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BI-(φ , ψ) CONVEX SETS

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Abstract: In this paper, we extend the notion of E-convex sets to bi-(φ, ψ) convex sets and study some properties of this class of sets.

1. The notion of bi-(φ , ψ) convex set

Let X and Y be linear real spaces, $\varphi : X \to X$ and $\psi : Y \to Y$ two maps.

Definition 1.1. Let $T \subseteq \mathbb{R}$, $T \neq \emptyset$. A subset M of $X \times Y$ is said to be *bi-*(φ, ψ) *convex* with respect to T, either if $M = \emptyset$ or, if for all (x, y), (x, v), (u, y) of M, and all $t \in T$, we have

$$(\varphi(x), (1-t)\psi(y) + t\psi(v)) \in M \tag{1}$$

and

$$((1-t)\varphi(x) + t\varphi(u), \psi(y)) \in M. \tag{2}$$

Example 1.1. Let $X = Y = \mathbb{R}$, $\varphi : \mathbb{R} \to \mathbb{R}$,

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$$\varphi(x) = x^2, \text{ for all } x \in \mathbb{R},$$

and $\psi : \mathbb{R} \to \mathbb{R}$,

$$\psi(x) = x^2, \text{ for all } x \in \mathbb{R}.$$

Let

$$M = ([0,1] \times [0,1]) \bigcup ([-1,0] \times [-1,0]) \subseteq \mathbb{R}^2 \quad \text{and} \quad T = [0,1].$$

Obviously, if (x,y), (x,v), $(u,y) \in M$ and $t \in [0,1]$, we have $\psi(x) \in [0,1]$, $\psi(y) \in [0,1]$, $(1-t)\psi(y) + t\psi(v) = (1-t)y^2 + tv^2 \in [0,1]$, $(1-t)\varphi(x) + t\varphi(u) = (1-t)x^2 + tu^2 \in [0,1]$. Hence $(\varphi(x), (1-t)\psi(y) + t\psi(v)) \in [0,1] \times [0,1] \subseteq M$, and $((1-t)\varphi(x) + t\varphi(u), \psi(y)) \in [0,1] \times [0,1] \subseteq M$. Therefore, M is bi-(φ, ψ) convex with respect to $[0,1]$. Evidently, the set M is not convex.

Remark 1.1. If $M \subseteq X \times Y$ is convex (in the classical sens), then M is bi-$(1_X, 1_Y)$ convex with respect to $T = [0,1]$.

The converse is not necessarily true as seen in the following

Example 1.2. Let $X = Y = \mathbb{R}$, and

$$M = \{(x,0) \mid x \in [0,1]\} \bigcup \{(1,y) \mid y \in [0,1]\}.$$

We show that the set M is bi-$(1_\mathbb{R}, 1_\mathbb{R})$ convex with respect to $T = [0,1]$. Let (x,y), (x,v), $(u,y) \in M$ and $t \in [0,1]$. It is easy to see that, if $x = u$, or $y = v$, then $(\varphi(x), (1-t)\psi(y) + t\psi(v)) \in M$, and $((1-t)\varphi(x) + t\varphi(u), \psi(y)) \in M$. Let now $x \neq u$, and $y \neq v$. It follows that $(x,y) = (1,0)$. Then $(\varphi(x), (1-t)\psi(y) + t\psi(v)) = (1, tv) \in M$, and $((1-t)\varphi(x) + t\varphi(u), \psi(y)) = ((1-t) + tu, 0) \in M$. Therefore M is bi-$(1_\mathbb{R}, 1_\mathbb{R})$ convex with respect to $[0,1]$. But the set M is not convex (we have $(0,0) \in M$, $(1,1) \in M$, and $\frac{1}{2}(0,0) + \frac{1}{2}(1,1) = (\frac{1}{2}, \frac{1}{2}) \notin M$).

Remark 1.2. If $M \subseteq X \times Y$ is an affine set, then M is bi-$(1_X, 1_Y)$ convex with respect to $T = \mathbb{R}$. It follows that the notion of bi-(φ, ψ) convex set with respect to $T = \mathbb{R}$ is a generalization of affine set.

Remark 1.3. Let us suppose that $\Delta \subseteq [0,1]$, with $\{0,1\} \subseteq \Delta$. We remember that a set $M \subseteq \mathbb{R}^n$ is said to be quasi-convex (see [4]) if for each $x, y \in M$, one has $\{tx + (1-t)y \mid t \in \Delta\} \subseteq M$. It is easy to see that if $M \subseteq \mathbb{R}^n \times \mathbb{R}^m$, where n, m are natural numbers, is a quasi-convex set, then M is bi-$(1_X, 1_Y)$ convex with respect to $T = \Delta$. For $\Delta = \{0, \frac{1}{2}, 1\}$ we obtain the midpoint convexity (see [5]).

Remark 1.4. Let $\varphi : X \to X$, $\psi : Y \to Y$ be two functions and let $T_1 \subseteq T_2 \subseteq \mathbb{R}$, $T_1 \neq \emptyset$. If a set $M \subseteq X \times Y$ is bi-(φ, ψ) convex with respect to T_2, then M is also bi-(φ, ψ) convex with respect to T_1.

The converse is not necessarily true as seen in the following

Example 1.3. Let $X = Y = \mathbb{R}$, and

$$M = [0, +\infty[\times[0, +\infty[.$$

It is easy to see that the set M is bi-$(1_{\mathbb{R}}, 1_{\mathbb{R}})$ convex with respect to $T_1 = [0, 1]$. But M is not bi-$(1_{\mathbb{R}}, 1_{\mathbb{R}})$ convex with respect to $T_2 = [0, 2]$, because if we take $(x, y) = (1, 1)$, $(x, v) = (1, 0)$, $(u, y) = (0, 1)$, and $t = 2$, we have $((1 - 2)1 + 2 \cdot 0, 1) = (-1, 1) \notin M$.

Remark 1.5. Let $A \subseteq X$, $B \subseteq Y$, and $\varphi : X \to X$, $\psi : Y \to Y$ be two functions. If $\varphi(A) \subseteq A$, $\psi(B) \subseteq B$, and the sets $\varphi(A)$ and $\psi(B)$ are convex, then the set $A \times B$ is bi-(φ, ψ) convex with respect to $T = [0, 1]$.

Proposition 1.1. *Let $T \subseteq \mathbb{R}$, $T \neq \emptyset$. If $M \subseteq X \times Y$ is bi-(φ, ψ) convex with respect to T, then for each $(x, y) \in M$ we have $(\varphi(x), \psi(y)) \in M$.*

Proof. Let be $(x, y) \in M$, and $t \in T$. If we take $u = x$, $v = y$, from Def. 1.1 we get

$$(\varphi(x), \psi(y)) = (\varphi(x), (1 - t)\psi(y) + t\psi(v)) \in M. \ \Diamond$$

Let $M \subseteq X \times Y$. We denote

$$M_1 = \{x \in X \mid \exists y \in Y \text{ such that } (x, y) \in M\},$$

$$M_2 = \{y \in Y \mid \exists x \in X \text{ such that } (x, y) \in M\}.$$

Throughout the paper M_1 and M_2 will always have this meaning.

Proposition 1.2. *Let $T \subseteq \mathbb{R}$, with $0 \in T$, or $1 \in T$, and let $M \subseteq X \times Y$. If the set M is bi-(φ, ψ) convex with respect to T, then*

$$\varphi(M_1) \times \psi(M_2) \subseteq M. \tag{3}$$

Proof. If $M = \emptyset$, then $M_1 = \emptyset$, and $M_2 = \emptyset$. Therefore (3) is true. Let now $M \neq \emptyset$, and let $(u, v) \in \varphi(M_1) \times \psi(M_2)$. Then there are $x \in M_1$, with $u = \varphi(x)$, and $y \in M_2$, with $v = \psi(y)$. If we take $t = 0$, or $t = 1$, we obtain

$$(u, v) = (\varphi(x), \psi(y)) = ((\varphi(x), (1 - t)\psi(y) + t\psi(y)) \in M. \ \Diamond$$

In the following, for each $x \in M_1$, we put

$$M_x = \{y \in Y \mid (x, y) \in M\},$$

and, for each $y \in M_2$, we put

$$M_y = \{x \in X \mid (x, y) \in M\}.$$

Proposition 1.3. *If the set $M \subseteq X \times Y$ is bi-(φ, ψ) convex with respect to T and $[0, 1] \subseteq T$, then*

i) *for each $x \in M_1$, the set $\{(\varphi(x), \psi(y)) \mid y \in M_x\}$ is convex;*

ii) *for each $y \in M_2$, the set $\{(\varphi(x), \psi(y)) \mid x \in M_y\}$ is convex.*

Proof. i) Let be $x \in M_1$, y, $v \in M_x$, and $t \in [0,1]$. As $(x,y) \in M$, $(x,v) \in M$, we get

$$(\varphi(x), (1-t)\psi(y) + t\psi(v)) \in M, \text{ for each } t \in [0,1].$$

It follows that

$$((1-t)\varphi(x)+t\varphi(x), (1-t)\psi(y)+t\psi(v)) = (\varphi(x), (1-t)\psi(y)+t\psi(v)) \in M,$$

for each $t \in [0,1]$. Therefore, for each $x \in M_1$, the set $\{(\varphi(x), \psi(y)) \mid y \in \in M_x\}$ is convex.

In the same way we can prove ii). $\Diamond$

We remark two particular cases: $\varphi = 1_X$, and $\psi = 1_Y$.

Proposition 1.4. *Let $\varphi : X \to X$, and $\psi : Y \to Y$, be two functions, and let $T \subseteq \mathbb{R}$, $T \neq \emptyset$. If the set $M \subseteq X \times Y$ is bi-$(1_X, \psi)$ convex with respect to T and bi-$(\varphi, 1_Y)$ convex with respect to T, then M is bi-(φ, ψ) convex with respect to T.*

Proof. If $M = \emptyset$, then the conclusion is true. Let now $M \neq \emptyset$. Let (x,y), (x,v), $(u,y) \in M$ and $t \in T$. We have

$$(x, (1-t)\psi(y) + t\psi(v)) \in M, \text{ and } ((1-t)\varphi(x) + t\varphi(u), y) \in M.$$

Then, applying Prop. 1.1, because M is bi-$(\varphi, 1_Y)$ convex, we obtain

$$(\varphi(x), (1-t)\psi(y) + t\psi(v)) \in M,$$

and, because M is $(1_X, \psi)$ convex, we have

$$((1-t)\varphi(x) + t\varphi(u), \psi(y)) \in M.$$

Therefore, the set M is bi-(φ, ψ) convex with respect to T. $\Diamond$

The converse is not necessarily true as seen in the following

Example 1.4. Let $X = Y = \mathbb{R}$, $T = [0,1]$, $\varphi : \mathbb{R} \to \mathbb{R}$,

$$\varphi(x) = 1, \text{ for all } x \in \mathbb{R},$$

$\psi = 1_{\mathbb{R}}$, and

$$M = \{(1, \lambda) \mid \lambda \in [0,1]\} \bigcup \{(2,0),\ (2,1)\}.$$

For each (x,y), (x,v), $(u,y) \in M$, we have

$$x \in \{1,2\},\ u \in \{1,2\},\ y \in [0,1],\ v \in [0,1].$$

It follows that

$$(\varphi(x), (1-t)\psi(y)+t\psi(v)) = (1, (1-t)y+tv)) = (1,z) \in \{1\} \times [0,1] \subseteq M,$$

and

$$((1-t)\varphi(x) + t\varphi(u), \psi(y)) = (1,y)) \in \{1\} \times [0,1] \subseteq M.$$

Hence the set M is bi-(φ, ψ) convex with respect to T. But the set M is not bi-$(1_X, \psi)$ convex with respect to $T = [0,1]$, because $(2,0) \in M$, $(2,1) \in M$, and

$$\left(2, \frac{1}{2}1 + \frac{1}{2}0\right) \notin M.$$

Proposition 1.5. *Let $\varphi : X \to X$, and $\psi : Y \to Y$ be two functions, and let $T \subseteq \mathbb{R}$, $T \neq \emptyset$. If $M = A \times B \subseteq X \times Y$, then M is bi-(φ, ψ) convex with respect to T, if and only if M is bi-$(1_X, \psi)$ convex with respect to T and bi-$(\varphi, 1_Y)$ convex with respect to T.*

Proof. The sufficiency results from Prop. 1.4. Now, let $M = A \times B$ bi-(φ, ψ) convex with respect to T. Let (x, y), (x, v), $(u, y) \in M$ and $t \in T$. We have (1) and (2). Then

$$(1-t)\psi(y) + t\psi(v) \in B, \text{ and } (1-t)\varphi(x) + t\varphi(u) \in A.$$

From $(x, y) \in A \times B$, it results that

$$x \in A \text{ and } y \in B.$$

Therefore

$(x, (1-t)\psi(y)+t\psi(v)) \in A\times B = M$, $((1-t)\varphi(x)+t\varphi(u), y) \in A\times B = M$.

Hence $M = A \times B$ is bi-$(1_X, \psi)$ convex and bi-$(\varphi, 1_Y)$ convex with respect to T. ◊

Remark 1.6. Let X and Y be linear real spaces, $\varphi : X \to X$, and $\psi : Y \to Y$ two maps. We take

$$S = \{\{(x, y), (x, v), (u, y)\} \,|\, x, u \in X, \ y, v \in Y\} \subseteq 2^{X\times Y},$$

and we consider the functions $s : S \to 2^{X\times Y}$,

$$s(\{(x, y), (x, v), (u, y)\}) =$$
$$= \{(\varphi(x), (1-t)\psi(y) + t\psi(v)), ((1-t)\varphi(x) + t\varphi(u), \psi(y)) \,|\, t \in T\},$$

for each $\{(x, y), (x, v), (u, y)\} \in S$. It is easy to see that a subset M of $X \times Y$ is bi-(φ, ψ) convex if and only if

$$s(D) \subseteq M, \text{ for all } D = \{(x, y), (x, v), (u, y)\} \in S, \ D \subseteq M.$$

In view of [3], §8.3, the bi-(φ, ψ) convex convexity is a (S, s) convexity.

2. The connection with the *E*-convexity

The class of convex sets has been extended recently to the class of E-convex sets. We recall the Youness definition (see [7]): a set $M \subseteq \mathbb{R}^n$ is said to be E-convex iff there is a map $E : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$(1-\lambda)E(x) + \lambda E(y) \in M, \text{ for each } x, y \in M, \text{ and } 0 \leq \lambda \leq 1.$$

It is obvious that, according to this definition, each set $M \subseteq \mathbb{R}^n$ is E-convex (the empty set is evidently E-convex for any map $E : \mathbb{R}^n \to \mathbb{R}^n$, and, if $M \neq \emptyset$, and $m \in M$, then we can take the map E given by $E(x) = m$, for each $x \in \mathbb{R}^n$). Therefore, we shall restrict this concept to the following: let W be a linear real space, and $E : W \to W$ be a given map; we say that a subset M of W is E-convex if

$$(1-\lambda)E(x) + \lambda E(y) \in M, \text{ for each } x,\ y \in M, \text{ and } 0 \leq \lambda \leq 1.$$

Remark 2.1. Obviously a subset M of W is E-convex if and only if the set $E(M)$ is convex.

A natural question, which arises concerning to the concept of bi-(φ, ψ) convexity with respect to $[0,1]$ is the following: is this notion a new one or it is an E-convexity, with E a vectorial function. In this short section we shall prove that the E-convexity, where $E = (\varphi, \psi)$ is a vectorial function, is a particular case of bi-(φ, ψ)-convexity with respect to $T = [0,1]$.

Remark 2.2. Let $E = (E_1, E_2) : X \times Y \to X \times Y$ be a given function. If $M \subseteq X \times Y$ is $E = (E_1, E_2)$-convex, then M is bi-(E_1, E_2) convex with respect to $[0,1]$. The converse is not necessarily true, as seen in the following example.

Example 2.1. Let be $X = Y = \mathbb{R}$, and

$$M = \{(x,0) \,|\, x \in [0,1]\} \bigcup \{(1,y) \,|\, y \in [0,1]\}.$$

The set M is $(1,1)$ convex with respect to $T = [0,1]$, (see Ex. 1.2), but it is not E-convex with respect to $E = (1_{\mathbb{R}}, 1_{\mathbb{R}})$ (we have $(0,0) \in M$, $(1,1) \in M$, and $\frac{1}{2}(0,0) + \frac{1}{2}(1,1) = (\frac{1}{2}, \frac{1}{2}) \notin M$).

There are cases when the converse of Remark 2.2 is true.

Proposition 2.1. *If $M = A \times B \subseteq X \times Y$, then M is bi-(φ, ψ) convex with respect to $[0,1]$ if and only if it is E-convex, where $E = (\varphi, \psi)$).*

Proof. The sufficiency results from Remark 2.2. Let M be bi-(φ, ψ) convex with respect to $[0,1]$, and let $(x,y) \in M$, $(u,v) \in M$, $t \in [0,1]$. Then $(x,v) \in M$, and $(u,y) \in M$. By applying Prop. 1.5, we get that M is bi-$(1_X, \psi)$ convex with respect to $[0,1]$ and bi-$(\varphi, 1_Y)$ convex with respect to $[0,1]$. It follows that

$$(x, (1-t)\psi(y) + t\psi(v)) \in M,\ (u, (1-t)\psi(y) + t\psi(v)) \in M,$$

and so,

$$((1-t)\varphi(x) + t\varphi(u), (1-t)\psi(y) + t\psi(v)) \in M.$$

Therefore the set M is $E = (\varphi, \psi)$-convex. $\Diamond$

Corollary 2.1. *If $A \subseteq X$, $B \subseteq Y$, and the set $M = A \times B$ is bi-(φ, ψ) convex with respect to T and if $[0,1] \subseteq T$, then the set $\varphi(A) \times \psi(B)$ is convex.*

Proof. From Remark 2.1 and Prop. 2.1, we get that the set M is E-convex, where $E = (\varphi, \psi)$. In view of Remark 2.1, $E(M)$ is convex. But $E(M) = \varphi(A) \times \psi(B)$. Therefore the set $\varphi(A) \times \psi(B)$ is convex. $\diamond$

Remark 2.3. It is obvious that, when $M = A \times B$, there are bi-(φ, ψ) convexity with respect to T that cannot be reduced to E-convexity. For example, $T = \mathbb{Q}$(the set of rational numbers) is not isomorphic to $[0,1]$. Therefore, a bi-(φ, ψ) convexity with respect to $T = \mathbb{Q}$ is not equivalent to an E-convexity, for $E = (\varphi, \psi)$.

Using Remark 2.1 we can prove that the notion of E-convexity is a particular case of induced 2-strongly convexity. For this, we remember Defin. 6 of [2]. Let be U and V two arbitrary sets, Λ a nonvoid subset of V, $f : U \to V$ a function, and $s : 2^V \to 2^V$ a set valued mapping. A subset A of U is called 2-strongly convex with respect to s, f, and Λ if

$$s(\{f(a_1), f(a_2)\}) \bigcap V \subseteq f(A), \text{ for any } a_1, a_2 \in A.$$

If W is a linear real space and if we take $U = V = \Lambda = W$, and we suppose that the map $s : 2^V \to 2^V$ has the property that

$$s(\{v_1, v_2\}) = \{(1-t)v_1 + tv_2 \,|\, t \in [0,1]\}, \text{ for any } \{v_1, v_2\} \in 2^V,$$

it is easy to see that a set $M \subseteq W$ is 2-strongly convex with respect to s, f, and V if and only if the set $s(M)$ is convex. Therefore, from Remark 2.1 we get that a subset M of W is $E = f$ convex if and only if it is 2-strongly convex with respect to s, f, and V.

3. Properties of bi-(φ, ψ) convex sets with respect to T

Let X and Y be linear real spaces, $\varphi : X \to X$, and $\psi : Y \to Y$, two maps.

Theorem 3.1. *Let L and M be two subsets of $X \times Y$, and $T \subseteq \mathbb{R}$, $T \neq \emptyset$. If L and M are bi-(φ, ψ) convex sets with respect to T, then $L \cap M$ is a bi-(φ, ψ) convex set with respect to T.*

The proof is obvious and is omitted.

Remark 3.1. Let L and M be two subsets of $X \times Y$, and $T \subseteq \mathbb{R}$, $T \neq \emptyset$. If L and M are bi-(φ, ψ) convex sets with respect to T, then $L \bigcup M$ is not necessarily a bi-(φ, ψ) convex set, as seen in the following example.

Example 3.1. Consider the functions $\varphi : \mathbb{R} \to \mathbb{R}$, $\psi : \mathbb{R} \to \mathbb{R}$, defined by $\varphi(t) = \psi(t) = t$, for all $t \in \mathbb{R}$, and consider the sets $L = [1,2] \times [0,1]$

and $M = [1, 2] \times [-2, -1]$.

Obviously, the sets L and M are bi-(φ, ψ) convex with respect to $[0, 1]$, but $L \cup M$ is not bi-(φ, ψ) convex with respect to $[0, 1]$, because for $(x, y) = (1, 0)$, $(x, v) = (1, -1)$, $(u, y) = (2, 0)$ and $t = 1/2$ we have

$$(\varphi(x), (1-t)\varphi(y) + t\varphi(v)) = \left(1, -\frac{1}{2}\right) \notin M.$$

Theorem 3.2. *Let L, $M \subseteq X \times Y$, $\varphi : X \to X$, $\psi : Y \to Y$ be two additive functions, and $T \subseteq \mathbb{R}$, $T \neq \emptyset$. If L and M are bi-(φ, ψ) convex with respect to T, then $L + M$ is bi-(φ, ψ) convex with respect to T.*

Proof. Let (x, y), (x, v), $(u, y) \in L + M$ and $t \in T$. Then there exist (x_1, y_1), (x_1, v_1), $(u_1, y_1) \in L$ and (x_2, y_2), (x_2, v_2), $(u_2, y_2) \in M$ such that

$$(x, y) = (x_1, y_1) + (x_2, y_2), \qquad (x, v) = (x_1, v_1) + (x_2, v_2),$$
$$(u, y) = (u_1, y_1) + (u_2, y_2).$$

Since the set L is bi-(φ, ψ) convex with respect to T, we get that $(\varphi(x_1), (1-t)\psi(y_1) + t\psi(v_1)) \in L$, $((1-t)\varphi(x_1) + t\varphi(u_1), \psi(y_1)) \in L$.

Analogously, since the set M is bi-(φ, ψ) convex with respect to T, we have

$(\varphi(x_2), (1-t)\psi(y_2) + t\psi(v_2)) \in M$, $((1-t)\varphi(x_2) + t\varphi(u_2), \psi(y_2)) \in M$.

From here and from the fact that φ and ψ are additive, we deduce

$$(\varphi(x), (1-t)\psi(y) + t\psi(v)) =$$
$$= (\varphi(x_1 + x_2), (1-t)\psi(y_1 + y_2) + t\psi(v_1 + v_2)) =$$
$$= (\varphi(x_1) + \varphi(x_2), (1-t)[\psi(y_1) + \psi(y_2)] + t[\psi(v_1) + \psi(v_2)]) =$$
$$= (\varphi(x_1), (1-t)\psi(y_1) + t\psi(v_1)) + (\varphi(x_2), (1-t)\psi(y_2) + t\psi(v_2)) \in L + M$$

and analogously

$$((1-t)\varphi(x) + t\varphi(u), \psi(y)) = ((1-t)\varphi(x_1) + t\varphi(u_1), \psi(y_1)) +$$
$$+ ((1-t)\varphi(x_2) + t\varphi(u_2), \psi(y_2)) \in L + M.$$

Hence the set $L + M$ is bi-(φ, ψ) convex with respect to T. $\Diamond$

Theorem 3.3. *Let $M \subseteq X \times Y$, $a \in \mathbb{R}$ and $\varphi : X \to X$, $\psi : Y \to Y$ be two homogeneous functions and $T \subseteq \mathbb{R}$, $T \neq \emptyset$. If M is bi-(φ, ψ) convex with respect to T, then the set aM is bi-(φ, ψ) convex with respect to T.*

Proof. Let $(x, y), (x, v), (u, y) \in aM$ and $t \in T$. Then there exist $(\widetilde{x}, \widetilde{y}), (\widetilde{x}, \widetilde{v}), (\widetilde{u}, \widetilde{y}) \in M$ such that

$$(x, y) = a(\widetilde{x}, \widetilde{y}) = (a\widetilde{x}, a\widetilde{y}), \quad (x, v) = a(\widetilde{x}, \widetilde{v}) = (a\widetilde{x}, a\widetilde{v}),$$
$$(u, y) = a(\widetilde{u}, \widetilde{y}) = (a\widetilde{u}, a\widetilde{y}).$$

Because the set M is bi-(φ, ψ) convex with respect to T, we have $(\varphi(\widetilde{x}), (1-t)\psi(\widetilde{y}) + t\psi(\widetilde{v})) \in M$, $((1-t)\varphi(\widetilde{x}) + t\varphi(\widetilde{u}), \psi(\widetilde{y}) \in M$.

From here and from the fact that φ and ψ are homogeneous, we deduce

$$(\varphi(x), (1-t)\psi(y) + t\psi(v)) == (\varphi(a\widetilde{x}), (1-t)\psi(a\widetilde{y}) + t\psi(a\widetilde{v})) =$$
$$= (a\varphi(\widetilde{x}), (1-t)a\psi(\widetilde{y}) + ta\psi(\widetilde{v})) = a(\varphi(\widetilde{x}), (1-t)\psi(\widetilde{y}) + t\psi(\widetilde{v})) \in aM,$$

and analogously

$$((1-t)\varphi(x) + t\varphi(u), \psi(y)) == a((1-t)\varphi(\widetilde{x}) + t\varphi(\widetilde{u}), \psi(\widetilde{y})) \in aM.$$

Hence the set aM is bi-(φ, ψ) convex with respect to T. $\Diamond$

Theorem 3.4. *Let $M \subseteq X \times Y$, $\varphi, \alpha : X \to X$, $\psi, \beta : Y \to Y$, and $T \subseteq \mathbb{R}$, $T \neq \emptyset$. If M is bi-(φ, ψ) convex with respect to T and bi-(α, β) convex with respect to T, then M is bi-$(\varphi \circ \alpha, \psi \circ \beta)$ convex with respect to T and bi-$(\alpha \circ \varphi, \beta \circ \psi)$ convex with respect to T.*

Proof. Let $(x, y), (x, v), (u, y) \in M$ and $t \in T$. From Prop. 1.1 we have $(\varphi(x), \psi(y)), (\varphi(x), \psi(v)), (\varphi(u), \psi(y)) \in M$. Then, from the fact that M is bi-(α, β) convex with respect to T, we have

$$(\alpha(\varphi(x)), (1-t)\beta(\psi(y)) + t\beta(\psi(v))) \in M,$$

$$((1-t)\alpha(\varphi(x)) + t\alpha(\varphi(u)), \beta(\psi(y))) \in M,$$

and hence

$$((\alpha \circ \varphi)(x), (1-t)(\beta \circ \psi)(u) + t(\beta \circ \psi)(v)) \in M,$$

$$((1-t)(\alpha \circ \varphi)(x) + t(\alpha \circ \varphi)(u), (\beta \circ \psi)(y) \in M,$$

which means that the set M is bi-$(\alpha \circ \varphi, \beta \circ \psi)$ convex with respect to T.

Analogously, we prove that M is $(\varphi \circ \alpha, \psi \circ \beta)$ convex with respect to T. $\Diamond$

4. Other properties of bi-(φ, ψ) convex sets with respect to T with consequences for E-convex sets

Let now $(X, +, \cdot, || \cdot ||)$ and $(Y, +, \cdot, || \cdot ||$ be real Hilbert spaces.

Theorem 4.1. *Let $\varphi : X \to X$ and $\psi : Y \to Y$ be two continuous maps, and let $T \subseteq \mathbb{R}$, $T \neq \emptyset$.*

If the set $M = A \times B \subseteq X \times Y$ is bi-(φ, ψ) convex with respect to T, then the set $\operatorname{cl} M$, *is also bi-(φ, ψ) convex with respect to T.*

Proof. If $\operatorname{cl} M = \emptyset$, then the proposition is true. Suppose that $\operatorname{cl} M \neq \emptyset$, and let (x, y), (x, v), $(u, y) \in \operatorname{cl} M$. Then there are sequences $(x_j, y_j)_{j \in \mathbb{N}}$, $(x_j, v_j)_{j \in \mathbb{N}}$, $(u_j, y_j)_{j \in \mathbb{N}}$, with $(x_j, y_j) \in M$, $(x_j, v_j) \in M$, $(u_j, y_j) \in M$, for each integer number $j \geq 1$, such that

$$\lim_{j \to +\infty} (x_j, y_j) = (x, y), \quad \lim_{j \to +\infty} (x_j, v_j) = (x, v), \quad \lim_{j \to +\infty} (u_j, y_j) = (u, y).$$

Since the set M is bi-(φ, ψ) convex, for each $t \in T$ and $j \in \mathbb{N}$ we have

(4)

$$((1-t)\varphi(x_j) + t\varphi(u_j), \psi(y_j)) \in M, \text{ and } (\varphi(x_j), (1-t)\psi(y_j) + t\psi(v_j)) \in M.$$

From the continuity of the maps φ and ψ, it results

$$\lim_{j \to \infty} ((1-t)\varphi(x_j) + t\varphi(u_j), \psi(y_j)) = ((1-t)\varphi(x) + t\varphi(u), \psi(y)),$$

and

$$\lim_{j \to \infty} (\varphi(x_j), (1-t)\psi(y_j) + t\psi(v_j)) = (\varphi(x), (1-t)\psi(y) + t\psi(v)).$$

Hence

$$((1-t)\varphi(x) + t\varphi(u), \psi(y)) \in \operatorname{cl} M, \text{ and } (\varphi(x), (1-t)\psi(y) + t\psi(v)) \in \operatorname{cl} M,$$

for each $t \in T$. Therefore the set $\operatorname{cl} M$ is bi-(φ, ψ) convex with respect to T. ◊

Theorem 4.2. *Let $\varphi : X \to X$, $\psi : Y \to Y$ be two continuous maps, and let $T \subseteq \mathbb{R}$, $[0, 1] \subseteq T$. If the set $M = A \times B \subset X \times Y$ is bi-(φ, ψ) convex with respect to T, then the sets* $\operatorname{int}(\varphi(A) \times \psi(B))$, *and* $\operatorname{cl}(\varphi(A) \times \psi(B))$, *are convex.*

Proof. From Cor. 2.1 we get that the set $\varphi(A) \times \psi(B)$ is convex. Then, the sets $\operatorname{int}(\varphi(A) \times \psi(B))$, and $\operatorname{cl}(\varphi(A) \times \psi(B))$, are convex. ◊

Let $(W, +, \cdot, \|\cdot\|)$ be a real Hilbert space, V a nonempty subset of W, $w^0 \in W$. We remember that *a point $x^0 \in V$ is called an element of the best approximation of w^0 by elements of V if*

$$\| w^0 - x^0 \| \leq \| w^0 - x \|, \quad \textit{for all} \quad x \in V. \tag{5}$$

Theorem 4.3. *Let $(X, +, \cdot, \|\cdot\|)$ and $(Y, +, \cdot, \|\cdot\|)$ be two real Hilbert spaces, $\varphi : X \to X$, and $\psi : Y \to Y$, two maps, and $T \subseteq \mathbb{R}$, $[0, 1] \subseteq T$. If $M = A \times B \subseteq X \times Y$ is a bi-(φ, ψ) convex set with respect to T, and if w^0 is a given point of $X \times Y$, then there exists at most one element of the best approximation of w^0 by elements of $V = \varphi(A) \times \psi(B)$. If in addition the set V is closed, then there is one element of the best approximation of w^0 by elements of V, and only one.*

Proof. From Cor. 2.1 we get that the set $\varphi(A) \times \psi(B)$ is convex. Then there exists at most one element of the best approximation of w^0 by elements of $V = \varphi(A) \times \psi(B)$.

If V is closed, then there exists one element of the best approximation of w^0 by elements of V, and only one. $\diamondsuit$

In view of Prop. 2.1, Ths. 4.1–4.3 highlight new properties of E-convex sets.

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