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A NEW KUROSH–AMITSUR RADICAL THEORY FOR PROPER SEMIFIELDS II.

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Abstract: We continue Part I of this paper (Mathematica Pannonica, **14/1** (2003), 5–28) with unified numbering of sections and references. In Part I, consisting of the Sections 1–8, a new Kurosh–Amitsur radical theory for proper semifields has been developed. Like a former theory presented in [12], also this new theory deals with subsemifields and with groups as kernels of proper semifields, but restricts the use of groups to necessity. In Section 9 we present the old theory as a special case of the new one including some improvements and a solution of a problem posed in [12]. In the final Section 10 we deal with the question whether semisimple classes are hereditary. The answer is negative in the new theory and it remains open for the old one.

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9. Comparing both radical theories

At first we repeat the basic definitions of the redical theory given in [12], where the axioms (Ra), (Rb), (Rc), (Sa), (Sb), (ϱb), (ϱc) and (ϱd) have already been introduced in Sections 4–7.

Definition 9.1. The frame of all the following considerations is a *universal class* $\mathfrak{H}$ of $\mathfrak{S}^* \cup \mathfrak{G}$ as already defined in Section 3, that is, an n-universal class $\mathfrak{H}$ which satisfies also

III) $(A,+,\cdot) \in \mathfrak{H} \Rightarrow (A,\cdot) \in \mathfrak{H}$ for all $(A,+,\cdot) \in \mathfrak{S}^*$.

a) A subclass $\mathbb{R}$ of $\mathfrak{H}$ is called a *radical class of* $\mathfrak{H}$ if $\mathbb{R}$ satisfies (Ra), (Rb), (Rc) and the following axiom

(Rk) For all $(A,+,\cdot) \in \mathfrak{H} \cap \mathfrak{S}^*$, if $|\mathfrak{K}(A,+,\cdot) \cap \mathbb{R}| = 1$ then $|\mathfrak{K}(A,\cdot) \cap \mathbb{R}| = 1$.

b) An operator ϱ which assigns to each $A \in \mathfrak{H}$ a kernel $\varrho A \in \mathfrak{K}(A)$ is called a *radical operator in* $\mathfrak{H}$ if it satisfies, for all $A \in \mathfrak{H}$, the axioms (ϱb), (ϱc), (ϱd) and the following one:

(ϱa) $\varphi(\varrho A) \subseteq B$ holds for all surjective morphisms $\varphi : A \to B$ of the types 1), 2) and 3).

c) A subclass $\mathbb{S}$ of $\mathfrak{H}$ is called a *semisimple class* of $\mathfrak{H}$, if $\mathbb{S}$ satisfies (Sa), (Sb) and the following axiom

(Sc) For all $(A,+,\cdot) \in \mathfrak{H} \cap \mathfrak{S}^*$, if $(A,+,\cdot) \in \mathbb{S}$ then $(A,\cdot) \in \mathbb{S}$.

Remark 9.2. A glance to Def. 9.1 suggests the following question posed as a problem in [12]. Is a subclass $\mathbb{R}$ of $\mathfrak{H}$ which satisfies (Ra), (Rb) and (Rc), called a *weak radical calss* in that paper, already a radical class of $\mathfrak{H}$? (In other words: is an n-radical class $\mathbb{R}$ of a universal class $\mathfrak{H}$ already a radical class of $\mathfrak{H}$?) Here we solve this problem in the negative by a counterexample at the end of this section. In fact, it is just the other way around, and we state as a useful improvement of [12] the following

Proposition 9.3. *In the definition of a radical class $\mathbb{R}$ of a universal class $\mathfrak{H}$, condition (Rc) is a consequence of (Ra), (Rb) and (Rk), and hence superfluous.*

Proof. Suppose $(A,\cdot) \in \mathbb{R}$ for some $(A,+,\cdot) \in \mathfrak{H} \cap \mathfrak{S}^*$. To show $(A,+,\cdot) \in \mathbb{R}$ by (Ra), we use that every homomorphism $(A,+,\cdot) \twoheadrightarrow (B,+,\cdot)$ induces a homomorphism $(A,\cdot) \twoheadrightarrow (B,\cdot)$. Applying (Rb) to $(A,\cdot) \in \mathbb{R}$, we get a kernel $C \triangleleft (B,\cdot)$ which satisfies $C \in \mathbb{R}$, that is, $|\mathfrak{K}(B,\cdot) \cap \mathbb{R}| \neq 1$. Now (Rk) yields $|\mathfrak{K}(B,+,\cdot) \cap \mathbb{R}| \neq 1$. This shows the existence of a kernel $D \triangleleft (B,+,\cdot)$ satisfying $D \in \mathbb{R}$ and thus $(A,+,\cdot) \in \mathbb{R}$

by (Ra). $\Diamond$

Our next point is to show that, only *with the exception of* Remark 7.4, *all of our statements on n-radical classes, n-radical operators and n-semisimple classes are true for the concepts introduced in* Def. 9.1, which establishes again various results obtained in [12]. By the following lemma, this is obvious for every statement in which such concepts occur only in the assumptions and not in the conclusions, as for instance in Prop. 5.5 and Th. 5.7.

Lemma 9.4. a) *Each radical class* $\mathbb{R}$ *of a universal class* $\mathfrak{H}$ *is an n-radical class of* $\mathfrak{H}$ *which satisfies* (Rk), *and conversely.*

b) *Each semisimple class* $\mathbb{S}$ *of* $\mathfrak{H}$ *is an n-semisimple class of* $\mathfrak{H}$ *which satisfies* (Sc), *and conversely.*

c) *Each radical operator* ϱ *in* $\mathfrak{H}$ *is an n-radical operator of* $\mathfrak{H}$ *which satisfies*

$$(9.1) \qquad \varrho(A,\cdot) \subseteq \varrho(A,+,\cdot) \text{ for all } (A,+,\cdot) \in \mathfrak{H},$$

and conversely.

Proof. Comparing Def. 9.1 with Defs. 5.1 and 7.1, one obtains a) and b), the latter because (Sc) implies obviously $(S\gamma)$. To show c), let ϱ be a radical operator and $\varphi : (A,\cdot) \to (B,+,\cdot)$ a surjective morphism of type 3. Then (ϱa) states $\varphi(\varrho(A,\cdot)) \subseteq \varrho(B,+,\cdot)$. Applying this to the morphism $\varphi : (A,\cdot) \to (A,+,\cdot)$ given by the identical mapping on A, we obtain (9.1). The latter yields $(\varrho\omega)$ since $\varrho(A,\cdot) = (A,\cdot)$ and $\varrho(A,\cdot) \subseteq \varrho(A,+,\cdot)$ by (9.1) imply $\varrho(A,+,\cdot) = (A,+,\cdot)$. Hence each radical operator ϱ satisfies (9.1) and is an n-radical operator. Conversely, assume (9.1) for an n-radical operator. We have to show that ϱ satisfies (ϱa) for each surjective morphism $\varphi : (A,\cdot) \to (B,+,\cdot)$ of type 3. For this end we apply $(\varrho\alpha)$ for ϱ to the surjective morphism $\varphi : (A,\cdot) \to (B,\cdot)$ of type 2, and we obtain $\varphi(\varrho(A,\cdot)) \subseteq \varrho(B,\cdot)$, whereas $\varrho(B,\cdot) \subseteq \varrho(B,+,\cdot)$ holds by (9.1). $\Diamond$

Now, we show that the bijective correspondence between n-radical classes, n-radical operators and n-semisimple classes transfers to the concepts of Def. 9.1.

Proposition 9.5. a) *For every semisimple class* $\mathbb{S}$ *of a universal class* $\mathfrak{H}$, *the n-radical class* $\mathbb{R} = \mathcal{U}\mathbb{S}$ *determined by* $\mathbb{S}$ *is a radical class of* $\mathfrak{H}$.

b) *For every radical class* $\mathbb{R}$ *of* $\mathfrak{H}$, *the n-radical operator* $\varrho_{\mathbb{R}}$ *determined by* $\mathbb{R}$ *is a radical operator of* $\mathfrak{H}$.

c) *For every radical operator* ϱ *in* $\mathfrak{H}$, *the n-semisimple class* $\mathbb{S}$ *determined by* ϱ *according to* $\mathbb{S} = \{A \in \mathfrak{H} \mid \varrho A \in \mathfrak{T}\}$ *is a semisimple* *class of* $\mathfrak{H}$.

Proof. a) Considering $\mathbb{S}$ and $\mathbb{R} = \mathcal{U}\mathbb{S}$ as n-semisimple and n-radical classes of $\mathfrak{H}$, we have $|\mathfrak{K}(A) \cap \mathbb{R}| = 1 \Leftrightarrow A \in \mathbb{S}$ for each $A \in \mathfrak{H}$. Hence $(A,+,\cdot) \in \mathbb{S} \Rightarrow (A,\cdot) \in \mathbb{S}$ by (Sc) for $\mathbb{S}$ yields $|\mathfrak{K}(A,+,\cdot) \cap \mathbb{R}| = 1 \Rightarrow$ $\Rightarrow |\mathfrak{K}(A,\cdot) \cap \mathbb{R}| = 1$, that is, (Rk) for $\mathbb{R}$. So $\mathbb{R}$ is a radical class by Lemma 9.4 a).

b) We have to show that $\varrho_{\mathbb{R}}$ satisfies (9.1). By way of contradiction, we assume $\varrho_{\mathbb{R}}(A,\cdot) \not\subseteq \varrho_{\mathbb{R}}(A,+,\cdot)$ for some $(A,+,\cdot) \in \mathfrak{H}$ and use at first only properties which $\mathbb{R}$ shares with each n-radical class. The homomorphism $(A,+,\cdot) \to (A,+,\cdot)/\varrho_{\mathbb{R}}(A,+,\cdot) = (B,+,\cdot)$ maps $(A,\cdot)$ onto $(B,\cdot)$, and so $(K,\cdot) = \varrho_{\mathbb{R}}(A,\cdot) \in \mathbb{R}$ is mapped onto a subgroup $(C,\cdot) \in \mathbb{R}$ of $(B,\cdot)$. Further, $(K,\cdot) \not\subseteq \varrho_{\mathbb{R}}(A,+,\cdot)$, as assumed, yields $(C,\cdot) \notin \mathfrak{T}$, and so $|\mathfrak{K}(B,\cdot) \cap \mathbb{R}| \neq 1$. However, (ϱb) for $\varrho_{\mathbb{R}}$ yields $\varrho_{\mathbb{R}}(B,+,\cdot) \in \mathfrak{T}$, that is, $|\mathfrak{K}(B,+,\cdot) \cap \mathbb{R}| = 1$. This contradicts the assumption that $\mathbb{R}$ is a radical class which satisfies (Rk).

c) By Lemma 9.4 c), the radical operator ϱ satisfies (9.1). This yields, in particular, $\varrho(A,+,\cdot) \in \mathfrak{T} \Rightarrow \varrho(A,\cdot) \in \mathfrak{T}$, and so $(A,+,\cdot) \in$ $\in \mathbb{S} \Rightarrow (A,\cdot) \in \mathbb{S}$ for every $(A,+,\cdot) \in \mathfrak{H}$, that is, (Sc) for $\mathbb{S}$. Hence $\mathbb{S}$ is a semisimple class. $\Diamond$

Proposition 9.6. a) *Each of the characterizations* i)–iv) *of n-radical classes given in* Th. 5.9 *yields a corresponding characterization of radical classes of a universal class $\mathfrak{H}$ if one replaces (Rc) by (Rk), and the same holds for* Prop. 6.3 b*)*.

b) *Similarly, each characterization* i)–v) *of n-semisimple classes in* Th. 7.9 *yields a characterization of semisimple classes of $\mathfrak{H}$ if one replaces $(S\gamma)$ by (Sc) in* i)–iv) *and adds (Sc) in* v)*. A further characterization of semisimple classes can be obtained from* v) *if one writes "$\mathcal{U}\mathbb{S}$ is a radical class" in place of "$\mathcal{U}\mathbb{S}$ is an n-radical class".*

c) *Also* Th. 8.3 *characterizes subclasses $\mathbb{R}$ and $\mathbb{S}$ of a universal class $\mathfrak{H}$ as a radical class and the corresponding semisimple class if one formulates* e) *with (Rk) and (Sc).*

Proof. a) If a set of properties characterizes an n-radical class, the same set and (Rk) characterize a radical class $\mathbb{R}$ of $\mathfrak{H}$ by Lemma 9.4 a). In the set (Ra), (Rb), (Rc) and (Rk) obtained from i) in this way, (Rc) is superfluous since (Ra), (Rb) and (Rk) imply (Rc) by Prop. 9.3. The same applies to the sets obtained from ii)–iv), since each of these sets implies (Ra) and (Rb) without making use of (Rc) (cf. Remark 5.10). The last statement is obvious by Prop. 6.3 a).

b) Again, one obtains characterizations of semisimple classes of $\mathfrak{H}$ if one adds (Sc) to i)–v) (or to the characterizations obtained from i)–v) by a)–d) in Th. 7.9). Since (Sc) implies $(S\gamma)$, the latter is superfluous everywhere it occurs. Only for the last assertion of b) one has to consult the proof of v)$\Rightarrow$i) in Th. 7.9 up to the statement that $\mathcal{SUS} = \mathbb{S}$ holds for $\mathcal{U}\mathbb{S}$, assumed to be an n-radical class in v). Strengthening the latter to a radical class, $\mathcal{SUS} = \mathbb{S}$ yields that $\mathbb{S}$ is even a semisimple class.

c) The properties a)–d) in Th. 8.3 imply (Ra) and (Rb) for $\mathbb{R}$ and (Sa) and (Sb) for $\mathbb{S} = \mathcal{S}\mathbb{R}$. Hence $\mathbb{R}$ becomes a radical class of $\mathfrak{H}$ and $\mathbb{S}$ a semisimple class if one adds (Rk) for $\mathbb{R}$ or (Sc) for $\mathbb{S}$ instead of e). ◊

Remark 9.7. a) Comparing the results of this section with those of the previous ones, we can state the following. *Each of our statements on n-radical classes, n-semisimple classes and n-radical operators of an n-universal class yields, apart from one exception, a corresponding statement on radical classes, semisimple classes and radical operators in a universal class $\mathfrak{H} \subseteq \mathfrak{S}^* \cup \mathfrak{G}$ if one likewise replaces (Rc) by (Rk) and $(S\gamma)$ by (Sc) and if one adds (ϱa) or (9.1) to the concept of an n-radical operator.*

b) The exception of this general rule is Remark 7.4, where we could not decide whether the corresponding statement is valid. So we formulate it as a problem:

Let $\mathbb{M}$ be a subclass of a universal class $\mathfrak{H}$ such that $\mathbb{M}$ satisfies (Sb) and (Sc). Is then $\mathbb{R} = \mathcal{U}\mathbb{M}$ a radical class of $\mathfrak{H}$?

Clearly, $\mathbb{R} = \mathcal{U}\mathbb{M}$ is a radical class if one assumes also (Sa) for $\mathbb{M}$ (cf. Prop. 9.5 a)). Otherwise, we know only by Remark 7.4 that $\mathbb{R} = \mathcal{U}\mathbb{M}$ is an n-radical class of $\mathfrak{H}$, and that $\mathbb{R}' \subseteq \mathbb{R}$ holds even for every n-radical class $\mathbb{R}'$ of $\mathfrak{H}$ which satisfies $\mathbb{R}' \cap \mathbb{M} \subseteq \mathfrak{T}$.

Finally, as announced in Remark 9.2, we solve a problem posed in [12].

Theorem 9.8. *Let $\mathbb{R}$ be a subclass of a universal class $\mathfrak{H}$ which satisfies (Ra), (Rb) and (Rc), that is, a weak radical class or an n-radical class of $\mathfrak{H}$. Then, by the following example, $\mathbb{R}$ need not be a radical class of $\mathfrak{H}$.*

Example 9.9. a) The key of this example is a semifield $(A, +, \cdot)$ constructed from two simple idempotent semifields. The first one is the semifield $(B, +, \cdot)$ considered in Ex. 2.9 b) which can be obvoiusly linearly ordered by $b^i \leq b^j \Leftrightarrow i \leq j$. The second one is any semifield $(G, +, \cdot) \in \mathfrak{S}^{\mathrm{idp}}$ such that $(G, \cdot)$ is a simple non-commutative group.

(For each group $(G,\cdot)$ one obtains a semifield $(G,+,\cdot) \in \mathfrak{S}^{\mathrm{idp}}$ by defining $g+h = g$ for all $g,h \in G$. This addition is non-commutative, but there are also additively commutative and idempotent semifields $(G,+,\cdot)$ for suitable simple non-commutative groups $(G,\cdot)$, see Ex. 10.15.) On the direct product $(A,\cdot) = (B,\cdot) \times (G,\cdot)$ we define an addition by

$$(9.2) \qquad (b^i,g) + (b^j,h) = \begin{cases} (b^i,g) & \text{for} \quad i > j \\ (b^i,g+h) & \text{for} \quad i = j \\ (b^j,h) & \text{for} \quad i < j. \end{cases}$$

As one easily checks, in this way we obtained a semifield $(A,+,\cdot)$ which is additively commutative iff $(G,+,\cdot)$ is so. Moreover, $(A,+,\cdot)$ has exactly three kernels

$$(9.3) \quad \{(b^0,e)\} \in \mathfrak{T}, \quad A, \text{ and } L = (\{(e,g) \mid g \in G\},+,\cdot) \cong (G,+,\cdot).$$

The construction of $(A,+,\cdot)$ and the statement about its kernels follow from the more general considerations in [13, Ex. 3.4 amd Th. 3.5]. For the sake of completeness, we give a short direct proof of the latter. First of all, $(b^i,g) \mapsto b^i$ defines a surjective homomorphism $\varphi : (A,+,\cdot) \twoheadrightarrow (B,+,\cdot)$ with kernel L. Now, let K be any kernel of A. Since $(L,+,\cdot)$ is simple, $K \subseteq L$ implies $K \in \mathfrak{T}$ or $K = L$. Otherwise, K contains an element (b^i,g) for some $i \neq 0$ and $g \in G$. We may choose $i > 0$, and thus $i > 1$. Then we have

$$(b^0,e) + (b^{1-i},h) = (b^0,e) \quad \text{for all} \quad h \in G$$

by (9.2), which yields by (2.2)

$$(b^0,e) + (b^{1-i},h)(b^i,g) = (b^0,e) + (b^1,hg) = (b^1,hg) \in K.$$

This shows that $(b^1,\tilde{g}) \in K$ for all $\tilde{g} \in G$, and so $K = A$.

b) Let $\mathfrak{H} \subseteq \mathfrak{S}^* \cup \mathfrak{G}$ be any universal class which contains $(A,+,\cdot)$, and hence also $(G,+,\cdot),(A,\cdot)$ and $(G,\cdot)$. We claim that $\mathfrak{H}$ *contains a subclass* $\mathbb{R}$ *which satisfies (Ra), (Rb) and (Rc) but not (Rk)*. For this end we consider the subset

$$\mathbb{M} = \{(A,+,\cdot),\ (G,+,\cdot),\ (G,\cdot)\}$$

and the subclass

$$\mathbb{R} = \mathcal{U}\mathbb{M} = \{X \in \mathfrak{H} \mid \forall Y(X \twoheadrightarrow Y \Rightarrow Y \notin \mathbb{M})\}$$

of $\mathfrak{H}$. In view of (9.3) and since $(G,\cdot)$ is simple, the set $\mathbb{M}$ satisfies (Sb). Moreover, also (Sγ) holds for $\mathbb{M}$ because $(G,\cdot)$ and $(A,\cdot)$ are not contained in $\mathcal{U}\mathbb{M}$, the latter by $(A,\cdot) \twoheadrightarrow (G,\cdot)$ and $(G,\cdot) \in \mathbb{M}$. Hence

$\mathbb{R} = \mathcal{U}\mathbb{M}$ is – by Remark 7.4 – an n-universal class of $\mathfrak{H}$ such that $\mathbb{R}$ satisfies *(Ra)*, *(Rb)* and *(Rc)*.

To disprove *(Rk)*, we take into account that all non-trivial kernels of $(A, +, \cdot)$ are contained in $\mathbb{M}$. This yields that $|\mathfrak{K}(A, +, \cdot) \cap \mathbb{R}| = 1$. But the group $(A, \cdot) = (B, \cdot) \times (G, \cdot)$ has a kernel isomorphic to $(B, \cdot)$, and also $(B, \cdot) \in \mathbb{R}$ because all homomorphic images of this cyclic group are commutative and so not in $\mathbb{M}$. This shows that $|\mathfrak{K}(A, \cdot) \cap \mathbb{R}| \neq 1$, and so (Rk) fails to be true for $\mathbb{R}$.

c) Note that $\mathbb{R}$ is an n-radical class of the universal class $\mathfrak{H}$ which is not a radical class. Hence $\mathbb{S} = \mathcal{S}\mathbb{R}$ is an n-semisimple class of $\mathfrak{H}$ which is not a semisimple class, that is, $\mathbb{S}$ is an example of a class satisfying *(Sa)*, *(Sb)* and $(S\gamma)$ but not *(Sc)*. Indeed, as a consequence of b), we have $(A, +, \cdot) \in \mathbb{S}$ and $(A, \cdot) \notin \mathbb{S}$. In particular, as far as $(\mathrm{S}\gamma)$ is concerned, $(A, \cdot) \notin \mathcal{U}\mathbb{S} = \mathbb{R}$ was stated above, and $(G, \cdot)$ is a group such that $(A, \cdot) \twoheadrightarrow (G, \cdot) \in \mathbb{S}$.

10. Hereditariness of semisimple and n-semisimple classes

As already mentioned in the introduction, the claim in [12] that every semisimple class of a universal class is hereditary has not been substantiated, and it remains open as whether this assertion is true or not. In the sequel we present sufficient conditions for the hereditariness of semisimple classes $\mathbb{S}$ of a universal class $\mathfrak{H} \subseteq \mathfrak{S}^* \cup \mathfrak{G}$, and we deal with the same question also in the more general case of n-semisimple classes of an n-universal class. As in other radical theories, the hereditariness of an n-semisimple class $\mathbb{S}$ is guaranteed if the corresponding class $\mathbb{R} = \mathcal{U}\mathbb{S}$ satisfied the following property named after Anderson, Divinsky and Suliński [1].

Definition 10.1. Let ϱ be an n-radical operator in an n-universal class $\mathfrak{H}$ of $\mathfrak{S}^* \cup \mathfrak{G}$ and $\mathbb{R} = \mathbb{R}_\varrho$ the corresponding n-radical class. We say that ϱ or $\mathbb{R}$ have the *ADS-property* if

(10.1) $\qquad K \in \mathfrak{K}(A)$ implies $\varrho K \in \mathfrak{K}(A)$ for all $K, A \in \mathfrak{H}$.

In [12] we posed also the question as whether there are radical classes $\mathbb{R}$ of a universal class $\mathfrak{H}$ which do not have the ADS-property. We still cannot answer this question, although we shall give examples of n-semisimple classes which are not hereditary and n-radical classes which do not satisfy the ADS-property. Prior to this, however, some conditions and statements will be discussed which concern merely semifields, tacitly assumed to be proper ones.

Definition 10.2. A semifiled A satisfies the *restricted transitivity condition for the kernel relation,* briefly *denoted by* (RT), if

(10.2) $N \triangleleft K \triangleleft A$ and $(N, \cdot) \triangleleft (A, \cdot)$ imply $N \triangleleft A$ for all $N, K \in \mathfrak{S}^* \cup \mathfrak{G}$.

Moreover, A *satisfies the condition* (RT') if (10.2) holds under the supplementing assumption that N is also a semifield.

Since the trivial kernel N of $K \in \mathfrak{K}(A)$ satisfies $N \in \mathfrak{K}(A)$, we avoid in (10.2) the somewhat clumsy formulation

$$N \in \mathfrak{K}(K),\ K \in \mathfrak{K}(A) \quad \text{and} \quad (N, \cdot) \in \mathfrak{K}(A, \cdot) \quad \text{imply} \quad N \in \mathfrak{K}(A),$$

and proceed similarly in corresponding situations. For instance, we may write $K \triangleleft A$ instead of $K \in \mathfrak{K}(A)$ in (10.1).

Proposition 10.3. a) *Every additively commutative semifield A satisfies condition (RT'), but not necessarily (RT).*

b) *Every additively commutative and idempotent semifield satisfies (RT).*

Proof. a) We assume $N \triangleleft K \triangleleft A$ and $(N, \cdot) \triangleleft (A, \cdot)$ as well as $N \in \mathfrak{S}^*$. Since $(A, +)$ is commutative, the same holds for $(K, +)$. Therefore $(K, +, \cdot, \leq)$ is a partially ordered semifield with respect to its natural order relation

(10.3) $$k_1 \leq k_2 \Leftrightarrow k_1 = k_2 \text{ or } k_1 + x = k_2 \text{ for some } x \in K,$$

which yields that each kernel N of K is a convex subset of $(K, \leq)$ (cf. Th. 4.2 and Cor. 4.7 in [6]). We use this to show that $T = \mathrm{hull}_A(N)$, the smallest kernel of A which contains N, coincides with N. By Prop. 2.13, an arbitrary element $t = t_1$ of $T = \mathrm{hull}_A(N)$ is a sum

$$t_1 = s_1 n_1 + s_2 n_2 + \cdots + s_r n_r$$

of elements $n_i \in N$ and $s_i \in A$ satisfying $s_1 + \cdots + s_r = e$. Hence also

$$t_2 = s_1 n_2 + s_2 n_3 + \cdots + s_r n_1$$

$$\cdots$$

$$t_r = s_1 n_r + s_2 n_1 + \cdots + s_r n_{r-1}$$

are elements of T. Using again that $(A, +)$ is commutative and $n_1 + \cdots + n_r = n \in N$ holds, we get

$$t_1 + t_2 + \cdots + t_r = (s_1 + \cdots + s_r)(n_1 + \cdots + n_r) = e \cdot n = n \in N.$$

Now $N \subseteq K \triangleleft A$ implies $T = \mathrm{hull}_A(N) \subseteq K$, and so $t_2 + \cdots + t_r = k \in K$. This shows that for each $t = t_1 \in T$ there are elements $k \in K$

and $n \in N$ satisfying $t + k = n$, that is, $t \le n$ by (10.3). This yields $t^{-1} \le m$ for some $m \in N$, and so $m^{-1} \le t$ for $m^{-1} = n' \in N$. Hence each $t \in T \subseteq K$ satisfies $n' \le t \le n$ for elements $n', n \in N$. Since $N \in \mathfrak{K}(K)$, N is convex in K, and therefore $t \in N$ follows. Thus we have shown $N = T = \mathrm{hull}_A(N)$, and so $N \lhd A$. For the last statement of a) we refer to the following Ex. 10.4.

b) All kernels of an idempotent semifield are again idempotent semifields (cf. Cor. 2.8 a)). Hence $N \lhd K \lhd A \in \mathfrak{S}^{\mathrm{idp}}$ yields $N \in \mathfrak{S}^{\mathrm{idp}} \subseteq \mathfrak{S}^*$. Taking into account that A is also additively commutative, (RT') for A, as shown above, implies (RT) for A. ◊

Example 10.4. Consider the factor ring $R = \mathbb{Q}[z]/(z^2)$ of the rational polynomial ring $\mathbb{Q}[z]$ and denote the elements of R by $a + b\overline{z}$ for $a, b \in \mathbb{Q}$. One easily checks that the subset

$$A = \{\alpha + b\overline{z} \mid \alpha \in \mathbb{H}, b \in \mathbb{Q}\}$$

is a semifield $(A, +, \cdot)$, where the inverse of $\alpha + b\overline{z}$ is $\alpha^{-1} - \alpha^{-2}b\overline{z} \in A$. Obviously, $\alpha + b\overline{z} \longmapsto \alpha$ yields a homomorphism $\varphi : A \twoheadrightarrow \mathbb{H}$. The kernel $K = \{1 + b\overline{z} \mid b \in \mathbb{Q}\}$ of φ is a group because of $\mathbb{H} \in \mathfrak{S}^\circ$, and $(K, \cdot)$ is isomorphic to $(\mathbb{Q}, +)$. The subset $N = \{1 + g\overline{z} \mid g \in \mathbb{Z}\}$ is a kernel $(N, \cdot)$ of $(K, \cdot)$. Thus we have $N \lhd K \lhd A$ *for an additively commutative semifield A and clearly $(N, \cdot) \lhd (A, \cdot)$, but N is not a kernel of A.* The latter follows from (2.2), because of $1/2 + 1/2 = 1$ and $1/2 + (1/2)(1 + 1\overline{z}) = 1 + (1/2)\overline{z} \notin N$.

To obtain additively non-commutative semifields which satisfy (RT), we consider rectangular semifields as introduced in Ex. 8.5. Every semifield $(M, +, \cdot)$ of this kind is a direct product $(G, +, \cdot) \times (G_2, +, \cdot)$ of (uniquely determined) semifields $(G_i, +, \cdot)$, where the group $(G_1, \cdot)$ is endowed with the addition $a_1 + b_1 = a_1$ for all $a_1, b_1 \in G_1$ and the group $(G_2, \cdot)$ with the addition $a_2 + b_2 = b_2$. Conversely, starting from arbitrary groups $(G_i, \cdot)$, one obtains a rectangular semifield $(M, +, \cdot)$ in this way (cf. [10]). The following lemma has not been published so far.

Lemma 10.5. *Let $(M, +, \cdot) = (G_1, +, \cdot) \times (G_2, +, \cdot)$ be a rectangular semifield and N a normal subgroup of $(M, \cdot)$. Then the following statements are equivalent:*

a) *N is additively closed, that is, $(N, +, \cdot)$ is a subsemifield of $(M, +, \cdot)$.*

b) *$(N, \cdot) = (N_1, \cdot) \times (N_2, \cdot)$ holds for normal subgroups $(N_i, \cdot)$ of $(G_i, \cdot)$.*

c) *N is a kernel of $(M, +, \cdot)$.*

Proof. Note that $(a_1, a_2) + (b_1, b_2) = (a_1, b_2)$ holds for the addition in $(M, +, \cdot)$. Assuming a), if $(n_1, n_2) \in N$ and $(e_1, e_2) \in N$ for the identity of M, then

$$(n_1, n_2) + (e_1, e_2) = (n_1, e_2) \in N$$

and likewise $(e_1, n_2) \in N$, which clearly yields b). To show b)$\Rightarrow$c) by (2.2), assume $(a_1, a_2) + (b_1, b_2) = (e_1, e_2)$ and $(n_1, n_2) \in N$. The former implies $a_1 = e_1$ and $b_2 = e_2$, and

$$(e_1, a_2) + (b_1, e_2)(n_1, n_2) = (e_1, n_2) \in N$$

by b) proves c). The latter yields a) since $(M, +, \cdot)$ is idempotent. $\Diamond$

Proposition 10.6. *Let M be a rectangular semifield. Then the kernel relation in M is transitive, a statement which clearly yields (RT).*

Proof. We apply Lemma 10.5 to $N \triangleleft K \triangleleft M$. Then $N \triangleleft K$ implies that N is a subsemifield of K and hence one of M, which yields $N \triangleleft M$. $\Diamond$

Now, we focus our attention to statements on hereditariness and the ADS-property. *We start with those which are even true for n-semisimple classes and n-radical operators of an n-universal class, and hence by* Lemma 9.4 *all the more for semisimple classes and radical operators of a universal class.*

Proposition 10.7. *Let $\mathfrak{H} \subseteq \mathfrak{S}^* \cup \mathfrak{G}$ be an n-universal class and ϱ an n-radical operator in $\mathfrak{H}$, and suppose $\varrho K = N \subseteq K \triangleleft A \in \mathfrak{H}$. Then $(N, \cdot)$ is a normal subgroup of $(A, \cdot)$ regardless whether these groups are contained in $\mathfrak{H}$ or not.*

This yields $K \triangleleft A \Rightarrow \varrho K \in \mathfrak{K}(A)$ whenever $A \in \mathfrak{H}$ is a group, recovering the validity of the ADS-property for groups.

Proof. Since K is (at least) a normal subgroup of $(A, \cdot)$, each $a \in A$ defines an automorphism φ_a of K by $\varphi_a(k) = a^{-1}ka$. This automorphism φ_a is of type 2 for $K \in \mathfrak{G}$ and of type 1 for $K \in \mathfrak{S}^*$. In both cases it holds $K \in \mathfrak{H}$ and we can apply $(\varrho\alpha)$ to φ_a and obtain $\varphi_a(\varrho K) \subseteq \varrho K$. Hence $N = \varrho K$, considered as a group $(N, \cdot) \in \mathfrak{G}$, is a normal subgroup of $(A, \cdot) \in \mathfrak{G}$. $\Diamond$

Theorem 10.8. *Let $\mathfrak{H} \subseteq \mathfrak{S}^* \cup \mathfrak{G}$ be an n-universal class.*

a) *If an n-radical class $\mathbb{R}$ of $\mathfrak{H}$ does not contain non-trivial groups, the ADS-property holds for every $(A, +, \cdot) \in \mathfrak{H}$ which satisfies (RT'). In particular, the n-semisimple class $\mathbb{S} = \mathcal{S}\mathbb{R}$ is hereditary if each $(A, +, \cdot) \in \mathbb{S}$ satisfies (RT').*

b) *For an arbitrary n-radical class $\mathbb{R}$ of $\mathfrak{H}$, the ADS-property holds for each $(A, +, \cdot) \in \mathfrak{H}$ which satisfies (RT), and an n-semisimple class*

$\mathbb{S}$ of $\mathfrak{H}$ is hereditary if each $(A,+,\cdot) \in \mathbb{S}$ satisfies (RT).

c) *If each $(A,+,\cdot) \in \mathfrak{H}$ satisfies (RT), the ADS-property holds for all n-radical classes $\mathbb{R}$ of $\mathfrak{H}$ and every n-semisimple class $\mathbb{S}$ of $\mathfrak{H}$ is hereditary.*

Proof. We start with b), and suppose $\varrho K = N \subseteq K \triangleleft A \in \mathfrak{H}$. Then $(N,\cdot) \in \mathfrak{K}(A,\cdot)$ holds in $\mathfrak{G}$ by Prop. 10.7. Hence, if $A \in \mathfrak{H}$ is a semifield which satisfies (RT), we get $\varrho K \in \mathfrak{K}(A)$, and so the ADS-property for all semifields with (RT). Now, assume that every semifield contained in $\mathbb{S}$ satisfies (RT). Then $\varrho K \in \mathfrak{K}(A)$ holds for all $K \triangleleft A \in \mathbb{S}$, regardless whether A is a group or a semifield. Consequently we have $\varrho K \subseteq \varrho A$ by Th. 5.7, and $\varrho A \in \mathfrak{T}$ implies $K \in \mathbb{S}$, that is, $\mathbb{S}$ is hereditary.

Since b) clearly implies c), it remains to show a). Again for $\varrho K = N \subseteq K \triangleleft A \in \mathfrak{H}$, the assumption on $\mathbb{R}$ yields $\varrho K = \varrho_{\mathbb{R}} K \in \mathfrak{S}^* \cup \mathfrak{T}$. Hence it is enough to assume (RT') instead of (RT) for a semifield $A \in \mathfrak{H}$ to obtain $\varrho K \in \mathfrak{K}(A)$ and to complete the proof of a) in the above pattern. ◊

Combining Th. 10.8 with Props. 10.3 and 10.6, we obtain

Corollary 10.9. *Let $\mathfrak{H} \subseteq \mathfrak{S}^* \cup \mathfrak{G}$ be an n-universal class.*

a) *If an n-radical class $\mathbb{R}$ of $\mathfrak{H}$ does not contain non-trivial groups, the ADS-property holds for each semifield $(A,+,\cdot) \in \mathfrak{H}$ which is either additively commutative or rectangular. In particular, the n-semisimple class $\mathbb{S} = S\mathbb{R}$ is hereditary if each $(A,+,\cdot) \in \mathbb{S}$ is either additively commutative or rectangular.*

b) *For an arbitrary n-radical class $\mathbb{R}$ of $\mathfrak{H}$, the ADS-property holds for each semifield $(A,+,\cdot) \in \mathfrak{H}$ which is idempotent and either commutative or rectangular, and an n-semisimple class $\mathbb{S}$ of $\mathfrak{H}$ is hereditary if each $(A,+,\cdot) \in \mathbb{S}$ has the properties just described.* ◊

We emphasize again that all these results apply, in particular, to the radical theory given in [12] which deals with radical classes and semisimple classes in a universal class $\mathfrak{H} \subseteq \mathfrak{S}^* \cup \mathfrak{G}$. Stronger results, which are in fact only true in this theory, are prepared by the following

Lemma 10.10. *Let $\mathfrak{H} \subseteq \mathfrak{S}^* \cup \mathfrak{G}$ be a universal class, ϱ a radical operator in $\mathfrak{H}$ and $(A,+,\cdot) \in \mathfrak{H}$. If the radical $\varrho(A,+,\cdot)$ of $(A,+,\cdot)$ happens to be a group, then $\varrho(A,+,\cdot)$ coincides with the radical $\varrho(A,\cdot)$ of the group $(A,\cdot)$, say $\varrho(A,+,\cdot) = \varrho(A,\cdot)$.*

Proof. Assume $\varrho(A,+,\cdot) = (C,\cdot) \in \mathfrak{G}$. Then $(C,\cdot)$ is contained in the radical class $\mathbb{R} = \mathbb{R}_{\varrho}$, and is a kernel of $(A,\cdot)$, which yields $(C,\cdot) \subseteq \varrho(A,\cdot)$, that is, $\varrho(A,+,\cdot) \subseteq \varrho(A,\cdot)$. The converse inclusion holds by

(9.1) in Lemma 9.4 since ϱ is now a radical operator. ◊

Theorem 10.11. *Let $\mathfrak{H} \subseteq \mathfrak{S}^* \cup \mathfrak{G}$ be a universal class.*

a) *If each semifield $(A, +, \cdot)$ in a semisimple class $\mathbb{S}$ of $\mathfrak{H}$ satisfies (RT'), then $\mathbb{S}$ is hereditary.*

b) *If each $(A, +, \cdot) \in \mathfrak{H}$ satisfies (RT'), then all semisimple classes $\mathbb{S}$ of $\mathfrak{H}$ are hereditary.*

c) *If each $(A, +, \cdot) \in \mathfrak{H}$ is either additively commutative or rectangular, then all semisimple classes $\mathbb{S}$ of $\mathfrak{H}$ are hereditary.*

Proof. The implications a)⇒b)⇒c) are obvious, the latter by Props. 10.3 a) and 10.6. To show a) we assume again $\varrho K = N \subseteq K \triangleleft A \in \in \mathbb{S}$. This implies $(N, \cdot) \in \mathfrak{K}(A)$ by Prop. 10.7, and so $\varrho K \in \mathfrak{K}(A)$ if A is a group. Now, let A be a semifield satisfying (RT'). Then we get $\varrho K \in \mathfrak{K}(A)$ if $\varrho K = N$ is a semifield, too. Hence in both of these cases $\varrho K \in \mathfrak{K}(A)$ implies $\varrho K \subseteq \varrho A \in \mathfrak{T}$, and so $K \in \mathbb{S}$. It remains to consider the case $A \in \mathfrak{S}^*$ and $\varrho K \in \mathfrak{G}$. If also $K \in \mathfrak{G}$ holds, then $\varrho K = = \varrho(K, \cdot)$ is clear. If $K \in \mathfrak{S}^*$, then $\varrho K = \varrho(K, +, \cdot) = \varrho(K, \cdot)$ follows from Lemma 10.10. Now, we use $\varrho(K, \cdot) \subseteq \varrho(A, \cdot)$ for the group $(A, \cdot)$ and $\varrho(A, \cdot) \subseteq \varrho(A, +, \cdot)$ by (9.1). Hence $\varrho K = \varrho(K, \cdot) \subseteq \varrho(A, +, \cdot) \in \mathfrak{T}$ shows $K \in \mathbb{S}$ also in the remaining case, that is, $\mathbb{S}$ is hereditary if every $(A, +, \cdot) \in \mathbb{S}$ satisfies (RT'). ◊

In contrast to these results of the old theory, all statements of Th. 10.11 fail to be true in the new more general theory.

Theorem 10.12. *Even if all semifields $(A, +, \cdot)$ contained in an n-universal class $\mathfrak{H} \subseteq \mathfrak{S}^* \cup \mathfrak{G}$ satisfy (RT'), in particular, if all these semifields are additively commutative, an n-semisimple class $\mathbb{S}$ of $\mathfrak{H}$ need not be hereditary. This yields that an n-radical class $\mathbb{R}$ of $\mathfrak{H}$ need not satisfy the ADS-property.*

The statement follows from Ex. 10.15 which is based on

Proposition 10.13. *Let $\mathbb{S}$ be an n-semisimple class of an n-universal class $\mathfrak{H}$ which is not semisimple. Hence there is a semifield $(A, +, \cdot) \in \mathbb{S}$ such that $(A, \cdot) \notin \mathbb{S}$ holds. Assume that $\mathbb{S}$ contains a direct product $(H, +, \cdot) = (A, +, \cdot) \times (C, +, \cdot)$ of such a semifield $(A, +, \cdot)$ and a non-idempotent semifield $(C, +, \cdot)$. Then $\mathbb{S}$ is not hereditary.*

Proof. By Ex. 2.4 the natural projection $H \twoheadrightarrow C \in \mathfrak{S}^\diamond$ yields a kernel K of $(H, +, \cdot) \in \mathbb{S}$ which satisfies $K = (K, \cdot) \cong (A, \cdot) \notin \mathbb{S}$. Hence $\mathbb{S}$ is not hereditary. ◊

Corollary 10.14. *Let $\mathfrak{H}$ be an n-universal class which is closed under taking finite direct products and $\mathbb{S}$ an n-semisimple, but not semisimple* *class of $\mathfrak{H}$. If $\mathbb{S}$ contains a non-idempotent semifield, then $\mathbb{S}$ is not hereditary.*

Proof. As assumed, $\mathbb{S}$ contains a semifield $(A,+,\cdot)$ such that $(A,\cdot) \notin \mathbb{S}$ holds. Now $(C,+,\cdot) \in \mathbb{S}$ for some $(C,+,\cdot) \in \mathfrak{S}^\diamond$ yields $(A,+,\cdot) \times \times (C,+,\cdot) \in \mathbb{S}$ by the assumption on $\mathfrak{H}$ and Prop. 7.7. Hence $\mathbb{S}$ is not hereditary by Prop. 10.13. $\Diamond$

Example 10.15. *We present an n-universal class $\mathfrak{H} \subseteq \mathfrak{S}^* \cup \mathfrak{G}$ with the following properties:*

1) *Every semifield in $\mathfrak{H}$ is additively commutative.*

2) *There are an n-semisimple class $\mathbb{S}$ and a semifield $H = A \times C$ in $\mathfrak{H}$ which satisfy the assumptions of Prop. 10.13, and hence $\mathbb{S}$ is not hereditary.*

a) At first we define a suitable semifield $(H,+,\cdot)$. For this end we need an additively commutative and idempotent semifield $(G,+,\cdot)$ such that $(G,\cdot)$ is a non-commutative simple group. Now each additively commutative and idempotent semifield $(G,+,\cdot)$ is determined by a lattice-ordered group $(G,\cdot)$ and the addition $g+h = g \vee h$ (cf. Prop. 4.1 in [13]). So we need such a group which is non-commutative and simple, for instance the (linearly ordered) Chehata group (cf. [2]).

Now let $(A,+,\cdot)$ be the semifield constructed in Ex. 9.9 a) from the semifield $(B,+,\cdot)$ used there and a semifield $(G,+,\cdot)$ as described above. The addition of $(A,+,\cdot)$ is defined by (9.2) and so A is an additively commutative and idempotent semifield. Moreover, in view of (9.3) the semifield A has exactly three kernels, and we also recall

$$A \twoheadrightarrow B \cong A/L \text{ where } L = \{(e,g) \mid g \in G\} \cong (G,+,\cdot).$$

Finally, we define $(H,+,\cdot)$ as the direct product of $(A,+,\cdot)$ and the simple semifield $C = (\mathbb{P},+,\cdot) \in \mathfrak{S}^\diamond$ of positive real numbers (cf. Ex. 2.9). Clearly, $(H,+,\cdot) = (A,+,\cdot) \times (\mathbb{P},+,\cdot)$ is additively commutative.

b) For $\mathfrak{H}$ we can take any n-universal class satisfying condition 1) and containing $(H,+,\cdot)$ and so also $(\mathbb{P},+,\cdot)$, but not the group $(\mathbb{P},\cdot)$. Later we shall prove that such a class $\mathfrak{H}$ really exists and contains the group $(A,\cdot)$ as well as the set

$$\mathbb{M} = \{(A,+,\cdot),(G,+,\cdot),(G,\cdot),(\mathbb{P},+,\cdot)\}.$$

Similarly to Ex. 9.9 b) one can verify that $\mathbb{M}$ fulfils (Sb) and $(S\gamma)$, (for the latter one needs $(\mathbb{P},\cdot) \notin \mathfrak{H}$ to obtain $(\mathbb{P},\cdot) \notin \mathcal{U}\mathbb{M}$). Hence $\mathbb{R} = \mathcal{U}\mathbb{M}$ is an n-radical class of $\mathfrak{H}$, and we show that the corresponding n-semisimple class $\mathbb{S} = \mathcal{S}\mathbb{R}$ satisfies the assumptions of Prop. 10.13.

Firstly, $\mathbb{S}$ contains obviously $(A,+,\cdot)$ but not $(A,\cdot)$. The latter follows since $(A,\cdot) = (B,\cdot) \times (G,\cdot)$ has a kernel isomorphic to the infinite cyclic group $(B,\cdot)$, where $(B,\cdot) \in \mathbb{R}$ holds because of $(B,\cdot)$ cannot be mapped homomorphically onto a member of $\mathbb{M}$. Secondly, $\mathbb{S}$ contains the direct product $(H,+,\cdot)$ of $(A,+,\cdot)$ and $(\mathbb{P},+,\cdot) \in \mathfrak{S}^{\diamond}$. This follows from $A, \mathbb{P} \in \mathbb{S}$ and $H \in \mathfrak{H}$, again by Prop. 7.7.

c) To obtain an n-universal class $\mathfrak{H}$ as required in b) we gather information on semifields and groups which occur if we start with $H = (H,+,\cdot)$ and proceed step by step to homomorphic images and kernels. We write

$$(b^i, g, \alpha) = ((b^i, g), \alpha) \quad \text{with } i \in \mathbb{Z},\ g \in G \text{ and } \alpha \in \mathbb{P}$$

for the elements of $(H,+,\cdot) = (A,+,\cdot) \times (\mathbb{P},+,\cdot)$. This direct product has at least the following homomorphic images, listed together with their kernels:

$$H \twoheadrightarrow A \cong H/K_1 \in \mathfrak{S}^{\mathrm{idp}} \qquad K_1 = \{(b^0, e, \alpha) \mid \alpha \in \mathbb{P}\},$$

$$H \twoheadrightarrow \mathbb{P} \cong H/K_2 \in \mathfrak{S}^{\diamond} \qquad K_2 = \{(b^i, g, 1) \mid i \in \mathbb{Z}, g \in G\},$$

$$H \twoheadrightarrow B \cong H/K_3 \in \mathfrak{S}^{\mathrm{idp}} \qquad K_3 = \{(b^0, g, \alpha) \mid g \in G, \alpha \in \mathbb{P}\},$$

$$H \twoheadrightarrow B \times \mathbb{P} \cong H/K_4 \in \mathfrak{S}^{\diamond} \qquad K_4 = \{(b^0, g, 1) \mid g \in G\}.$$

Note that K_1 and K_3 are semifields isomorphic to $(\mathbb{P},+,\cdot)$ and $(G,+,\cdot) \times (\mathbb{P},+,\cdot)$, respectively, and that K_2 and K_4 are groups isomorphic to $(A,\cdot)$ and $(G,\cdot)$, respectively. Now we use the following assertion which will be proved in d):

i) *The above list contains already all proper kernels of* $(H,+,\cdot)$, that is, all except of H and $\{(b^0, e, 1)\}$, and hence all proper homomorphic images up to isomorphism.

So, we can go on, and consider kernels and homomorphic images of the semifields and groups obtained so far. Recall that $(\mathbb{P},+,\cdot)$, $(B,+,\cdot)$ and $(G,\cdot)$ are simple and that $(A,+,\cdot)$ has only the proper kernel $(L,+,\cdot) \cong (G,+,\cdot)$, where A/L is isomorphic to $(B,+,\cdot)$. For the remaining semifields we shall prove under e):

ii) *All proper kernels of the semifields* $B \times \mathbb{P}$ *and* $G \times \mathbb{P} \cong K_3$ *are isomorphic to* $(B,\cdot)$, $(G,\cdot)$ *and* $(\mathbb{P},+,\cdot)$, and so their proper homomorphic images are isomorphic to $(\mathbb{P},+,\cdot)$, $(B,+,\cdot)$ and $(G,+,\cdot)$.

Since all semifields occuring in ii) are simple, it remains to deal with groups obtained from $(A,\cdot) = (B,\cdot) \times (G,\cdot)$. Here we come also to an end by the following statement which can be verified straightforwardly.

iii) *Let* $(Z, \cdot)$ *be any cyclic group and* $(G, \cdot)$ *a simple non-commutative group. Then all kernels and homomorphic images of* $(Z, \cdot) \times (G, \cdot)$ *are cyclic groups or isomorphic to* $(G, \cdot)$ *or to a direct product of two such groups.*

Based on these preparations, we obtain an n-universal class $\mathfrak{H}$ as desired. Let $\mathfrak{H}'$ be the n-universal (in fact, universal) class consisting of all finite and countable additively commutative semifields and groups. Define a class $\mathfrak{H}$ by extending $\mathfrak{H}'$ with isomorphic copies of the semifields H, A, $\mathbb{P}$, B, G, $B \times \mathbb{P}$, $G \times \mathbb{P}$ and of the groups described in iii). Then, by the above considerations, $\mathfrak{H}$ is an n-universal class which satisfes 1) and which contains all objects used in b), but not the group $(\mathbb{P}, \cdot)$.

d) We show i) in c) by the following steps:

α) *For each* $K \lhd H$, *either* $K \subseteq K_2$ *or* $K \supseteq K_1$. If K is not contained in K_2, then there is some $(b^i, g, \alpha) \in K$ such that $\alpha \neq 1$. Then $(b^0, e, \sigma) + (b^j, e, \tau) = (b^0, e, 1)$ holds for $\sigma + \tau = 1$ and every $j < 0$. Assuming also $j + i < 0$, we get by (2.2)

$$(b^0, e, \sigma) + (b^j, e, \tau) \cdot (b^i, g, \alpha) = (b^0, e, \sigma + \tau\alpha) \in K$$

for $\sigma + \tau\alpha \neq 1$. Hence $K \cap K_1$ is a non-trivial kernel of H, and so of K_1. Since $K_1 \cong (\mathbb{P}, +, \cdot)$ is a simple semifield, this yields $K \cap K_1 = K_1$ and so $K \supseteq K_1$.

β) *If* $K \lhd H$ *and* $K \subseteq K_2$, *then* K *equals* K_2 *or* K_4. First, let $K \subseteq K_2$ contain some $(b^i, g, 1)$ such that $i \neq 0$. We may assume $i > 0$ and so $i > 1$. Then, for $\sigma + \tau = 1$ in $\mathbb{P}$, we obtain

$$(b^0, e, \sigma) + (b^{1-i}, h, \tau) = (b^0, e, 1) \text{ for all } h \in G, \tag{10.4}$$

and so by (2.2)

$$(b^0, e, \sigma) + (b^{1-i}, h, \tau)(b^i, g, 1) = (b^1, h \cdot g, 1) \in K. \tag{10.5}$$

This yields $K = K_2$. Otherwise, $K \lhd H$ is contained in $K_4 \cong (G, \cdot)$, which is a minimal kernel of H since $(G, \cdot)$ is simple. Hence $K = K_4$ follows.

γ) *If* $K \lhd H$ *and* $K_1 \subseteq K$, *then* K *equals* K_1 *or* K_3 *or* H. Assume that $K \neq K_1$. Then K contains some $(b^i, g, \alpha) \notin K_1$, and so by $(b^0, e, \alpha^{-1}) \in K$, also $(b^i, g, 1) \neq (b^0, e, 1)$. Hence $K \cap K_2$ is a non-trivial kernel of H, and therefore $K \cap K_2 = K_2$ or $K \cap K_2 = K_4$ by β). In both cases K contains K_4, and so $K_1 \cdot K_4 = K_3$. But K_3 is a maximal kernel of H because of $H/K_3 \cong (\mathbb{P}, +, \cdot)$ is simple (cf. Remark 2.11). Hence $K \supseteq K_3$ implies $K = K_3$ or $K = H$.

e) First, we show ii) in c) for the semifield $S = B \times \mathbb{P}$. Since $(\mathbb{P}, +, \cdot)$

is simple, $L_1 = \{(b^0, \alpha) \mid \alpha \in \mathbb{P}\} \cong (\mathbb{P}, +, \cdot)$ is a minimal kernel of S, and $L_2 = \{(b^i, 1) \mid i \in \mathbb{Z}\}$ is a maximal one because of $S/L_2 \cong$ $\cong (\mathbb{P}, +, \cdot)$. Let L be any proper kernel of S and assume $(b^i, \beta) \in L$ with $i \neq 0$. Choose $i > 1$, and replace (10.4) by $(b^0, \sigma) + (b^{1-i}, \tau) =$ $= (b^0, 1)$. Then $(b^1, 1) \in L$ follows in view of (10.5). This shows $L \supseteq L_2$ and thus $L = L_2$ since L_2 is maximal. Otherwise L is contained in L_1 which yields $L = L_1$ since L_1 is minimal.

Turning to the semifield $S = G \times \mathbb{P}$, both kernels $L_1 = \{(e, \alpha) \mid \alpha \in$ $\in \mathbb{P}\}$ and $L_2 = \{(g, 1) \mid g \in G\}$ are minimal as well as maximal. Hence each element $(g, \alpha) \neq (e, 1)$ of a further proper kernel L of S should satisfy $g \neq e$ and $\alpha \neq 1$. But such a kernel does not exist. Choosing $g > e$, we get $(e, \sigma) + (g^{-1}, \tau) = (e, 1)$ because of $g^{-1} < e$, and so by (2.2) $(e, \sigma) + (g^{-1}, \tau)(g, \alpha) = (e, \sigma + \tau\alpha) \in L$, where $\sigma + \tau\alpha \neq 1$ contradicts the above statement on L.

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