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WEAK STABILIZABILITY OF A NONAUTONOMOUS AND NON-LINEAR SYSTEM

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Abstract: We consider in a real separable Hilbert space a class of nonautonomous and nonlinear systems. Under a natural condition (more general than the controllability condition used in [2], [3] and [4]) and using LaSalle invarince principle, we prove that these systems are weakly stabilizable. This paper aims to generalize and unify the main results of the papers [2], [3] and [4].

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1. Statement of the Theorem

1.1 Throughout this paper, H will be a real separable Hilbert space with inner product $< . >$ and associated norm $\|.\|$. Let β be a real function defined on $\mathbb{R}$. Let A be a possibly unbounded operator on H and denote $D(A)$ for its domain. Let $b \in H$ and let $t \mapsto C(t)$ be a mapping from $[0, \infty[$ to $\mathcal{L}(H)$ the Banach algebra of bounded operators in H, and let $B \in \mathcal{L}(H)$. We consider the following linear nonautonomous and nonlinear system:

$$(\mathrm{S}) \quad \begin{cases} \dot{y}(t) = [A + C(t)]y(t) + u(t)(By(t) + b), & t \geq 0 \\ y(0) = y_0 \in H, \quad u(t) = -\beta(< b, y(t) >). \end{cases}$$

The aim of this paper is to establish some stabilization results concerning the system (S) under the following assumptions:

(H 1) The linear operator A is closed, densely defined, dissipative (i.e. $< Ax, x > \leq 0$, for all $x \in D(A)$).

(H 2) The operator B is skew-adjoint on H.

(H 3) $b \in D(A^*)$ (where A^* is the adjoint of the operator A) and if for all $t \geq 0$ $< b, e^{tA}x_0 >= 0$ then $x_0 = 0$.

(H 4) Every $C(t)$ is dissipative and the mapping $t \mapsto C(t)$ is continuously differentiable from $[0, \infty[$ to $\mathcal{L}(H)$, and verifies $\lim_{t\to\infty} \|C(t)\| = 0$.

(H 5) β is a real C^1-function on the real line satisfying $\beta(0) = 0$ and for which there exist a positive number ϵ and a positive integer k such that $t\beta(t) \geq \epsilon t^{2k}$ for all real number t.

The assumption (H 3) is more general than the controllability condition used in the papers [2], [3] and [4]. We recall that the systems studied in these papers are particular cases of the system (S) considered here. The aim of this paper is to generalize and unify all these papers. The main result of this paper is the following theorem:

Theorem 1.2. *Suppose that* (H 1), (H 2), (H 3), (H 4), *and* (H 5) *are satisfied. Then we have:*

(i) *The system* (S) *has a unique mild solution* y *defined on the infinite interval* $[0, \infty[$.

(ii) *The system* (S) *is weakly stabilizable, and the feedback* $u(t) = -\beta(< b, y(t) >)$ *(which may be nonlinear) is a weakly stabilizing control law.*

2. Existence and uniqueness of a mild solution for (S)

2.1. In this section, we suppose that (H 1), (H 2), (H 4) and (H 5) are satisfied. We shall prove the existence and uniqueness of mild solutions of the system (S). We observe that assumption (H 1) ensures that A is the generator of a C_0 semigroup of contractions which will be denoted by $(e^{tA})_{t\geq 0}$.
2.2. Let us define the map f from $[0,\infty[\times H$ to H by setting for all $t \in [0,\infty[$ and all $v \in H$,

$$f(t,v) = C(t)v - \beta(< b, v >)(Bv + b).$$

It is easy to see that f satisfies a local Lipschitz condition in v, uniformly in t on bounded intervals, that is for every $T \geq 0$ and constant $c \geq 0$ there is a constant $M(c,T)$ such that

$$\|f(t,v) - f(t,u)\| \leq M(c,T)\|v - u\|$$

holds for all $u, v \in H$ with $\|v\| \leq c$, $\|u\| \leq c$, and $t \in [0,T]$. Thus we may apply Th. 1.4. in [7], p. 185, to obtain that there is a $t_{\max} \leq \infty$ such that (S) has a unique *mild* solution y on $[0,\infty[$ and that moreover, if $t_{\max} < \infty$ then $\lim_{t\to t_{\max}} \|y(t)\| = \infty$.
2.3. We recall (see for example [7], [4]) that for every $T > 0$ a mild solution of the initial value problem (S) on $[0,T]$ is any continuous function $y(t)$ defined on $[0,T]$ satisfying the integral equation

$$y(t) = e^{tA}y_0 + \int_0^t e^{(t-s)A} f(s, y(s))\, ds, \quad \forall t \in [0,T]$$

Since f is continuously differentiable from $[0,T] \times H$ into H then (see [7]) this mild solution $y(t)$ becomes a classical solution of initial value problem (S) on $[0,T]$ when the initial value $y_0 \in D(A)$.
2.4. Let $y_0 \in H$ and consider a sequence $(y_0^n)_n$ of elements in H converging to y_0. For each $T > 0$, let $y(t)$ and $y^n(t)$ be the mild solutions of (S) associated respectively to the initial values y_0 and y_0^n. Then one can prove that for each $t \in [0,T]$, the sequence $(y^n(t))_n$ converges in H to $y(t)$.
Theorem 2.5. *Suppose that* (H 1), (H 2), (H 4), *and* (H 5) *are satisfied. Then the system* (S) *has a unique mild solution* y *defined on the infinite interval* $[0,\infty[$.
Proof. It is sufficient to prove that for each $T > 0$ the mild solution $y(t)$ is bounded by a constant independent of T. To do this we discuss two cases:

i) If the initial value $y_0 \in D(A)$ then $y(t)$ becomes a classical solution and a differentiation of the function $v(t) := \frac{1}{2}\|y(t)\|^2$ defined for all $t \in [0,T]$ will give the following inequality:

$$\frac{dv(t)}{dt} =< y(t), \frac{dy(t)}{dt} >=$$
$$=< [A + C(t)]y(t), y(t) > - < b, y(t) > \beta(< b, y(t) >) \leq 0,$$

from which we deduce that $\|y(t)\| \leq \|y_0\|$ for all $t \in [0,T]$.

ii) If $y_0 \notin D(A)$ then we can find a sequence $(y_0^n)_n$ of elements in $D(A)$ converging to y_0 in H. For all $t \in [0,T]$ and all integer n we know from i) that $\|y_0^n(t)\| \leq \|y_0^n\|$. Know, we conclude by 2.4. that $\|y(t)\| \leq \|y_0\|$, for all $t \in [0,T]$. $\Diamond$

3. Weak stabilizability of the system (S)

We suppose as before that (H 1), (H 2), (H 3), (H 4), and (H 5) are satisfied. The purpose of this section is to study the weak stabilizability of the system (S). Our result will be based on several lemmas listed below.

Lemma 3.1. $\lim_{t\to\infty} < b, y(t) >= 0$.

Proof. As in the proof of Th. 2.5., two cases will be distinguished:

(i) If the initial value $y_0 \in D(A)$ then the function $v(t) := \frac{1}{2}\|y(t)\|^2$ is continuously differentiable and we can write for all $t \geq 0$, the following:

$$v(t) =$$
$$= v(0) + \int_0^t < [A + C(s)]y(s), y(s) > \, ds - < b, y(t) > \beta(< b, y(t) >).$$

By using assumption (H 5), we deduce from the previous equality the following inequality

$$\int_0^t < b, y(s) >^{2k} ds \leq \frac{1}{2\epsilon}\|y_0\|^2, \; \forall t \geq 0.$$

This ensures the convergence of the integral $\int_0^\infty < b, y(t) >^{2k} dt$. Since the derivative $\frac{d}{dt} < b, y(t) >^{2k}$ is bounded on $[0, \infty[$, our claim in this case is proved.

(ii) If $y_0 \notin D(A)$ then there is a sequence $(y_0^n)_n$ of elements in $D(A)$ converging to y_0 in H. Let $y^n(t)$ (resp. $y(t)$) be the mild solution of (S) defined on $[0, \infty[$ associated to y_0^n (resp. to y_0). Since $(y^n(t))_n$ converges in H to $y(t)$ for all $t \geq 0$, and the following inequality holds

$$\int_0^{\infty} < b, y^n(s) >^{2k} ds \leq \frac{1}{2\epsilon} \|y_0^n\|^2.$$

An application of Fatou's theorem will help us to obtain

$$\int_0^{\infty} < b, y(s) >^{2k} ds \leq \frac{1}{2\epsilon} \|y_0\|^2.$$

In this case too, we verify easily that the derivative $\frac{d}{dt} < b, y(t) >^{2k}$ is bounded on $[0, \infty[$. So we can assert that $\lim_{t\to\infty} < b, y(t) >= 0$. $\diamond$

3.2. Now for each initial value $y_0 \in H$, we will denote by $\Gamma(y_0)$ the weak positive $\omega-$limit set of the trajectory of the state $y(t)$. This set given by

$$\{z \in H : \exists \quad \text{a sequence} \quad t_n \to \infty \quad \text{such that} \quad w - \lim_{n\to\infty} y(t_n) = z\}.$$

From Lemma 3.1, we deduce that $\Gamma(y_0)$ is included in $b^{\perp}$ the orthogonal of b. The following lemma is needed to finish the proof of Th. 1.2.

Lemma 3.3. *We have* $e^{tA}(\Gamma(y_0)) \subseteq \Gamma(y_0)$, *for all* $t \geq 0$.

Proof. Let $t \geq 0$. and let $z \in \Gamma(y_0)$. Consider a sequence $(t_n)_n$ such that $t_n \to \infty$ for which the sequence $(y(t_n))_n$ converges weakly to z. We know that the state $y(t + t_n)$ has the following expression:

$$e^{tA} y(t_n) + \int_0^t e^{(t-s)A} \Big[C(s + t_n)(y(s + t_n)) - $$
$$ - \beta(< b, y(s + t_n) >)(By(s + t_n) + b) \Big] \, ds.$$

Now, for any arbitrary vector $\xi \in H$, we can write the following inequality:

$$|< y(t+t_n), \xi > - < e^{tA} z, \xi >| \leq |< y(t + t_n), \xi > - < e^{tA} y(t_n), \xi >| +$$
$$+ \ |< e^{tA} y(t_n), \xi > - < e^{tA} z, \xi >|$$

Moreover, we can find two positive constants M_1 and M_2 such that the following holds

$$|< y(t+t_n), \xi > - < e^{tA} y(t_n), \xi >| \leq M_1 \int_0^t |\ \beta(< b, y(s + t_n) >)\ | \ ds+$$
$$+ M_2 \int_0^t \|C(s + t_n)\| \, ds.$$

Using the fact that $\lim_{n\to\infty} < b, y(t + t_n) >= 0$, the assumptions

(H 4) and (H 5) (made respectively on on the functions C and β) and Lebesgue theorem of bounded convergence, we obtain

$$\lim_{n\to\infty}\int_0^t \mid \beta(< b, y(s+t_n) >) \mid ds = 0,$$

and

$$\lim_{n\to\infty}\int_0^t \|C(s+t_n)\| \, ds = 0.$$

Since

$$\lim_{n\to\infty} \mid < e^{tA}y(t_n), \xi > - < e^{tA}z, \xi > \mid = 0,$$

then we have

$$\lim_{n\to\infty} \mid < y(t+t_n), \xi > - < e^{tA}z, \xi > \mid = 0.$$

We conclude that we have proved the existence of a sequence $(s_n)_n$ such that $s_n \to \infty$ for which $(y(s_n))_n$ converges weakly to $e^{tA}z$, so our lemma is proved. $\Diamond$

3.4. End of the proof of Th. 1.2. Let $y_0 \in H$ and let $z \in \Gamma(y_0)$. By Lemma 3.3, we have $< e^{tA}z, b >= 0$, $\forall t \geq 0$. By using assumption (H 4), we deduce that $z = 0$ and conclude that $\Gamma(y_0) = \{0\}$. Consequently, the state $y(t)$ converges weakly to 0 when $t \to \infty$.

We conclude that the system (S) is weakly stabilizable, since there exists a stabilizing feedback law $u(t) := -\beta(< b, y(t) >)$ so that the whole closed loop system

$$\dot{y}(t) = [A + C(t)]y(t) - \beta(< b, y(t) >)(By(t) + b)$$

is weakly asymptotically stable. $\Diamond$

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