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AFFINE COMPLETE ALGEBRAS GENERALIZING KLEENE AND STONE ALGEBRAS

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Abstract: In this paper we describe affine complete and locally affine complete members of the variety generated by all Kleene and Stone algebras. We also characterize local polynomial functions of the algebras of that variety. These results generalize the results earlier known for Kleene and Stone algebras.

1. Preliminaries

Let $\mathbf{A}$ be a universal algebra. A function $f : A^n \to A$ is called *compatible* if, for any congruence ρ of $\mathbf{A}$, $(a_i, b_i) \in \rho$, $i = 1, \dots, n$, implies

$$(f(a_1, \dots, a_n), f(b_1, \dots, b_n)) \in \rho\,.$$

An algebra $\mathbf{A}$ is called *affine complete* if every compatible function on $\mathbf{A}$ is a polynomial. Furthermore, an algebra $\mathbf{A}$ is said to be *locally affine complete*, if for every $n \geq 1$, every n-ary compatible function on $\mathbf{A}$ can be interpolated on any finite subset $F \subseteq A^n$ by a polynomial of $\mathbf{A}$.

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Originally, the problem of characterization of affine complete algebras was formulated in [3]. For various varieties of algebras affine completeness has already been investigated. In [4] affine completeness of algebras abstracting Kleene and Stone algebras was studied. In particular, it was shown there that a finite Kleene algebra is affine complete if and only if it is Boolean. In [6] a characterization of (locally) affine complete Stone algebras was presented. In [5] a description of local polynomial functions as well as a characterization of affine completeness and local affine completeness for Kleene algebras was given. In [7] an alternative approach to the affine completeness problems of Kleene algebras together with examples was presented. In [8] local polynomial functions of Stone algebras were described. The aim of this paper is to generalize these results to the variety generated by all Kleene and Stone algebras.

A distributive *Ockham algebra* is an algebra $\langle L; \vee, \wedge, ^*, 0, 1\rangle$, where $\langle L; \vee, \wedge, 0, 1\rangle$ is a bounded distributive lattice and * is a unary operation such that $0^* = 1$, $1^* = 0$ and for all $x, y \in L$,

$$(x \wedge y)^* = x^* \vee y^* \,, \tag{1}$$

$$(x \vee y)^* = x^* \wedge y^* \,. \tag{2}$$

More information about these algebras can be found in [1].

It is well known that Kleene and Stone algebras are distributive Ockham algebras. The variety $\mathcal{K} \vee \mathcal{S}$ generated by the class of all Kleene and Stone algebras is the subvariety of the variety of distributive Ockham algebras defined by the following additional identities:

$$x \leq x^{**} \,; \tag{3}$$

$$x \wedge x^* x^{**} \wedge x^* \,; \tag{4}$$

$$(x \wedge x^*) \vee y \vee y^* = y \vee y^* \,; \tag{5}$$

$$x \vee y^* \vee y^{**} \geq x^{**} \,. \tag{6}$$

It is known that the only subdirectly irreducible algebras in the variety $\mathcal{K} \vee \mathcal{S}$ are $\mathbf{K}_3 = \{0, a, 1\}$, $\mathbf{S}_3 = \{0, b, 1\}$ and $\mathbf{B}_2 = \{0, 1\}$ where $0 < a < 1$, $0 < b < 1$, $a^* = a$, $b^* = 0$. Thus, given an algebra $\mathbf{A} \in \mathcal{K} \vee \mathcal{S}$, we may write

$$\mathbf{A} \leq_{\text{s.d.}} \prod_{i \in I} \mathbf{A}_i \,,$$

where $\mathbf{A}_i \in \{\mathbf{B}_2, \mathbf{K}_3, \mathbf{S}_3\}$. The next lemma is a direct consequence from this subdirect decomposition. It will be crucial in the proofs of our main results because it reduces all problems to Kleene algebras and to functions with range contained in $\mathbf{A}^{\vee}$ defined below.

Lemma 1.1. *The following identity holds in the variety $\mathcal{K} \vee \mathcal{S}$:*

$$x = x^{**} \wedge (x \vee x^*)\,.$$

For every algebra $\mathbf{A} \in \mathcal{K} \vee \mathcal{S}$ we denote

$$\mathbf{A}^{\vee} = \{x \vee x^* : | : x \in A\}\,.$$

It is easy to prove that $\mathbf{A}^{\vee}$ is a filter of the lattice $\mathbf{A}$.

For any subset $H \subseteq A$ we define

$$H^{**} = \{x^{**} \mid x \in H\}\,.$$

Note that the set A^{**} is a subuniverse of $\mathbf{A}$; in fact $\mathbf{A}^{**}$ is a Kleene algebra. It is easy to observe that the operation ** is an idempotent endomorphism of $\mathbf{A}$ with range $\mathbf{A}^{**}$. The kernel Φ of this homomorphism is called the *Glivenko congruence* of $\mathbf{A}$ in the literature. Given an element $u \in A$ we denote by $[u]_{\Phi}$ the Φ-block containing u. Obviously $[u]_{\Phi}$ is a distributive lattice with greatest element u^{**}.

Lemma 1.2. *Let $\mathbf{A} \in \mathcal{K} \vee \mathcal{S}$. For every $x \in (A^{**})^{\vee}$ and $y \in A$ with $y \geq x$ we have $y \in A^{**}$.*

Proof. Let $x \in (A^{**})^{\vee}$ and $y \in A$ with $y \geq x$. Since $x \in A^{\vee}$, (2) and (5) imply $x^* \leq x$ and $x = x^{**}$ because of $x \in A^{**}$. Hence, by (3) we have $x^* \leq x^{**}$. Thus (6) implies $y^{**} \leq y \vee x^{**} = y \vee x = y$. Consequently $y = y^{**}$ and $y \in A^{**}$. $\Diamond$

Now we define an important binary relation on an arbitrary algebra $\mathbf{A} \in \mathcal{K} \vee \mathcal{S}$. This relation generalizes one which for Kleene algebras was introduced by M. Haviar, K. Kaarli and M. Ploščica in [5].

The *uncertainty order* of the algebra $\mathbf{A}$ is a binary relation $\sqsubseteq$, defined by

$$x \sqsubseteq y : \Leftrightarrow : x \wedge s \leq y \leq x \vee s^* \text{ for some } s \in A^{\vee}\,.$$

The next two lemmas list some properties of the uncertainty order relation. Originally similar results were proved and used in [5] for Kleene algebras, but one can easily generalize them to algebras from the variety $\mathcal{K} \vee \mathcal{S}$.

Lemma 1.3. *The restriction of the uncertainty order to $A^{\vee}$ coincides with the reverse order relation of the lattice $\mathbf{A}^{\vee}$.*

Lemma 1.4. *If $\mathbf{A} \leq_{\text{s.d.}} \prod_{i \in I} \mathbf{A}_i$ and $x, y \in A$ then $x \sqsubseteq y$ if and only if $x_i \sqsubseteq y_i$ for every $i \in I$.*

Now we recall necessary results on affine completeness of distributive lattices, Kleene algebras and Stone algebras. For that we need the notion of almost principal filter.

A filter F of a lattice $\mathbf{L}$ is said to be *principal* if it is of the form $\uparrow u = \{x \in L \mid x \geq u\}$, for some $u \in L$. We say that F is *almost principal* if its intersection with every principal filter of $\mathbf{L}$ is a principal filter of $\mathbf{L}$. Note that Lemma 1.2 implies that $(A^{**})^{\vee}$ is a filter of the lattice $\mathbf{A}^{\vee}$. Moreover, $(A^{**})^{\vee}$ is an almost principal filter of $\mathbf{A}^{\vee}$ with $\uparrow x \cap (A^{**})^{\vee} = \uparrow x^{**}$ for all $x \in A^{\vee}$. Any almost principal filter F of a lattice $\mathbf{L}$ defines a function $f_F : L \to L$ such that $\uparrow f_F(x) = \uparrow x \cap F$ for every $x \in L$. In particular, $f_{(A^{**})^{\vee}}(x) = x^{**}$ for all $x \in A^{\vee}$.

Theorem 1.5. ([2])

1. *A distributive lattice is locally affine complete if and only if it does not contain nontrivial Boolean intervals.*

2. *A function on a distributive lattice is a local polynomial if and only if it is compatible and order preserving.*

Theorem 1.6. ([6]) *Let D be a filter of a bounded distributive lattice $\mathbf{L}$. The lattice $\mathbf{D}$ is affine complete in $\mathbf{L}$, that is, every compatible function of the lattice $\mathbf{D}$ is a restriction of a polynomial function of the lattice $\mathbf{L}$, if and only if the following conditions are satisfied:*

1. *$\mathbf{D}$ has no nontrivial Boolean intervals;*

2. *if F is an almost principal filter of $\mathbf{D}$ then there exists $a \in L$ such that $F = \uparrow a \cap D$.*

Theorem 1.7. ([5], [6] and [8])

1. *A function on a Kleene (Stone) algebra $\mathbf{A}$ is a local polynomial function if and only if it preserves the congruences of $\mathbf{A}$ and the uncertainty order.*

2. *A Kleene (Stone) algebra $\mathbf{A}$ is locally affine complete if and only if the lattice $\mathbf{A}^{\vee}$ does not contain nontrivial Boolean intervals.*

3. *A Kleene (Stone) algebra $\mathbf{A}$ is affine complete if and only if it satisfies the following two conditions:*

(a) *the lattice $\mathbf{A}^{\vee}$ does not contain nontrivial Boolean intervals;*

(b) *for every almost principal filter F of the lattice $\mathbf{A}^{\vee}$, there exists $b \in A$ such that $F = \uparrow b \cap A^{\vee}$.*

We will use several times the following general lemma. Its proof can be found in [8].

Lemma 1.8. *Let $\mathbf{A}$ be an algebra and e be a unary idempotent compatible function on $\mathbf{A}$ such that $C = e(A)$ is a subuniverse of some reduct of $\mathbf{A}$. Then, if f is an n-ary compatible function of that subreduct then the function*

$$g(x_1, \dots, x_n) = f(e(x_1), \dots, e(x_n))$$

is a compatible function on $\mathbf{A}$ and extends f.

Throughout the paper we assume that $\mathbf{A}$ is an algebra of the variety $\mathcal{K} \vee \mathcal{S}$ and there is an embedding

$$\mathbf{A} \leq \prod_{i \in I} \mathbf{A}_i \text{ where } \mathbf{A}_i \in \{\mathbf{S}_3, \mathbf{K}_3\}, \tag{7}$$

for some index set I. We denote by $\pi_i : A \to A_i$ the projection map to the ith subdirect factor of $\mathbf{A}$. We write the elements of $\mathbf{A}$ in the form $x = (x_i)_{i \in I}$. It is not difficult to see that if $f : A^n \to A$ is a compatible function of $\mathbf{A}$ and $\mathbf{x}, \mathbf{y} \in A^n$ then $\mathbf{x}_i = \mathbf{y}_i$ implies $f(\mathbf{x})_i = f(\mathbf{y})_i$. This means that every compatible function f of $\mathbf{A}$ determines the coordinate functions f_i of $\pi_i(\mathbf{A})$ such that $f_i(\mathbf{x}_i) = f(\mathbf{x})_i$ for all $\mathbf{x} \in A^n$. Obviously, the family $(f_i)_{i \in I}$ completely determines f, so we may identify f with this family.

2. Local polynomial functions

In this section we describe local polynomials of algebras $\mathbf{A} \in \mathcal{K} \vee \mathcal{S}$.

Lemma 2.1. *Let f be a compatible function on $\mathbf{A}$ which preserves the uncertainty order. Then the restriction $g = f^{**}|_{A^{**}}$ is a compatible function of the Kleene algebra $\mathbf{A}^{**}$ which preserves the uncertainty order relation of $\mathbf{A}^{**}$.*

Proof. The fact that g is a compatible function of $\mathbf{A}^{**}$ follows easily from the observation that every congruence ρ of $\mathbf{A}^{**}$ is a restriction of a suitable congruence τ of $\mathbf{A}$. That congruence τ is defined by

$$(x, y) \in \tau \iff (x^{**}, y^{**}) \in \rho .$$

Let $\sqsubseteq$ be the uncertainty order of $\mathbf{A}$ and $\sqsubseteq_{**}$ be the uncertainty order of $\mathbf{A}^{**}$. Suppose that $\mathbf{x} = (x_1, \ldots, x_n), \mathbf{y} = (y_1, \ldots, y_n) \in (A^{**})^n$ and $\mathbf{x} \sqsubseteq_{**} \mathbf{y}$. Then, obviously, $\mathbf{x} \sqsubseteq \mathbf{y}$. Thus, since f preserves the uncertainty order of $\mathbf{A}$, we have $f(\mathbf{x}) \sqsubseteq f(\mathbf{y})$, that is,

$$f(\mathbf{x}) \wedge s \leq f(\mathbf{y}) \leq f(\mathbf{x}) \vee s^* \text{ for some } s \in A^{\vee} .$$

Hence

$$(f(\mathbf{x}) \wedge s)^{**} \leq f(\mathbf{y})^{**} \leq (f(\mathbf{x}) \vee s^*)^{**}$$

and

$$f(\mathbf{x})^{**} \wedge s^{**} \leq f(\mathbf{y})^{**} \leq f(\mathbf{x})^{**} \vee (s^{**})^* .$$

The latter implies $f(\mathbf{x})^{**} \sqsubseteq_{**} f(\mathbf{y})^{**}$ and thus $g(\mathbf{x}) \sqsubseteq_{**} g(\mathbf{y})$. ◊

Lemma 2.2. *Assume that f is a compatible function on $\mathbf{A}$ and there exists $s \in A^{\vee}$ such that f preserves $\uparrow s$. Then the restriction $f|_{\uparrow s}$ is a compatible function of the lattice $\uparrow s$.*

Proof. It is sufficient to prove that, for any congruence ρ of the lattice $\uparrow s$, there exists a congruence τ of the algebra $\mathbf{A}$ such that $\tau|_{\uparrow s} = \rho$.

We define an equivalence relation τ on A by

$$(x, y) \in \tau$$
$$\Updownarrow$$
$$(x \vee s, y \vee s) \in \rho \text{ and } (x^* \vee s, y^* \vee s) \in \rho \text{ and } (x^{**} \vee s, y^{**} \vee s) \in \rho.$$

It is easy to verify that τ is a congruence of the algebra $\mathbf{A}$ and obviously $\tau|_{\uparrow s} \subseteq \rho$. Assume that $x, y \in \uparrow s$ and $(x, y) \in \rho$. Then clearly

$$(x \vee s, y \vee s) = (x, y) \in \rho \quad \text{and} \quad (x^* \vee s, y^* \vee s) = (s, s) \in \rho.$$

Hence $(x, y) \in \tau$ as soon as we show that $(x^{**} \vee s, y^{**} \vee s) = (x^{**}, y^{**}) \in \rho$. It is easy to see that $x \vee (x \wedge y)^{**} \leq x^{**}$. By Lemma 1.2 we have $x \leq x \vee (x \wedge y)^{**} \in A^{**}$. Thus $x \vee (x \wedge y)^{**} \geq x^{**}$ implying $x^{**} = x \vee (x \wedge y)^{**}$. Now, obviously, $(x, y) \in \rho$ implies $(x^{**}, y^{**}) \in \rho$. $\Diamond$

Lemma 2.3. *If the lattice* $\mathbf{A}^{\vee}$ *does not contain nontrivial Boolean intervals then also the lattice* $(\mathbf{A}^{**})^{\vee}$ *does not contain nontrivial Boolean intervals.*

Proof. Assume that $(\mathbf{A}^{**})^{\vee}$ contains a nontrivial Boolean interval

$$L(u, v) = \{x \in A^{**} \mid u \leq x \leq v\}.$$

By Lemma 1.2 we have

$$\{x \in A^{**} \mid u \leq x \leq v\} = \{y \in A \mid u \leq y \leq v\}.$$

Thus $L(u, v)$ is also a Boolean interval in the lattice $\mathbf{A}^{\vee}$ and we have a contradiction. $\Diamond$

Consider the set $\mathcal{Q}$ of all ordered pairs $\alpha = (\alpha_1, \alpha_2)$, where α_1 and α_2 are disjoint subsets of the set $\underline{n} = \{1, \dots, n\}$. We assign to every $\alpha \in \mathcal{Q}$ the n-ary term

$$T^{\alpha}(\mathbf{x}) = \bigvee_{k \in \alpha_1} x_k^{**} \vee \bigvee_{k \in \alpha_2} x_k^* \vee \bigvee_{k \in \underline{n} \setminus (\alpha_1 \cup \alpha_2)} (x_k \vee x_k^*) .$$

Let $f : A^n \to A$ be a compatible function. For every $\alpha \in \mathcal{Q}$ we define a unary function $f_{\alpha} : A \to A$ by the rule

$$f_{\alpha}(y) = f(\mathbf{y}^{\alpha}), \quad \text{where } y_k^{\alpha} = \begin{cases} 0 & \text{if } k \in \alpha_1 ; \\ 1 & \text{if } k \in \alpha_2 ; \\ y & \text{otherwise} . \end{cases}$$

The following theorem describes local polynomial functions on algebras of the variety $\mathcal{K} \vee \mathcal{S}$.

Theorem 2.4. *A function on $\mathbf{A} \in \mathcal{K} \vee \mathcal{S}$ is a local polynomial function if and only if it preserves the congruences of* $\mathbf{A}$ *and the uncertainty order.*

Proof. The congruences of $\mathbf{A}$ as well as the uncertainty order are the subuniverses of $\mathbf{A}^2$ containing the diagonal. Thus they are preserved by all local polynomial functions of $\mathbf{A}$.

For the converse, let f be an n-ary compatible function on $\mathbf{A}$ which preserves the uncertainty order. We have to prove that f is a local polynomial. Let X be a finite subset of A^n. By Lemma 1.1 we have the identity

$$f(\mathbf{x}) = f(\mathbf{x})^{**} \wedge (f(\mathbf{x}) \vee f(\mathbf{x})^*) .$$

By Lemma 2.1 and Th. 1.7 the restriction $f^{**}|_{A^{**}}$ is a local polynomial function of the Kleene algebra $\mathbf{A}^{**}$. Thus there exists a polynomial function q_1 of the Kleene algebra $\mathbf{A}^{**}$ such that for any $\mathbf{x} \in X^{**}$ we have $q_1(\mathbf{x}) = f(\mathbf{x})^{**}$. Let $q(\mathbf{x}) = q_1(\mathbf{x}^{**})$. Since f preserves Φ, for every $\mathbf{x} \in X$

$$q(\mathbf{x}) = q_1(\mathbf{x}^{**}) = f(\mathbf{x}^{**})^{**} = f(\mathbf{x})^{**} .$$

It remains to show that also $h(\mathbf{x}) = f(\mathbf{x}) \vee f(\mathbf{x})^*$ is a local polynomial function. Because h preserves $A^\vee$, there exists $u \in A^\vee$ such that $h(X) \subseteq \uparrow u$. Let $r(\mathbf{x}) = h(\mathbf{x}) \vee u$; clearly the restrictions of h and r to X coincide. We are going to prove that r coincides with the polynomial function

$$p(\mathbf{x}) = u \vee \Big(\bigwedge_{\alpha \in \mathcal{Q}} (s_\alpha \vee T^\alpha(\mathbf{x})) \Big) ,$$

where $s_\alpha = r_\alpha(u) = r(\mathbf{u}^\alpha)$. In view of the embedding (7), it suffices to show that $p_i(\mathbf{x}_i) = r_i(\mathbf{x}_i)$ for every $\mathbf{x} \in A^n$ and every $i \in I$.

If $u_i = 1$ then both r_i and p_i are constantly 1, thus they are equal. Since the range of r is contained in $A^\vee$, the remaining possibility is $u_i \in \{a, b\}$. Now we have to distinguish between two cases.

1. $r_i(\mathbf{x}_i) = 1$. We have to prove that $(s_\alpha)_i \vee T^\alpha(\mathbf{x}_i) = 1$ for every $\alpha \in \mathcal{Q}$. Assume $T^\alpha(\mathbf{x}_i) \neq 1$, then

$$(x_k)_i \in \begin{cases} \{0, u_i = a\} & \text{if } k \in \alpha_1 \\ \{1, u_i\} & \text{if } k \in \alpha_2 \\ \{u_i\} & \text{otherwise.} \end{cases}$$

Note that $(x_k)_i = u_i = a$ implies $\pi_i(\mathbf{A}) = \mathbf{K}_3$. Let $\tilde{\mathbf{x}} = (\tilde{x}_1, \ldots, \tilde{x}_n)$, where

$$\tilde{x}_k = \begin{cases} x_k \wedge x_j^* & \text{if } k \in \alpha_1 \\ x_k \vee u & \text{if } k \in \alpha_2 \\ u & \text{otherwise.} \end{cases}$$

It is not difficult to see that $\mathbf{u}^\alpha \sqsubseteq \tilde{\mathbf{x}}$. Thus $s_\alpha \sqsubseteq r(\tilde{\mathbf{x}})$. It is also easy to see that $\tilde{\mathbf{x}}_i = \mathbf{x}_i$. Hence by Lemma 1.3 we have $(s_\alpha)_i \geq r_i(\tilde{\mathbf{x}}_i) = r_i(\mathbf{x}_i) = 1$, which proves that $(s_\alpha)_i \vee T^\alpha(\mathbf{x}_i) = 1$ for every $\alpha \in \mathcal{Q}$.

2. $r_i(\mathbf{x}_i) = u_i$. Let $\alpha_1 = \{k \mid (x_k)_i = 0\}$ and $\alpha_2 = \{k \mid (x_k)_i = 1\}$. Then $(s_\alpha)_i = r_i(\mathbf{x}_i) = u_i$ and $T^\alpha(\mathbf{x}_i) \leq u_i$. Hence $p_i(\mathbf{x}_i) = u_i$. Thus the polynomial $q \wedge p$ interpolates f on X. ◊

Having a characterization of local polynomial functions, it is not difficult to describe the locally affine complete algebras of the variety $\mathcal{K} \vee \mathcal{S}$.

Theorem 2.5. *An algebra $\mathbf{A} \in \mathcal{K} \vee \mathcal{S}$ is locally affine complete if and only if the lattice $\mathbf{A}^\vee$ does not contain nontrivial Boolean intervals.*

Proof. Suppose $\mathbf{A}^\vee$ contains a nontrivial Boolean interval. Then by Th. 1.5 the lattice $\mathbf{A}^\vee$ has a compatible function $g(x)$ which does not preserve the order relation. Define a function $f : A \to A$ by $f(x) = g(x \vee x^*)$. Obviously the restriction of f to $A^\vee$ coincides with g, and it follows from Lemma 1.8 that f is a compatible function of $\mathbf{A}$. Now Lemma 1.3 implies that f does not preserve the uncertainty order. Thus, by Th. 2.4, f is not a local polynomial function.

For the converse, suppose that $\mathbf{A}$ has a compatible function f which is not a local polynomial. By Th. 2.4 this means that f does not preserve the uncertainty order. Similarly to the case of distributive lattices (see [8], Th. 5.3.9) we may assume that f is unary. Using Lemma 1.1, we have $f = f^{**} \wedge (f \vee f^*)$.

Now we distinguish two cases.

1. f^{**} is not a local polynomial. Then also $f^{**}|_{\mathbf{A}}^{**}$ is not a local polynomial function of the Kleene algebra $\mathbf{A}^{**}$ though by the argument regarding the compatibility in the proof of Lemma 2.1 it is a compatible function of the Kleene algebra $\mathbf{A}^{**}$. Thus the Kleene algebra $\mathbf{A}^{**}$ is not locally affine complete. Now Th. 1.7 and Lemma 2.3 imply that $\mathbf{A}^\vee$ contains a nontrivial Boolean interval.

2. $h = f \vee f^*$ is not a local polynomial function and hence h does not preserve the uncertainty order of $\mathbf{A}$. Let $u, v \in A$ be such that $u \sqsubseteq v$ but $h(u) \not\sqsubseteq h(v)$. In view of the embedding (7) and Lemma 1.4 there must exist $i \in I$ such that h_i does not preserve the uncertainty order of $\mathbf{A}_i$. If $\mathbf{A}_i = \mathbf{K}_3$ then also $f_i = f_i^{**}$ cannot preserve the uncertainty

order of $\mathbf{A}_i$ and consequently also f^{**} is not a local polynomial function of $\mathbf{A}$, hence the result follows by the first case.

Assume that $\mathbf{A}_i = \mathbf{S}_3$. Then, due to $h(A) \subseteq A^\vee$ we must have $(u_i, v_i) = (1, b)$ and $(h_i(u_i), h_i(v_i)) = (b, 1)$. Let $s = v \vee v^*$. Then $s \in A^\vee$ and $s_i = v_i = b$. Now let $\widetilde{h}(x) = h(x) \vee s$. Since $s_i = b$, we have $\widetilde{h}_i(x_i) = h_i(x_i)$. Then $s, 1 \in \uparrow s$, $1 \geq s$, but $\widetilde{h}(1) \not\geq \widetilde{h}(s)$ because $h_i(1) = h_i(u_i) = b$, $h_i(s_i) = h_i(v_i) = 1$. It follows that the function $\widetilde{h}|_{\uparrow s}$ does not preserve the order relation of the lattice $\uparrow s$. Now Lemma 2.2 and Th. 1.5 yield that the lattice $\uparrow s$, but then also the filter $A^\vee$ has a nontrivial Boolean interval. $\Diamond$

3. Affine completeness

In this section we characterize the affine complete algebras of the variety $\mathcal{K} \vee \mathcal{S}$. Our main tools are the almost principal filters of $\mathbf{A}^\vee$ and related compatible functions.

Similarly to the case of distributive lattices (see [9]) we can prove the following:

Lemma 3.1. *Let f be a unary compatible function on an algebra $\mathbf{A}$ which preserves the uncertainty order and whose range is contained in $A^\vee$. Then*

1. *the set $F = \uparrow f(A^\vee)$ is an almost principal filter of $\mathbf{A}^\vee$;*
2. *$f_F(x) = f(x) \vee x$ for every $x \in A^\vee$;*
3. *$f(x) = f_F(x) \wedge f(1)$ for every $x \in A^\vee$.*

Proof. First we will show that

$$\uparrow f(A^\vee) \cap \uparrow u = \uparrow (u \vee f(u))$$

for every $u \in A^\vee$. This proves, in particular, the second claim of the lemma. Obviously $\uparrow (u \vee f(u)) \subseteq \uparrow f(A^\vee) \cap \uparrow u$. To prove the reverse inclusion, let $x \in \uparrow f(A^\vee) \cap \uparrow u$. Since $x \in \uparrow u$, we only need to show that $x \geq f(u)$. Now use the embedding (7) and show that $x_i \geq f_i(u_i)$ for every $i \in I$. The latter is trivial if $x_i = 1$ or $f_i(u_i) \in \{a, b\}$. Assume that $x_i \in \{a, b\}$ and $f_i(u_i) = 1$. Then $x \in \uparrow u$ implies that $x_i = u_i$. Thus, by Lemma 1.3 we have $f_i(\pi_i(A^\vee)) = \{1\}$. Consequently $x \in \uparrow f(A^\vee)$ implies $x_i = 1$, a contradiction. This proves $f_i(u_i) \in \{a, b\}$, a contradiction. Thus $x \geq f(u)$.

To finish the proof of the first claim of the lemma it remains to show that $\uparrow f(A^\vee)$ is closed with respect to meets. Let $x, y \in \uparrow f(A^\vee)$

and $u = (x \wedge y) \vee f(x \wedge y)$. Then by the first part of the proof $x, y \geq u$ and clearly $u \geq x \wedge y$, implying $x \wedge y = u \in \uparrow f(A^{\vee})$.

Finally, we need to show $f(x) = f_F(x) \wedge f(1)$ for every $x \in A^{\vee}$. We already know that $f(x) \leq f_F(x)$ and by Lemma 1.3 $f(x) \leq f(1)$. Hence $f(x) \leq f_F(x) \wedge f(1)$ and we need to prove the reverse inequality. Since $f_F(x) \wedge f(1) = f(x) \vee (x \wedge f(1))$, our proof will be complete as soon as we show that $f(x) \geq x \wedge f(1)$. Considering the embedding (7) we see that the inequality $f(x) \geq x \wedge f(1)$ fails only if there exists $i \in I$ such that $x_i = f_i(1) = 1$ and $f_i(x_i) < 1$ but this is impossible. $\Diamond$

Lemma 3.2. *If F is an almost principal filter of the lattice $\mathbf{A}^{\vee}$ then F^{**} is an almost principal filter of $\mathbf{A}^{\vee}$ and $(f_F)^{**} = f_{F^{**}}$. On the other hand, every almost principal filter of the lattice $(\mathbf{A}^{**})^{\vee}$ is an almost principal filter of the lattice $\mathbf{A}^{\vee}$.*

Proof. Since $F^{**} \subseteq F$, Lemma 1.2 implies that F^{**} is a filter of $\mathbf{A}^{\vee}$. Take any $s \in A^{\vee}$ and consider $\uparrow s \cap F^{**}$. Since F is almost principal, there exists $t \in A^{\vee}$ such that $\uparrow t = \uparrow s \cap F$. Now $t \in F$ implies $t^{**} \in F^{**}$. Thus $\uparrow t^{**} \subseteq \uparrow s \cap F^{**}$. On the other hand, $\uparrow s \cap F^{**} \subseteq \uparrow s \cap F$ implies that for every $x = x^{**} \in \uparrow s \cap F^{**}$ we have $x \geq t$ and hence $x \geq t^{**}$. Thus $\uparrow s \cap F^{**} \subseteq \uparrow t^{**}$. This proves that F^{**} is an almost principal filter of $\mathbf{A}^{\vee}$ as well as the equality $(f_F)^{**} = f_{F^{**}}$.

The second claim follows easily from Lemma 1.2. $\Diamond$

Lemma 3.3. *Let F be an almost principal filter of the lattice $\mathbf{A}^{\vee}$ and $g(x) = f_F(x \vee x^*)$. Then the following conditions are equivalent.*

(1) *g is a polynomial function of $\mathbf{A}$.*

(2) *There exist $k, s \in A$, $k \leq s$, such that*
   (a) $F^{**} = \uparrow k \cap (A^{**})^{\vee}$,
   (b) $F \cap [u]_{\Phi} = \uparrow (s \wedge u) \cap [u]_{\Phi}$ *for every* $u \in F^{**}$.

(3) *There exist $k, s \in A$, $k \leq s$, such that for every $x \in A^{\vee}$,*

$$g(x) = (s \vee x) \wedge (k \vee x^{**}).$$

(4) *There exist $k, s \in A$, $k \leq s$, such that:*
   (a) $g(x)^{**} = k \vee x^{**}$ *for every* $x \in A^{\vee}$,
   (b) $g(x) = (s \wedge u) \vee x$ *for every* $u \in F^{**}$ *and* $x \in [u]_{\Phi}$.

Note that the same elements $k, s \in A$ can be taken in the last three conditions.

Proof. We first prove the equivalence of the conditions (1) and (3). That (3) implies (1), is obvious. To prove the opposite implication, assume that g is a polynomial function of $\mathbf{A}$. Then there exist $k_i \in A$ such that the polynomial

$$(k_1 \vee x) \wedge (k_2 \vee x^*) \wedge (k_3 \vee x^{**}) \wedge (k_4 \vee x \vee x^*) \wedge (k_5 \vee x^* \vee x^{**}) \wedge k_6$$

coincides with $g(x)$ on A. In particular, if $x \in A^\vee$ we have

$$g(x) = ((k_1 \wedge k_4) \vee x) \wedge (k_2 \vee x^*) \wedge ((k_3 \wedge k_5) \vee x^{**}) \wedge k_6 .$$

Further, since $g(1) = 1$ we have $k_2 = k_6 = 1$. Thus for $x \in A^\vee$

$$g(x) = ((k_1 \wedge k_4) \vee x) \wedge ((k_3 \wedge k_5) \vee x^{**}) =$$
$$= ((k_1 \wedge k_4) \wedge ((k_1 \wedge k_4 \wedge k_3 \wedge k_5) \vee x^{**})) \vee x = (s \vee x) \wedge (k \vee x^{**})$$

where $s = k_1 \wedge k_4$ and $k = k_1 \wedge k_4 \wedge k_3 \wedge k_5$.

(3) $\Rightarrow$ (4). Condition (3) implies

$$g(x^{**}) = (s \vee x^{**}) \wedge (k \vee x^{**}) = (s \wedge k) \vee x^{**} = k \vee x^{**}$$

for any $x \in A^\vee$. Now $g(x)^{**} = g(x^{**})^{**} = (k \vee x^{**})^{**}$. Since by Lemma 1.2 $k \vee x^{**} \in F^{**}$, we have $g(x)^{**} = (k \vee x^{**})^{**} = k \vee x^{**}$ for every $x \in A^\vee$. If $u \in F^{**}$ then $g(u)^{**} = f_F(u)^{**} = u$. But on the other hand, $g(u)^{**} = k \vee u$ implying $k \leq u$. Now, if $x \in [u]_\Phi$, then

$$g(x) = (s \vee x) \wedge (k \vee x^{**}) = (s \vee x) \wedge (k \vee u) = (s \vee x) \wedge u = (s \wedge u) \vee x .$$

(4) $\Rightarrow$ (2). Given $u \in F^{**}$, by (4)(a) we have $u = u^{**} = k \vee u$, hence $F^{**} \subseteq \uparrow k \cap (A^{**})^\vee$. If $x \in \uparrow k \cap (A^{**})^\vee$ then again using (4)(a), we get

$$g(x)^{**} = k \vee x^{**} = x^{**} = x .$$

Hence $x \in F^{**}$ and we have proved the equality (2)(a).

To prove (2)(b), take $u \in F^{**}$ and $x \in F \cap [u]_\Phi$. Then by (4)(b) we have $x = g(x) = (s \wedge u) \vee x$, implying $x \in \uparrow (s \wedge u)$. This proves the inclusion $\subseteq$ in (2)(b). To show the opposite inclusion, take any $x \in \uparrow (s \wedge u) \cap [u]_\Phi$. Then $g(x) = (s \wedge u) \vee x = x$ implying $x \in F$.

(2) $\Rightarrow$ (3). Since

$$(s \vee x) \wedge (k \vee x^{**}) = k \vee x \vee (s \wedge x^{**}) ,$$

the equality $y = g(x) = (s \vee x) \wedge (k \vee x^{**})$ is equivalent to the following 5 inequalities: 1) $y \geq x$; 2) $y \leq k \vee x^{**}$; 3) $y \geq k$; 4) $y \geq s \wedge x^{**}$ and 5) $y \leq s \vee x$.

The first of these is obvious. By Lemma 3.2 we have the equality $(f_F)^{**} = f_{F^{**}}$. Hence, y^{**} is the least element $z \in F^{**}$ such that $x \leq z$. It follows from (2)(a) that $z = k \vee x^{**}$. But then $y \leq y^{**} = k \vee x^{**}$, proving the inequality 2). To prove 3), we observe that $y \in F \cap [u]_\Phi$ where $u = y^{**}$. Hence, using (2)(b) and the equality $u = k \vee x^{**}$, we have $y \geq s \wedge u \geq k \wedge u = k$. This proves 3). The inequality 4) follows from $y \geq s \wedge u$ and $u \geq x^{**}$. To prove 5), we first observe that $(k \vee \vee x)^{**} = y^{**}$, that is, $k \vee x \in [u]_\Phi$. The inequality $y \geq s \wedge u$ implies $s \wedge y \geq s \wedge u \geq s \wedge y$, thus $s \wedge y = s \wedge u$.

We now prove the equality $g(k \vee x) = (s \wedge u) \vee (k \vee x)$. Since

$$g(k \vee x)^{**} = k \vee (k \vee x)^{**} = k^{**} \vee x^{**} = (k \vee x^{**})^{**} = k \vee x^{**},$$

we have $g(k \vee x) \in F \cap [u]_\Phi$. Thus, by formula (2)(b) we have that $g(k \vee x) \geq (s \wedge u) \vee (k \vee x)$. On the other hand,

$$((s \wedge u) \vee (k \vee x))^{**} = (s^{**} \wedge u) \vee (k \vee x)^{**} = (s^{**} \wedge u) \vee u = u.$$

We see that $(s \wedge u) \vee (k \vee x) \in [u]_\Phi$. Thus formula (2)(b) implies $(s \wedge u) \vee \vee (k \vee x) \in F$. By the definition of g we have $g(k \vee x) \leq (s \wedge u) \vee (k \vee x)$.

It remains to calculate:

$$y = g(x) \leq g(k \vee x) = (s \wedge u) \vee (k \vee x) = (s \wedge y) \vee (k \vee x) =$$
$$= (s \vee k \vee x) \wedge (y \vee k \vee x) = (s \vee x) \wedge y \leq s \vee x.$$

This completes the proof of the lemma. ◊

We need one more lemma which allows us to prove that all compatible functions are polynomial provided all unary compatible functions are so.

Lemma 3.4. *Assume that f and g are n-ary compatible functions of* $\mathbf{A}$. *Then $f = g$ if and only if $f_\alpha(x) = g_\alpha(x)$ for every $\alpha \in \mathcal{Q}$ and every $x \in A^\vee$.*

Proof. Of course we only have to prove the sufficiency. Suppose there exists $\mathbf{v} \in A^n$ such that $f(\mathbf{v}) \neq g(\mathbf{v})$. Then, having in mind the embedding (7), $f_i(\mathbf{v}_i) \neq g_i(\mathbf{v}_i)$ for some $i \in I$. Now we show that $(f_\beta)_i \neq (g_\beta)_i$ where $\beta = (\beta_1, \beta_2)$ is defined by

$$\beta_1 = \{j \in \underline{n} \mid (v_j)_i = 0\} \quad \text{and} \quad \beta_2 = \{j \in \underline{n} \mid (v_j)_i = 1\}.$$

If $\beta_1 \cup \beta_2 = \underline{n}$ then $(f_\beta)_i$ and $(g_\beta)_i$ are the constant functions $f_i(\mathbf{v}_i)$ and $g_i(\mathbf{v}_i)$, respectively. Hence $f_i(\mathbf{v}_i) \neq g_i(\mathbf{v}_i)$ implies $f_\beta(1) \neq g_\beta(1)$.

Let now $\beta_1 \cup \beta_2 \neq \underline{n}$. It is easy to see that if $j_1, j_2 \notin \beta_1 \cup \beta_2$ then $(v_{j_1})_i = (v_{j_2})_i \in \{a, b\}$. Let $y = v_j \vee v_j^*$ for some $j \notin \beta_1 \cup \beta_2$. Then $y \in A^\vee$ and

$$f_i(\mathbf{v}_i) = (f_\beta)_i(y_i) \quad \text{and} \quad g_i(\mathbf{v}_i) = (g_\beta)_i(y_i).$$

This implies $f_\beta(y) \neq g_\beta(y)$ and we are done. ◊

Now we are ready to describe the affine complete algebras of the variety $\mathcal{K} \vee \mathcal{S}$.

Theorem 3.5. *An algebra $\mathbf{A} \in \mathcal{K} \vee \mathcal{S}$ is affine complete if and only if it satisfies the following two conditions:*

(1) *the lattice $\mathbf{A}^\vee$ does not contain nontrivial Boolean intervals;*

(2) *for every almost principal filter* F *of the lattice* $\mathbf{A}^{\vee}$ *there exist* $k, s \in A$, $k \leq s$, *such that*

(a) $F^{**} = \uparrow k \cap (A^{**})^{\vee}$,

(b) $F \cap [u]_{\Phi} = \uparrow (s \wedge u) \cap [u]_{\Phi}$ *for every* $u \in F^{**}$.

Proof. First assume that $\mathbf{A}$ is affine complete. Then $\mathbf{A}$ is locally affine complete and by Th. 2.5 the lattice $\mathbf{A}^{\vee}$ does not contain nontrivial Boolean intervals.

Assume that F is an almost principal filter of the lattice $\mathbf{A}^{\vee}$. The function f_F induced by the filter F is a compatible function of the lattice $\mathbf{A}^{\vee}$. By Lemma 1.8 the function $g(x) = f_F(x \vee x^*)$ is a compatible function of $\mathbf{A}$. Since $\mathbf{A}$ is affine complete, $g(x)$ is a polynomial function, which is by Lemma 3.3 equivalent to conditions (2).

In order to prove the sufficiency of the two conditions, assume that $\mathbf{A}$ is an algebra satisfying these conditions. It is easy to observe that then the Kleene algebra $\mathbf{A}^{**}$ is affine complete. Indeed, if F is an almost principal filter of the lattice $(\mathbf{A}^{**})^{\vee}$ then by Lemma 3.2 F is also an almost principal filter of the lattice $\mathbf{A}^{\vee}$, so by our assumption there is $k \in A$ such that $F = F^{**} = \uparrow k \cap (A^{**})^{\vee}$. Since $\uparrow k \cap (A^{**})^{\vee} = \uparrow k^{**} \cap (A^{**})^{\vee}$, we see that Th. 1.7 applies to show that $\mathbf{A}^{**}$ is affine complete.

Let now f be an arbitrary n-ary compatible function on $\mathbf{A}$. As in the proof of Th. 2.5, we have that $f^{**}|_{A^{**}}$ is a compatible function of the Kleene algebra $\mathbf{A}^{**}$. Since $\mathbf{A}^{**}$ is affine complete, there is a polynomial q of $\mathbf{A}^{**}$ such that $f(\mathbf{x})^{**} = q(\mathbf{x})$ for every $\mathbf{x} \in (A^{**})^n$. But then $f(\mathbf{x})^{**} = q(\mathbf{x}^{**})$ for every $\mathbf{x} \in A^n$, showing that $f(\mathbf{x})^{**}$ is a polynomial function of $\mathbf{A}$.

Now Lemma 1.1 implies that f is a polynomial function if and only if so is the function $h = f \vee f^*$. Note that the range of h is contained in $A^{\vee}$ and in view of the above proof we know that h^{**} is a polynomial function of $\mathbf{A}$.

For every $\alpha \in \mathcal{Q}$ let $F_\alpha = \uparrow h_\alpha(A^{\vee})$. By Lemma 3.1 F_α is an almost principal filter of the lattice $\mathbf{A}^{\vee}$, for every $\alpha \in \mathcal{Q}$. Thus there exist constants $k_\alpha, s_\alpha \in A$ satisfying conditions (a) and (b) of (2) for $F = F_\alpha$, for any $\alpha \in \mathcal{Q}$. Then by Lemmas 3.3 and 3.1 we have

$$h_\alpha(x) = h_\alpha(x)^{**} \wedge (s_\alpha \vee x) \wedge h_\alpha(1) \tag{8}$$

for every $x \in A^{\vee}$.

Note that the constants s_α are not uniquely determined, in general. In particular, if $\alpha = (\alpha_1, \alpha_2) \in \mathcal{Q}$ is such that $\alpha_1 \cup \alpha_2 = \underline{n}$ then the function h_α is the constant function and therefore we may choose $s_\alpha = h_\alpha(1)$. We introduce an order relation on $\mathcal{Q}$ by $\alpha \preccurlyeq \beta$ iff $\alpha_1 \subseteq \beta_1$ and $\alpha_2 \subseteq \beta_2$. Now we are going to show that the constants s_α can be chosen so that

$$\alpha \preccurlyeq \beta \text{ and } \alpha_1 = \beta_1 \implies s_\alpha \le s_\beta . \tag{9}$$

Obviously, condition (9) is satisfied for all α, β such that $|\alpha_1 \cup \alpha_2| = n$. Suppose it is satisfied for all α, β with $|\alpha_1 \cup \alpha_2| > j$. Then it suffices to show that for every α with $|\alpha_1 \cup \alpha_2| = j$, the constant s_α may be replaced by

$$r_\alpha = \bigwedge_{\alpha \preccurlyeq \beta,\ \alpha_1 = \beta_1} s_\beta .$$

Clearly $\uparrow(s_\alpha \wedge u) \cap [u]_\Phi \subseteq \uparrow(r_\alpha \wedge u) \cap [u]_\Phi$, for every $u \in F_\alpha^{**}$. Thus we have to show the opposite inclusion. Take an arbitrary element $x \in \uparrow(r_\alpha \wedge u) \cap [u]_\Phi$. Now $x \vee (s_\beta \wedge u) \in \uparrow(s_\beta \wedge u) \cap [u]_\Phi$ for every $\beta \in \mathcal{Q}$. Since h preserves the uncertainty order, $\alpha \preccurlyeq \beta$, $\alpha_1 = \beta_1$ implies $F_\beta \subseteq F_\alpha$ and hence

$$\uparrow(s_\beta \wedge u) \cap [u]_\Phi \subseteq \uparrow(s_\alpha \wedge u) \cap [u]_\Phi .$$

Consequently, $x \vee (s_\beta \wedge u) \in \uparrow(s_\alpha \wedge u) \cap [u]_\Phi$ for every $\beta \in \mathcal{Q}$ such that $\alpha \preccurlyeq \beta$, $\alpha_1 = \beta_1$. Since $\uparrow(s_\alpha \wedge u) \cap [u]_\Phi$ is a filter of the lattice $[u]_\Phi$, the meet $\bigwedge_{\alpha \preccurlyeq \beta,\ \alpha_1 = \beta_1}(x \vee (s_\beta \wedge u))$ is also a member of $\uparrow(s_\alpha \wedge u) \cap [u]_\Phi$. However, because of $x \ge r_\alpha \wedge u$ and the distributivity of $\mathbf{A}$, this meet is precisely x. Thus in what follows we may assume that the constants s_α satisfy the condition (9).

Next we will show that

$$\alpha \preccurlyeq \beta \implies h_\alpha(y)^{**} \le s_\beta \vee y^{**} \tag{10}$$

for every $y \in A^\vee$. Let $\gamma = (\alpha_1, \beta_2)$. Since h preserves the uncertainty order, it follows that $h_\alpha^{**} \le h_\gamma^{**}$. The equality (8) implies that we have $h_\beta(y)^{**} \vee y^{**} \le s_\beta \vee y^{**}$ for every $y \in A^\vee$.

So it remains to prove that

$$h_\gamma(y)^{**} \le h_\beta(y)^{**} \vee y^{**} \tag{11}$$

for every $y \in A^\vee$. Here we again need to use the embedding (7). Clearly (11) holds in Stone factors $\mathbf{A}_i = \mathbf{S}_3$ because $y_i^{**} = 1$ there, but in Kleene factors $\mathbf{A}_i = \mathbf{K}_3$ the inequality (11) takes the form

$$(h_\gamma)_i(y_i) \le (h_\beta)_i(y_i) \vee y_i , \tag{12}$$

where we may assume that $y_i = a$. Now observe that

$$(\mathbf{y} \wedge \mathbf{y}^*)^\beta \sqsubseteq (\mathbf{y} \wedge \mathbf{y}^*)^\gamma.$$

This follows because $\gamma_2 = \beta_2$ and $0 \sqsubseteq y \wedge y^*$ for every $y \in A^\vee$. Since h preserves the uncertainty order, this implies

$$h_\beta(y \wedge y^*) = h((\mathbf{y} \wedge \mathbf{y}^*)^\beta) \sqsubseteq h((\mathbf{y} \wedge \mathbf{y}^*)^\gamma) = h_\gamma(y \wedge y^*)\,.$$

Thus $h_\gamma(y \wedge y^*) \leq h_\beta(y \wedge y^*)$ implying $(h_\gamma)_i(a) \leq (h_\beta)_i(a)$. This proves (12) but then also (11) and (10).

To finish the proof we are going to show that the polynomial

$$p(\mathbf{x}) = h(\mathbf{x})^{**} \wedge \bigwedge_{\alpha \in \mathcal{Q}} (s_\alpha \vee T^\alpha(\mathbf{x}))$$

coincides with $h(\mathbf{x})$. Lemma 3.4 implies that we only need to show $p_\alpha(y) = h_\alpha(y)$ for any $\alpha \in \mathcal{Q}$ and any $y \in A^\vee$. Note that if for some $\beta \in \mathcal{Q}$ we have $\alpha \not\preccurlyeq \beta$ then $T^\beta(\mathbf{y}^\alpha) = 1$. Thus

$$p_\alpha(y) = h_\alpha(y)^{**} \wedge \bigwedge_{\beta \in \mathcal{Q},\ \alpha \preccurlyeq \beta} (s_\beta \vee T^\beta(\mathbf{y}^\alpha))\,.$$

Now, if $\alpha_1 \cup \alpha_2 = \underline{n}$ then $T^\beta(\mathbf{y}^\alpha) \neq 1$ only if $\beta = \alpha$. In this case $T^\alpha(\mathbf{y}^\alpha) = 0$, $s_\alpha = h_\alpha(y)$ and thus $p_\alpha(y) = h_\alpha(y)^{**} \wedge h_\alpha(y) = h_\alpha(y)$.

Assume that $\alpha_1 \cup \alpha_2 \neq \underline{n}$. Note that in this case $T^\alpha(\mathbf{x}) \in A^\vee$. Let

$$\mathcal{Q}_1 = \{\beta \in \mathcal{Q} \mid \beta_1 = \alpha_1 \text{ and } \alpha_2 \subseteq \beta_2 \subset \underline{n} \setminus \alpha_1\}\,,$$
$$\mathcal{Q}_2 = \{\beta \in \mathcal{Q} \mid \alpha_1 \subset \beta_1 \text{ and } \alpha_2 \subseteq \beta_2\}\,.$$

Then

$$p_\alpha(y) = h_\alpha(y)^{**} \wedge s_{(\alpha_1, \underline{n} \setminus \alpha_1)} \wedge \bigwedge_{\beta \in \mathcal{Q}_1} (s_\beta \vee y) \wedge \bigwedge_{\beta \in \mathcal{Q}_2} (s_\beta \vee y^{**})\,.$$

In view of conditions (9) and (10), the right hand side of the latter formula is $h_\alpha(y)^{**} \wedge (s_\alpha \vee y) \wedge h_{(\alpha_1, \underline{n} \setminus \alpha_1)}(1)$ which, by (8), is equal to $h_\alpha(x)$. $\Diamond$

Corollary 3.5.1. *Let* $\mathbf{A} \in \mathcal{K} \vee \mathcal{S}$. *If* $\mathbf{A}^\vee$ *is affine complete in the lattice* $\mathbf{A}$ *then* $\mathbf{A}$ *is affine complete.*

Proof. Let $\mathbf{A}^\vee$ be affine complete in the lattice $\mathbf{A}$. Th. 1.6 implies that $\mathbf{A}^\vee$ has no nontrivial Boolean intervals. Let F be an almost principal filter of $\mathbf{A}^\vee$. By Th. 1.6 there exists $k \in A$ such that $F = \uparrow k \cap A^\vee$. It is not difficult to see that then $F^{**} = \uparrow k \cap (A^{**})^\vee$.

Let $s = k$. We show that $F \cap [u]_\Phi = \uparrow (k \wedge u) \cap [u]_\Phi$, for every $u \in F^{**}$. Note that $F^{**} = \uparrow k \cap (A^{**})^\vee$ implies $u \geq k$. If $x \in F \cap [u]_\Phi$ then $x \geq k$ and $x^{**} = u$. Thus $x \in \uparrow (k \wedge u) \cap [u]_\Phi$ and $F \cap [u]_\Phi \subseteq$ $\subseteq \uparrow (k \wedge u) \cap [u]_\Phi$. To prove that also the opposite inclusion holds let

$x \in \uparrow(k \wedge u) \cap [u]_\Phi$. Then $x \geq k$ and $x \in A^\vee$ implies $x \in F$. Thus by Th. 3.5 the algebra $\mathbf{A}$ is affine complete. ◊

Corollary 3.5.2. *The direct product of a (locally) affine complete Kleene algebra and a (locally) affine complete Stone algebra is a (locally) affine complete algebra in the variety $\mathcal{K} \vee \mathcal{S}$.*

Proof. Let $\mathbf{A}$ be the direct product of a locally affine complete Kleene algebra $\mathbf{K}$ and a locally affine complete Stone algebra $\mathbf{S}$. Obviously, $A^\vee = K^\vee \times S^\vee$.

Assume that the lattice $\mathbf{A}^\vee$ contains a nontrivial Boolean interval $I(x,y)$ and let $x = (x_K, x_S)$, $y = (y_K, y_S)$. If $x^{**} = y^{**}$ then $x_K = y_K$ and $I(x_S, y_S)$ is a nontrivial Boolean interval in $S^\vee$. If $x^{**} \neq y^{**}$ then $x_K \neq y_K$ and $I(x_K, y_K)$ is a nontrivial Boolean interval in $K^\vee$. Thus the local affine completeness of $\mathbf{S}$ and $\mathbf{K}$ imply the local affine completeness of $\mathbf{A}$.

Assume now that $\mathbf{K}$ and $\mathbf{S}$ are affine complete. Let F be an almost principal filter of the lattice $\mathbf{A}^\vee$. Then $F_K = \{x \mid (x,1) \in F\}$ and $F_S = \{y \mid (1,y) \in F\}$ are almost principal filters of the lattices $K^\vee$ and $S^\vee$, respectively. Moreover, $F = F_K \times F_S$. Since $\mathbf{K}$ and $\mathbf{S}$ are affine complete, there exist $k \in K$, $s \in S$ such that $F_K = \uparrow k \cap K^\vee$ and $F_S = \uparrow s \cap S^\vee$. Then $(k,s) \in A$ and $F = \uparrow(k,s) \cap A^\vee$. Thus $\mathbf{A}^\vee$ is affine complete in the lattice $\mathbf{A}$ and by Cor. 3.5.1 the algebra $\mathbf{A}$ is affine complete. ◊

Example 1. *An affine complete algebra $\mathbf{A} \in \mathcal{K} \vee \mathcal{S}$ such that the lattice $\mathbf{A}^\vee$ is not affine complete in the lattice $\mathbf{A}$.*

Let $\mathbf{K}$ be an affine complete Kleene algebra such that $K = K^\vee \cup K^\wedge$ and $K^\vee$ does not have a smallest element. (For an example of such an algebra see [7], Ex. 1.) Let $\mathbf{S}$ be an affine complete Stone algebra such that $S^\vee$ is bounded, $|S^\vee| > 1$ and $S = S^\vee \cup \{0\}$. By Cor. 3.5.2 we know that the direct product $\mathbf{A} = \mathbf{K} \times \mathbf{S} \in \mathcal{K} \vee \mathcal{S}$ is affine complete.

Consider $B = (K^\vee \times S^\vee) \cup (K^\wedge \times S^\wedge)$. It is easy to check that $\mathbf{B}$ is a subalgebra of $\mathbf{A}$. Observe that

$$B^\vee = K^\vee \times S^\vee, \quad (B^{**})^\vee = \{(x,1) \mid x \in K^\vee\}$$

and for every $(u_K, u_S) \in B^\vee$ we have $[(u_K, u_S)]_\Phi = \{(u_K, y) \mid y \in S^\vee\}$. Since $B^\vee = A^\vee$, the algebra $\mathbf{B}$ is locally affine complete. To see that $\mathbf{B}$ is affine complete consider an almost principal filter F of $\mathbf{B}^\vee$. Again, as in the proof of Cor. 3.5.2, we have that $F = F_K \times F_S$, where $F_K = \{x \mid (x,1) \in F\}$ and $F_S = \{y \mid (1,y) \in F\}$ are almost principal filters of lattices $K^\vee$ and $S^\vee$, respectively. Since $\mathbf{S}^\vee$ is bounded, there exist $s \in S^\vee$ such that $F_S = \uparrow s$. Take any $u \in F^{**}$. Then $u = (u_K, 1)$, where $u_K \in F_K$ and

$F \cap [u]_\Phi = \{(u_K, x) \mid x \in F_S\} = \uparrow (u_K, s) \cap [u]_\Phi = \uparrow ((1, s) \wedge u) \cap [u]_\Phi$.

Further, since the algebra $\mathbf{K}$ is affine complete, there exists $k \in K$ such that $F_K = \uparrow k \cap K^\vee$. Note that there exists $y \in S$ such that $(k, y) \in B$. Now $F^{**} = \{(x, 1) \mid x \in F_K\} = \uparrow (k, y) \cap (A_1^{**})^\vee$. Thus $(k, y) \wedge (1, s)$ and $(1, s)$ satisfy the condition (2) of Th. 3.5 and hence $\mathbf{B}$ is affine complete.

Finally, assume that the lattice $\mathbf{B}^\vee$ is affine complete in the lattice $\mathbf{B}$. Then there exists $(x_K, x_S) \in B$ such that

$$(B^{**})^\vee = \uparrow (x_K, x_S) \cap B^\vee.$$

Now, since $|S^\vee| > 1$, we must have $x_S = 1$ and thus $(x_K, x_S) \in B^\vee$, implying $x_K \in K^\vee$. Hence x_K is the smallest element of $K^\vee$, a contradiction. Thus the lattice $\mathbf{B}^\vee$ is not affine complete in the lattice $\mathbf{B}$.

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References

[1] BLYTH, T. S., VARLET, J. C.: Ockham Algebras, Oxford University Press, 1994.

[2] DORNINGER, D., EIGENTHALER, G.: On compatible and order-preserving functions on lattices, Universal Algebra and Appl. **9** (1982), Banach Center Publ., Warsaw, 97–104.

[3] GRÄTZER, G.: Universal Algebra, Van Nostrand, Toronto–London–Melbourne, 1968.

[4] HAVIAR, M.: Affine complete algebras abstracting Kleene and Stone algebras, *Acta Math. Univ. Comenianae* **2** (1993), 179–190.

[5] HAVIAR, M., KAARLI, K., PLOŠČICA, M.: Affine completeness of Kleene algebras, *Algebra Universalis* **37** (1997), 477–490.

[6] HAVIAR, M., PLOŠČICA, M.: Affine complete Stone algebras, *Algebra Universalis* **34** (1995), 355–365.

[7] HAVIAR, M., PLOŠČICA, M.: Affine complete Kleene algebras II, *Acta Univ. M. Belii* **5** (1997), 51–61.

[8] KAARLI K.. PIXLEY A. F.: Polynomial Completeness in Algebraic Systems, Chapman & Hall/CRC, 2000.

[9] PLOŠČICA, M., Affine complete distributive lattices, *Order* **11** (1994), 385–390.