = \Phi^{-1} \circ Q_{at} \circ \Phi, \qquad t \in \mathbb{R},$$

where $\Phi\colon S^1 \to S^1$ is an orientation preserving homeomorphism, $a \in \mathbb{R}$ and

$$Q_a(x) := e^{2\pi i a} \cdot x, \qquad x \in S^1.$$

From now on we assume that

$$(1) \qquad p(x) = 0 \quad \text{or} \quad p(x) = \infty \qquad \text{for an } x \in S^1.$$

Lemma 4. *Let $\{T^t, t \in \mathbb{R}\}$ be a quasi-continuous iteration group on S^1 satisfying condition (1). If there exists an $s \neq 0$ and an $x_0 \in S^1$ such that $T^s(x_0) = x_0$, then $T^t(x_0) = x_0$ for every $t \in \mathbb{R}$.*

Proof. From Remark 2, Lemmas 1, 2 and 3 it follows that either $C(x_0)$ is an arc or $C(x_0) = \{x_0\}$ or $C(x_0) = \{T^0(x_0)\}$. Clearly,

$$T^s(x_0) = T^{s+0}(x_0) = T^0(T^s(x_0)) = T^0(x_0),$$

since $T^s(x_0) = x_0$, so h_{x_0} is not a homeomorphism. Thus, by Lemma 3, $C(x_0)$ is not an arc. Finally, $C(x_0) = \{x_0\}$. $\Diamond$

Put

$$(2) \qquad A := \{x \in S^1 : \quad \forall t \in \mathbb{R} \quad T^t(x) = x\}.$$

By Lemma 4 we have

Remark 3. *Let $\{T^t, t \in \mathbb{R}\}$ be a quasi-continuous iteration group on S^1 and let condition (1) hold true. Then for every $t \in \mathbb{R} \setminus \{0\}$, $A_{T^t} = A$. Moreover, $A \subset T^0[S^1]$.*

We can now formulate

Theorem 3. *Let $\{T^t, t \in \mathbb{R}\}$ be an iteration group on S^1 satisfying (1) and let A be given by (2). Then $\{T^t, t \in \mathbb{R}\}$ is quasi-continuous if and only if either $T^0[S^1] = A$ or there exists a family of open pairwise* *disjoint arcs* $\{L_n : L_n \cap A = \emptyset, \quad n \in \mathcal{M}\}$, *where* $\emptyset \neq \mathcal{M} \subset \mathbb{N}$, *such that*

$$T^0[S^1] = \bigcup_{n\in\mathcal{M}} L_n \cup A \tag{3}$$

and for every $n \in \mathcal{M}$, $\{T^t|_{L_n}, t \in \mathbb{R}\}$ *is a continuous iteration group on* L_n *such that all* $T^t|_{L_n} : L_n \to L_n$ *are bijections.*

Proof. Suppose that $\{T^t, t \in \mathbb{R}\}$ is a quasi-continuous iteration group. In view of Lemmas 1, 2, 3, Remark 2 and Cor. 1, condition (1) shows that $C(x)$ is an open arc or a singleton for every $x \in S^1$. Assume that $T^0[S^1] \neq A$ and fix $x \in T^0[S^1] \setminus A$, $t \in \mathbb{R}$. By Lemmas 1, 2, 3 and Prop. 1, $C(x)$ is an arc with $x \in C(x)$, and therefore h_x is a homeomorphism. Thus, for every $y \in C(x)$ there exists an $s \in \mathbb{R}$ such that $s = h_x^{-1}(y)$. Moreover,

$$T^t(y) = T^t(h_x(s)) = T^t(T^s(x)) = T^{t+s}(x) = h_x(t+s),$$

and consequently $T^t(y) = h_x(t + h_x^{-1}(y))$. Hence we infer that $T^t|_{C(x)}$ is continuous and one-to-one. Clearly, $T^t[C(x)] = C(x)$. Consequently, putting

$$\{L_n, \quad n \in \mathcal{M}\} := \{C(x), \quad x \in T^0[S^1] \setminus A\},$$

we obtain, in view of Prop. 1(i), a family of open pairwise disjoint arcs such that for every $n \in \mathcal{M}$, $A \cap L_n = \emptyset$ and (3) holds true.

Conversely, we show that h_x is continuous for every $x \in S^1$. Indeed, we see at once that this is true for $x \in T^0[S^1]$. If $x \in S^1 \setminus T^0[S^1]$ then we have $h_x = h_y$ with $y := T^0(x) \in T^0[S^1]$. ◊

Now, on the base of Th. 3 we give the general construction of quasi-continuous iteration groups on S^1 satisfying condition (1).

Theorem 4. *The following construction gives the general form of quasi-continuous iteration groups on* S^1 *satisfying condition* (1).

1° *Let* $\{L_n, n \in \mathcal{M}\}$, *where* $\mathcal{M} \subset \mathbb{N}$ *(we admit* $\mathcal{M} = \emptyset$*) be a family of open pairwise disjoint arcs.*

2° *For every* $n \in \mathcal{M}$ *let* $\{F_n^t, t \in \mathbb{R}\}$ *be a continuous iteration group on* L_n *such that all functions* F_n^t *are one-to-one and* $F_n^0(x) = x$ *for* $x \in L_n$. *(Such groups are given by the formula:*

$$F_n^t(x) = h(t + h^{-1}(x)), \quad x \in L_n, \quad t \in \mathbb{R},$$

where $h : \mathbb{R} \to L_n$ *is a homeomorphism* (see [1], p. 248–9).*)*

3° *Let* A *be an arbitrary (if* $\mathcal{M} = \emptyset$ *then, moreover, non-empty) subset of* $S^1 \setminus \bigcup_{n\in\mathcal{M}} L_n$.

4° *Put*

$$J := \bigcup_{n\in\mathcal{M}} L_n \cup A$$

and let a *be an arbitrary function defined in* S^1 *such that* $a[S^1] = J$ *and* $a(x) = x$ *for* $x \in J$.

5° *Define*

$$(4) \qquad T^t(x) := \begin{cases} a(x) & \text{for } x \in a^{-1}[A], \quad t \in \mathbb{R}, \\ F_n^t(a(x)) & \text{for } x \in a^{-1}[L_n], \quad t \in \mathbb{R}, \quad n \in \mathcal{M}. \end{cases}$$

Proof. It is easy to check that the family of functions T^t defined by (4) is a quasi-continuous iteration group on S^1 for which (1) holds.

Conversely, we will show that every quasi-continuous iteration group satisfying (1) can be obtained in the above manner. Assume that $\{T^t, t \in \mathbb{R}\}$ is such a group and define A by (2) and $a := T^0$. From Th. 3 it follows that either $A = T^0[S^1] \neq \emptyset$ or there are a non-empty set $\mathcal{M} \subset \mathbb{N}$ and a family of open pairwise disjoint arcs $\{L_n, n \in \mathcal{M}\}$ such that (3) holds true and $A \subset S^1 \setminus \bigcup_{n\in\mathcal{M}} L_n$. If $T^0[S^1] = A$, then $T^t = T^0$ for $t \in \mathbb{R}$. Therefore (4) holds true with $\mathcal{M} := \emptyset$. In the later case, we put $F_n^t := T^t|_{L_n}$ for $t \in \mathbb{R}$, $n \in \mathcal{M}$. Prop. 1(iii) and Th. 3 complete the proof. $\Diamond$

We can now consider the problem of the embeddability of a given function into quasi-continuous iteration group. Recall that a function $T : L \to L$, for $L \subset S^1$, is said to be embeddable into a quasi-continuous (continuous) iteration group if there exists a quasi-continuous (continuous) iteration group, defined on L, $\{T^t, t \in \mathbb{R}\}$ with $T^1 = T$.

Theorem 5. *A function* $T : S^1 \to S^1$ *is embeddable into a quasi-continuous iteration group if and only if one of the following occurs*

(i) T *is an orientation – preserving homeomorphism and either* $T^m = Id_{S^1}$ *for a positive integer* m *or the set* $\{T^n(x), n \in \mathbb{N}\}$ *is dense in* S^1 *for every* $x \in S^1$,

(ii) *there exists a non-empty set* $\mathcal{M} \subset \mathbb{N}$ *and a family of open pairwise disjoint arcs* $\{L_n : L_n \cap A_T = \emptyset, n \in \mathcal{M}\}$ *such that*

$$T[S^1] = \bigcup_{n\in\mathcal{M}} L_n \cup A_T$$

and for every $n \in \mathcal{M}$, $T|_{L_n} : L_n \to L_n$ *is a continuous bijection,*

(iii) $T[S^1] = A_T$.

Proof. Let $\{T^t, t \in \mathbb{R}\}$ be a quasi-continuous iteration group such that $T^1 = T$. First, suppose that $0 < p(x_0) < \infty$ for an $x_0 \in S^1$. By Th. 2

we infer that the iteration group $\{T^t, t \in \mathbb{R}\}$ is continuous and T is either without fixed points or the identity mapping. Th. 3 in [5] now shows that (i) holds true. Next, assume that (1) is satisfied. By Prop. 1(ii), Remark 3 and Th. 3 we see that (ii) or (iii) holds true.

Conversely, assume first (i). Then, by Th. 3 in [5], T is embeddable into a continuous iteration group on S^1. Assume now that (ii) or (iii) occurs. In the first case fix, moreover, an $n \in \mathcal{M}$ and note that since $T|_{L_n}$ is a continuous bijection, $T|_{L_n}$ is embeddable into a continuous iteration group $\{F_n^t, t \in \mathbb{R}\}$ such that all functions F_n^t are one-to-one (see [1], p. 248–9, and [4]). In the later case, we define $\mathcal{M} := \emptyset$. Put $J := T[S^1]$ and $A := A_T$. Clearly, $T|_J : J \to J$ is a bijection and, if (iii) occurs, A is non-empty. Defining $a := (T|_J)^{-1} \circ T$ we see that a maps S^1 onto J and $a(x) = x$ for $x \in J$. By Th. 4 the family $\{T^t, t \in \mathbb{R}\}$ of functions T^t given by (4) is a quasi-continuous iteration group such that, as one can check, $T^1 = T$. $\Diamond$

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