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ON A CLASS OF ARITHMETIC CONVOLUTIONS INVOLVING ARBITRARY SETS OF INTEGERS

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Abstract: Let d, n be positive integers and S be an arbitrary set of positive integers. We say that d is an S-divisor of n if $d|n$ and $\gcd(d, n/d) \in S$. Consider the S-convolution of arithmetical functions given by (1.1), where the sum is extended over the S-divisors of n.

We determine the sets S such that the S-convolution is associative and preserves the multiplicativity of functions, respectively, and discuss other basic properties of it. We give asymptotic formulae with error terms for the functions $\sigma_S(n)$ and $\tau_S(n)$, representing the sum and the number of S-divisors of n, respectively, for an arbitrary S. We improve the remainder terms of these formulae and find the maximal orders of $\sigma_S(n)$ and $\tau_S(n)$ assuming additional properties of S. These results generalize, unify and sharpen previous ones.

We also pose some problems concerning these topics.

1. Introduction

Let $\mathbb{N}$ denote the set of positive integers and let S be an arbitrary subset of $\mathbb{N}$. For $n, d \in \mathbb{N}$ we say that d is an S-divisor of n if $d|n$

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and $\gcd(d, n/d) \in S$, notation $d|_S n$. Consider the S-convolution of arithmetical functions f and g defined by

$$(1.1) \qquad (f *_S g)(n) = \sum_{d|_S n} f(d)g(n/d) = \sum_{d|n} \rho_S((d, n/d))f(d)g(n/d),$$

where ρ_S stands for the characteristic function of S.

Let $\tau_S(n)$ and $\sigma_S(n)$ denote the number and the sum of S-divisors of n, respectively.

For $S = \mathbb{N}$ we obtain the Dirichlet convolution and the familiar functions $\tau(n)$ and $\sigma(n)$. For $S = \{1\}$ we have the unitary convolution and the functions $\tau^*(n)$ and $\sigma^*(n)$. These have been studied extensively in the literature, see for example [3] and its bibliography.

Among other special cases we mention here the following ones.

Let P be an arbitrary subset of the primes p and S be the multiplicative semigroup generated by $P \cup \{1\}$, i. e. $S = (P) \equiv \{1\} \cup \cup \{n > 1 : p|n \Rightarrow p \in P\}$. Then the (P)- convolution is the concept of the cross-convolution, see [7], which is a special regular convolution of Narkiewicz-type [4].

If S is the set of k-free integers, $k \geq 2$, i. e. $S = Q_k \equiv \{1\} \cup \{n > > 1 : p|n \Rightarrow p^k \nmid n\}$, then the Q_k-divisors are the k-ary divisors and (1.1) is the k-ary convolution, see [5], [6].

Let L_k denote the set of k-full integers, i. e. $L_k \equiv \{1\} \cup \{n > > 1 : p|n \Rightarrow p^k|n\}$, where $k \in \mathbb{N}, k \geq 2$. The L_k-convolution given by

$$(1.2) \qquad (f *_{L_k} g)(n) = \sum_{\substack{d|n \\ (d,n/d) \in L_k}} f(d)g(n/d)$$

seems to not have been investigated till now.

The aim of this note is to study some basic properties of the S-convolution, to give asymptotic formulae for the functions $\sigma_S(n)$ and $\tau_S(n)$ and to investigate the maximal orders of these functions.

Assuming that $1 \in S$ (then $1|_S n$ and $n|_S n$ for every $n \in \mathbb{N}$), we determine in Section 2 the subsets S such that the S-convolution is associative and preserves the multiplicativity of functions, respectively.

The most interesting property is that of associativity. It turns out that, for example, the Q_k-convolution with $k \geq 2$ is not associative, but the L_k-convolution is associative.

The L_k-convolution has also other nice properties, which are analogous to those of the Dirichlet convolution and of the unitary convolution. For example, the set of all complex valued arithmetical functions f with $f(1) \neq 0$ forms a commutative group under the L_k-convolution and the set of all nonzero multiplicative functions forms a subgroup of this group.

Furthermore, let μ_k denote the inverse with respect the L_k- convolution of the constant 1 function. We call it "k-full Möbius function", which is multiplicative and for every prime power p^a, $\mu_k(p^a) = -1$ for $1 \le a < 2k$ and $\mu_k(p^a) = \mu_k(p^{a-1}) - \mu_k(p^{a-k})$ for $a \ge 2k$.

Note that $\mu_1 \equiv \mu$ is the ordinary Möbius function. The function μ_2 takes the values $-1, 0, 1$.

We pose the following problems: Which are the values taken by μ_k? Investigate asymptotic properties of μ_k.

Note that the S-convolution is contained in the concept of the K-convolution to be defined in Section 2. Although there exist characterizations of basic properties of K-convolutions, see [2] and [3], Chapter 4, no study of (1.1) has been made in the literature.

Section 3 contains certain identities showing that for every S the S-convolution of two completely multiplicative functions can be expressed with the aid of their Dirichlet convolution and their unitary convolution, respectively.

Asymptotic formulae with error terms for the functions $\sigma_S(n)$ and $\tau_S(n)$, involving arbitrary subsets S, are given in Section 4. We show that the remainder terms can be sharpened assuming additional properties of S.

In Section 5 we determine the maximal order of $\sigma_S(n)$ assuming that S is multiplicative, i. e. $1 \in S$ and ρ_S is multitplicative, and give the maximal order of $\tau_S(n)$ for an arbitrary S with $1 \in S$.

What can be said on the maximal order of $\sigma_S(n)$ for an arbitrary subset S?

The results of Sections 4 and 5 are obtained by elementary methods, they generalize, unify and improve the corresponding known results concerning the functions $\sigma(n)$, $\tau(n)$, their unitary analogues $\sigma^*(n)$, τ^* $*(n)$, those involving k-ary divisors and the functions $\sigma_A(n)$, $\tau_A(n)$ associated with cross- convolutions, see [3], [5], [6], [7], [8].

2. Properties of the S-convolution

It is immediate that the S-convolution is commutative and distributive with respect ordinary addition for every S.

Assume in this section that $1 \in S$. Then $1|_S n$ and $n|_S n$ for every $n \in \mathbb{N}$ and denoting $\delta \equiv \rho_{\{1\}}$, i. e. $\delta(1) = 1$ and $\delta(n) = 0$ for $n > 1$, we have $f *_S \delta = f$ for every function f. This means that δ is the identity element for $*_S$.

We say that S is multiplicative if $1 \in S$ and its characteristic function ρ_S is multiplicative.

The K-convolution of arithmetical functions f and g is given by

$$(f *_K g)(n) = \sum_{d|n} K(n,d) f(d) g(n/d), \tag{2.1}$$

where K is a complex valued function defined on the set of all ordered pairs $\langle n, d\rangle$ with $n, d \in \mathbb{N}$ and $d|n$.

For $K(n,d) = \rho_S((d, n/d))$ (2.1) becomes (1.1), therefore the S-convolution is a special K-convolution.

Theorem 2.1. *The S-convolution preserves the multiplicativity of functions if and only if S is multiplicative.*

Proof. It is known ([3], Chapter 4) that the K- convolution preserves the multiplicativity if and only if

$$K(mn, de) = K(m,d)K(n,e)$$

holds for every $m, n, d, e \in \mathbb{N}$ such that $(m,n) = 1$ and $d|m, e|n$.

Hence the S-convolution has this property if and only if

$$\rho_S((de, mn/de)) = \rho_S((d, m/d))\rho_S((e, n/e)) \tag{2.2}$$

for every $m, n, d, e \in \mathbb{N}$ with $(m,n) = 1$ and $d|m, e|n$.

If S is multiplicative, then for every m, n, d, e given as above $(d, m/ /d)$ and $(e, n/e)$ are relatively prime, $(de, mn/de) = (d, m/d)(e, n/e)$ and we obtain (2.2).

Conversely, if (2.2) holds and $M, N \in \mathbb{N}$, $(M, N) = 1$ are given integers, then taking $d = M, m = M^2, e = N, n = N^2$ we obtain

$$\rho_S(MN) = \rho_S(M)\rho_S(N),$$

showing that S is multiplicative. $\diamondsuit$

Remark. It follows that all the convolutions mentioned in the Introduction preserve the multiplicativity.

Theorem 2.2. *The S-convolution is associative if and only if the following conditions hold:*

(1) S is multiplicative,

(2) for every prime p and for every $j \in \mathbb{N}$ if $p^j \in S$, then $p^\ell \in S$ for every $\ell > j$.

Remark. Condition (2) is equivalent with the following: for every prime p one of the next statements is true:

(i) $p^j \in S$ for every $j \in \mathbb{N}$,

(ii) $p^j \notin S$ for every $j \in \mathbb{N}$,

(iii) there exists $e = e(p) \in \mathbb{N}$ depending on p such that $p^j \notin S$ for every $1 \leq j < e$ and $p^j \in S$ for every $j \geq e$.

Proof. It is known ([3], Chapter 4) that the K- convolution is associative if and only if

$$K(n,d)K(d,e) = K(n,e)K(n/e, d/e)$$

holds for every $n, d, e \in \mathbb{N}$ with $d|n, e|d$.

Therefore the S-convolution is associative if and only if

$$\rho_S((d, n/d))\rho_S((e, d/e)) = \rho_S((e, n/e))\rho_S((d/e, n/d)) \tag{2.3}$$

for every $n, d, e \in \mathbb{N}$ with $d|n, e|d$.

First we show that if $*_S$ is associative, then ρ_S is multiplicative. Suppose that (2.3) is satisfied, let $M, N \in \mathbb{N}$, $(M,N) = 1$ and take $n = M^2N^2, d = MN, e = M$. Then we have

$$\rho_S((MN, MN))\rho_S((M,N)) = \rho_S((M, MN^2))\rho_S((N, MN)),$$

hence

$$\rho_S(MN) = \rho_S(M)\rho_S(N).$$

Assume now that S is multiplicative. Then, taking $n = p^a, d = p^b, e = = p^c$, (2.3) is equivalent to

$$\rho_S((p^b, p^{a-b}))\rho_S((p^c, p^{b-c})) = \rho_S((p^c, p^{a-c}))\rho_S((p^{b-c}, p^{a-b})) \tag{2.4}$$

for every prime p and for every $0 \leq c \leq b \leq a$. Note that it is sufficient to require (2.4) for every $0 < c < b < a$.

Suppose that $p^j \in S$, where $j \in \mathbb{N}$ and let $\ell > j$. We show that $p^\ell \in S$.

Case 1. $\ell < 2j$. Take $a = \ell + 2j, b = \ell + j, c = \ell$. From (2.4) we obtain

$$\rho_S((p^{\ell+j}, p^j))\rho_S((p^\ell, p^j)) = \rho_S((p^\ell, p^{2j}\rho_S((p^j, p^j)),$$

$$\rho_S(p^j)\rho_S(p^j) = \rho_S(p^\ell)\rho_S(p^j),$$

giving $\rho_S(p^\ell) = 1$.

Case 2. $\ell \geq 2j$. Now let $a = 2\ell, b = \ell, c = \ell - j$. From (2.4) we have

$$\rho_S((p^\ell, p^\ell))\rho_S((p^{\ell-j}, p^j)) = \rho_S((p^{\ell-j}, p^{\ell+j}))\rho_S((p^j, p^\ell)),$$

$$\rho_S(p^\ell)\rho_S(p^j) = \rho_S(p^{\ell-j})\rho_S(p^j),$$

thus

$$\rho_S(p^\ell) = \rho_S(p^{\ell-j}). \tag{2.5}$$

If $\ell = kj + r$, where $k \geq 2$ and $0 \leq r < j$, then applying (2.5) we have

$$\rho_S(p^\ell) = \rho_S(p^{\ell-j}) = \rho_S(p^{\ell-2j}) = ... = \rho_S(p^{j+r}) = 1,$$

where $j \leq j + r < 2j$ and we use the result of Case 1.

In order to complete the proof we show that if S is multiplicative and condition (2) holds, then we have (2.4) for every $0 < c < b < a$.

Cosider the cases of the Remark of above. For (i) and (ii) (2.4) holds trivially. In case (iii) if $p^j \notin S$ for every $1 \leq j \leq e-1$ and $p^j \in S$ for every $j \geq e$, then (2.4) means that the statements "[($b \geq e$ and $a - b \geq e$) and ($c \geq e$ and $b - c \geq e$)]" and "[($c \geq e$ and $a - c \geq e$) and ($b - c \geq e$ and $a - b \geq e$)]" are equivalent. A quick check shows that this is true. ◊

Remark. From Th. 2.2 we obtain that the Q_k-convolution ($k \geq 2$) is not associative, but the L_k-convolution and the (P)-convolution defined in the Introduction are associative.

Theorem 2.3. *If conditions* (1) *and* (2) *of* Th. 2.2 *hold, then the set of all complex valued arithmetical functions forms a commutative (and associative) ring with identity with respect to ordinary addition and S-convolution (in particular L_k convolution).*

This ring has no divisors of zero if and only if $S = \mathbb{N}$, i. e. $*_S$ is the Dirichlet convolution.

Proof. The first part of this result follows at once from Th. 2.2 and from the previous remarks.

Furthermore, it is well-known that for the Dirichlet convolution there are no divisors of zero. Conversely, suppose that $S \neq \mathbb{N}$ satisfies conditions (1) and (2) of Th. 2.2. Then there exists a prime p such that $p \notin S$ and the following functions are divisors of zero:

$$f(n) = g(n) = \begin{cases} 1, & \text{if } n = p, \\ 0, & \text{otherwise.} \end{cases} \qquad \diamondsuit$$

Theorem 2.4. *If conditions* (1) *and* (2) *of* Th. 2.2 *hold, then the set of all complex valued arithmetical functions f with $f(1) \neq 0$ forms a commutative group under S-convolution (in particular L_k-convolution) and the set of all nonzero multiplicative functions forms a subgroup of this group.*

Proof. This yields in a similar manner as in case of the Dirichlet convolution and unitary convolution or in general for certain K-convolutions, see [3], Chapter 4. $\Diamond$

Consider now the "k-full"-convolution corresponding to $S = L_k$, the set of k-full numbers. Let μ_k denote the "k-full Möbius function", representing the inverse of the function $I(n) = 1, n \in N$ with respect to this convolution. According to Th. 2.4 μ_k is multiplicative and a short computation shows that for every prime power p^a,

$$\mu_k(p^a) = -1, \quad 1 \le a < 2k$$

and

$$\mu_k(p^a) = \mu_k(p^{a-1}) - \mu_k(p^{a-k}), \quad a \ge 2k.$$

Observe that $\mu_1 \equiv \mu$ is the ordinary Möbius function.

For the "squarefull Möbius function" μ_2 (case $k = 2$) we have $\mu_2(p) = \mu_2(p^2) = \mu_2(p^3) = -1$ and

$$\mu_2(p^a) = \mu_2(p^{a-1}) - \mu_2(p^{a-2}), \quad a \ge 4.$$

Therefore, $\mu_2(p) = \mu_2(p^2) = \mu_2(p^3) = -1, \mu_2(p^4) = 0, \mu_2(p^5) = = \mu_2(p^6) = 1, \mu_2(p^7) = 0, \mu_2(p^8) = \mu_2(p^9) = -1, \mu_2(p^{10}) = 0, \dots$.

The values taken by μ_2 are $-1, 0, 1$. This is not true for μ_3, since $\mu_3(p^a) = -1$ for $1 \le a \le 5$, $\mu_3(p^6) = 0, \mu_3(p^7) = 1, \mu_3(p^8) = \mu_3(p^9) = = 2, \mu_3(p^{10}) = 1, \mu_3(p^{11}) = -1, \mu_3(p^{12}) = -3, \mu_3(p^{13}) = -4, \dots$.

We pose the following problems: Which are the values taken by μ_k? Investigate asymptotic properties of μ_k. Does it posses a mean value?

3. Identities

For an arbitrary $S \subseteq \mathbb{N}$ let μ_S be the Möbius function of S defined by

$$\sum_{d|n} \mu_S(n) = \rho_S(n), \quad n \in \mathbb{N}, \tag{3.1}$$

see [1], therefore, by Möbius inversion,

$$\mu_S(n) = \sum_{d|n} \rho_S(d)\mu(n/d), \quad n \in \mathbb{N}, \tag{3.2}$$

where $\mu \equiv \mu_{\{1\}}$ is the ordinary Möbius function.

The zeta function ζ_S is defined by

$$\zeta_S(z) = \sum_{n=1}^{\infty} \frac{\rho_S(n)}{n^z}.$$

It follows that $\zeta_{\mathbb{N}} \equiv \zeta$ is the Riemann zeta function and

$$\sum_{n=1}^{\infty} \frac{\mu_S(n)}{n^z} = \frac{\zeta_S(z)}{\zeta(z)} \quad (z > 1). \tag{3.3}$$

Theorem 3.1. *If $S \subseteq \mathbb{N}$ and f and g are completely multiplicative functions, then for every $n \in \mathbb{N}$,*

$$(f * sg)(n) = \sum_{d^2 \mid n} \mu_S(d) f(d) g(d) (f * g)(n/d^2), \tag{3.4}$$

where $* \equiv *_{\mathbb{N}}$ is the Dirichlet convolution and

$$(f * sg)(n) = \sum_{d^2 \mid n} \rho_S(d) f(d) g(d) (f \times g)(n/d^2), \tag{3.5}$$

where $\times \equiv *_{\{1\}}$ is the unitary convolution.
Proof. Using (3.1) we have for every $n \in \mathbb{N}$,

$$(f * sg)(n) = \sum_{de=n} \rho_S((d,e)) f(d) g(e) = \sum_{de=n} \left(\sum_{j \mid (d,e)} \mu_S(j) \right) f(d) g(e).$$

Hence with $d = ja, e = jb$,

$$(f * sg)(n) = \sum_{j^2 ab = n} \mu_S(j) f(ja) g(jb) = \sum_{j^2 ab = n} \mu_S(j) f(j) f(a) g(j) g(b) =$$

$$= \sum_{j^2 \ell = n} \mu_S(j) f(j) g(j) \sum_{ab = \ell} f(a) g(b) = \sum_{j^2 \ell = n} \mu_S(j) f(j) g(j) (f * g)(\ell),$$

which is (3.4).

Furthermore,

$$(f * sg)(n) = \sum_{de=n} \rho_S((d,e)) f(d) g(e) = \sum_{a \in S} \sum_{\substack{de=n \\ (d,e)=a}} f(d) g(e) =$$

$$= \sum_{a} \rho_S(a) \sum_{\substack{de=n \\ (d/a, e/a)=1}} f(d) g(e).$$

With $d = ai, e = bj$ we get

$$(f * sg)(n) = \sum_{\substack{a^2 ij=n \\ (i,j)=1}} \rho_S(a)f(a)g(a)f(i)g(j) =$$

$$= \sum_{a^2 b=n} \rho_S(a)f(a)g(a) \sum_{\substack{ij=b \\ (i,j)=1}} f(i)g(j) = \sum_{a^2 b=n} \rho_S(a)f(a)g(a)(f \times g)(b),$$

giving (3.5). ◊

Theorem 3.2. *If $S \subseteq \mathbb{N}$, then for every $n \in \mathbb{N}$,*

$$\tau_S(n) = \sum_{d^2|n} \mu_S(d)\tau(n/d^2) = \sum_{d^2|n} \rho_S(d)\tau^*(n/d^2), \tag{3.6}$$

$$\sigma_S(n) = \sum_{d^2|n} \mu_S(d)d\sigma(n/d^2) = \sum_{d^2|n} \rho_S(d)d\sigma^*(n/d^2). \tag{3.7}$$

Proof. This yields at once from Th. 3.1 applied for $f(n) = g(n) = 1$ and $f(n) = n, g(n) = 1$, respectively. ◊

Note that if S is multiplicative, then the functions $\tau_S(n)$ and $\sigma_S(n)$ are also multiplicative. The generalized Euler function $\phi_S(n) = \#\{k \in \in \mathbb{N} : k \leq n, (k,n) \in S\}$ was considered in [1] and one has $\phi_S = \mu_S * *E = \rho_S * \phi$, where $E(n) = n, n \in \mathbb{N}$ and $\phi \equiv \phi_{\{1\}}$ is the ordinary Euler function, see also [7].

4. Asymptotic formulae

The following asymptotic formulae generalize and improve the known formulae concerning the functions $\sigma(n)$, $\tau(n)$, their unitary analogues, those involving k-ary divisors and the functions $\sigma_A(n)$, $\tau_A(n)$ associated with cross-convolutions, cf. [3], Chapter. 6; [5], Cor. 3.1.1; [6], Cor. 3.1; [7], Th. 12; [7i], Th. 2; see also [9], Cor. 1.

Theorem 4.1. *If $S \subseteq \mathbb{N}$, then*

$$\sum_{n \leq x} \sigma_S(n) = \frac{\zeta(2)\zeta_S(3)}{2\zeta(3)} x^2 + R_S(x), \tag{4.1}$$

where the remainder term can be evaluated as follows:

(1) $R_S(x) = O(x \log^{8/3} x)$ *for an arbitrary S,*

(2) $R_S(x) = O(x \log^{5/3} x)$ *for an S such that $\sum_{n \in S} \frac{1}{n} < \infty$ (in particular for every finite S) and for every multiplicative S,*

(3) $R_S(x) = O(x\log^{2/3} x)$ *for every multiplicative* S *such that* $\sum\limits_{p\notin S} \frac{1}{p} < \infty$ *(in particular if the set* $\{p : p \notin S\}$ *is finite).*

Proof. We have from (3.7),

$$\sum_{n\le x} \sigma_S(n) = \sum_{d\le\sqrt{x}} \mu_S(d)d \sum_{e\le x/d^2} \sigma(e).$$

Applying now the well-known result of Walfisz [10],

$$\sum_{n\le x} \sigma(n) = \frac{\zeta(2)}{2}x^2 + O(x\log^{2/3} x)$$

we obtain

$$\sum_{n\le x} \sigma_S(n) = \sum_{d\le\sqrt{x}} \mu_S(d)d \left(\frac{\zeta(2)x^2}{2d^4} + O\left(\frac{x}{d^2}(\log\frac{x}{d^2})^{2/3}\right)\right) =$$

$$= \frac{\zeta(2)x^2}{2}\sum_{d=1}^{\infty} \frac{\mu_S(d)}{d^3} + O\left(x^2 \sum_{d>\sqrt{x}} \frac{|\mu_S(d)|}{d^3}\right) +$$

$$+ O\left(x(\log x)^{2/3} \sum_{d\le\sqrt{x}} \frac{|\mu_S(d)|}{d}\right).$$

For the main term apply (3.3) and the given error term yields from the next statements:

(a) For an arbitrary $S \subseteq \mathbb{N}$, $|\mu_S(n)| \le \sum\limits_{d|n} \rho_S(d) \le \tau(n)$ for every $n \in \mathbb{N}$ and

$$\sum_{n\le x} \frac{|\mu_S(n)|}{n} \le \sum_{d\le x} \frac{\rho_S(d)}{d} \sum_{e\le x/d} \frac{1}{e} =$$

$$= O\left(\log x \sum_{d\le x} \frac{\rho_S(d)}{d}\right) = \begin{cases} O(\log x), & \text{if } \sum\limits_{n=1}^{\infty} \frac{\rho_S(n)}{n} < \infty, \\ O(\log^2 x), & \text{otherwise.} \end{cases}$$

(b) If S is multiplicative, then μ_S is multiplicative too, $\mu_S(p^a) = \rho_S(p^a) - \rho_S(p^{a-1})$ for every prime power p^a ($a \ge 1$) and $\mu_S(n) \in \{-1, 0, 1\}$ for each $n \in \mathbb{N}$.

(c) Suppose S is multiplicative. Then

$$\sum_p \sum_{k=1}^{\infty} \frac{|\mu_S(p^k)|}{p^k} \le \sum_p \left(\frac{|\rho_S(p)-1|}{p} + \sum_{k=2}^{\infty} \frac{1}{p^k} \right) =$$

$$= \sum_{p\in S} \frac{1}{p(p-1)} + \sum_{p\notin S} \frac{1}{p-1} \le$$

$$\le 2\left(\sum_{p\in S} \frac{1}{p^2} + \sum_{p\notin S} \frac{1}{p} \right) < \infty \quad \text{if} \quad \sum_{p\notin S} \frac{1}{p} < \infty.$$

It follows that in this case the series $\sum_{n=1}^{\infty} \frac{|\mu_S(n)|}{n}$ is convergent. ◊

Theorem 4.2. *If S is an arbitary subset of $\mathbb{N}$, then*

$$\sum_{n\le x} \tau_S(n) =$$

$$(4.2) \quad = \frac{\zeta_S(2)}{\zeta(2)} x \left(\log x + 2\gamma - 1 + \frac{2\zeta_S'(2)}{\zeta_S(2)} - \frac{2\zeta'(2)}{\zeta(2)} \right) + O(\sqrt{x}\log^2 x),$$

where γ is the Euler constant and $\zeta_S'(z)$ is the derivative of $\zeta_S(z)$.

This result follows applying the first identity of (3.6) and using Dirichlet's formula

$$\sum_{n\le x} \tau(n) = x(\log x + 2\gamma - 1) + O(x^{\alpha}).$$

The remainder term of (4.2) can be improved assuming further properties of S. For example, if S is multiplicative, then the error term is $O(\sqrt{x}\log x)$ and if S (i. e. ρ_S) is completely multiplicative and $\{p : : p \notin S\}$ is a finite set, then the error term is $O(x^{\alpha})$. We do not go into details.

5. Maximal orders

Generalizing the result of Gronwall concerning the function $\sigma(n)$ we prove the following theorem.

Theorem 5.1. *Let S be an arbitrary multiplicative subset. Denote by P the set of primes p such that $p^j \in S$ for every $j \in \mathbb{N}$. For every $p \notin P$ let $s(p) \in \mathbb{N}$ denote the least exponent j such that $p^j \notin S$ (i. e. $p^j \in S$ for every $1 \le j < s(p)$ and $p^{s(p)} \notin S$).*

Then

$$\limsup_{n\to\infty} \frac{\sigma_S(n)}{n\log\log n} = e^{\gamma} \prod_{p\notin P} \left(1 - \frac{1}{p^{2s(p)}}\right).$$

Proof. For every $p \in P, a \in \mathbb{N}$ and for every $p \notin P, a < 2s(p)$ the S-divisors of p^a are all divisors $1, p, p^2, ..., p^a$. Hence $\sigma_S(p^a) = \sigma(p^a) = = 1 + p + p^2 + ... + p^a$.

For every $p \notin P$ and $a \geq 2s(p)$ the numbers $p^{s(p)}$ and $p^{a-s(p)}$ are certainly not S-divisors of p^a, since $(p^{a-s(p)}, p^{s(p)}) = p^{s(p)} \notin S$. Therefore $\sigma_S(p^a) < (1 + p + p^2 + ... + p^{a-s(p)-1}) + (p^{a-s(p)+1} + ... + +p^a) < p^{a-s(p)} + p^{a-s(p)+1} + ... + p^a \leq p^{a-2s(p)+1} + p^{a-2s(p)+2} + ... + p^a$.

We obtain that

$$\frac{\sigma_S(p^a)}{p^a} \leq 1 + \frac{1}{p} + \frac{1}{p^2} + ... + \frac{1}{p^{2s(p)-1}} \tag{4.3}$$

holds for every prime power p^a with $p \notin P$ with equality for $a = 2s(p) - - 1$.

Also, for every $p \in P, a \in \mathbb{N}$,

$$\frac{\sigma_S(p^a)}{p^a} < \left(1 - \frac{1}{p}\right)^{-1}. \tag{4.4}$$

We show that

$$\frac{\sigma_S(n)}{n} \leq e^{\gamma} \prod_{p\notin P} \left(1 - \frac{1}{p^{2s(p)}}\right) \log\log n(1 + o(1)) \qquad \text{as} \qquad n \to \infty.$$

Using (4.3) and (4.4) we have for every $n \geq 1$,

$$\frac{\sigma_S(n)}{n} \leq \prod_{\substack{p|n \\ p\in P}} \left(1 - \frac{1}{p}\right)^{-1} \prod_{\substack{p|n \\ p\notin P}} \left(1 + \frac{1}{p} + \frac{1}{p^2} + ... + \frac{1}{p^{2s(p)-1}}\right) =$$

$$= \prod_{\substack{p|n \\ p\leq \log n \\ p\in P}} \left(1 - \frac{1}{p}\right)^{-1} \prod_{\substack{p|n \\ p>\log n \\ p\in P}} \left(1 - \frac{1}{p}\right)^{-1} \times$$

$$\times \prod_{\substack{p|n \\ p\leq \log n \\ p\notin P}} \left(1 + \frac{1}{p} + \frac{1}{p^2} + ... + \frac{1}{p^{2s(p)-1}}\right) \times$$

$$\times \prod_{\substack{p|n \\ p>\log n \\ p\notin P}} \left(1+\frac{1}{p}+\frac{1}{p^2}+\ldots+\frac{1}{p^{2s(p)-1}}\right) \leq$$

$$\leq \prod_{\substack{p\leq\log n \\ p\in P}} \left(1-\frac{1}{p}\right)^{-1} \prod_{\substack{p\leq\log n \\ p\notin P}} \left(1+\frac{1}{p}+\frac{1}{p^2}+\ldots+\frac{1}{p^{2s(p)-1}}\right) \times$$

$$\times \prod_{\substack{p|n \\ p>\log n \\ p\in P}} \left(1-\frac{1}{p}\right)^{-1} \prod_{\substack{p|n \\ p>\log n \\ p\notin P}} \left(1-\frac{1}{p}\right)^{-1} =$$

$$= \prod_{\substack{p\leq\log n \\ p\notin P}} \left(1-\frac{1}{p^{2s(p)}}\right) \prod_{p\leq\log n} \left(1-\frac{1}{p}\right)^{-1} \prod_{\substack{p|n \\ p>\log n}} \left(1-\frac{1}{p}\right)^{-1} \leq$$

$$\leq \prod_{\substack{p\leq\log n \\ p\notin P}} \left(1-\frac{1}{p^{2s(p)}}\right) \prod_{p\leq\log n} \left(1-\frac{1}{p}\right)^{-1} \prod_{\substack{p|n \\ p>\log n}} \left(1-\frac{1}{\log n}\right)^{-1} =$$

$$= e^{\gamma} \prod_{p\notin P} \left(1-\frac{1}{p^{2s(p)}}\right) \log\log n(1+o(1)),$$

applying Mertens' theorem $\prod_{p\leq x}(1-\frac{1}{p}) = \frac{e^{-\gamma}}{\log x}(1+o(1))$ as $x\to\infty$, and the fact that $\#\{p : p|n, p>\log n\} \leq \frac{\log n}{\log\log n}$.

Now we show that this upper bound is asymptotically attained.

For a given $\varepsilon > 0$ choose t so large such that

$$\prod_{p>t}\left(1-\frac{1}{p^2}\right) \geq 1-\varepsilon.$$

For this t choose an exponent $a \geq 1$ such that

$$\prod_{p\leq t}\left(1-\frac{1}{p^a}\right) \geq 1-\varepsilon.$$

Consider the sequence $(n_k)_{k\geq 1}$ given by

$$n_k = \prod_{\substack{p\leq t \\ p\in P}} p^{a-1} \prod_{\substack{p\leq t \\ p\notin P}} p^{2s(p)-1} \prod_{t<p\leq e^k} p.$$

We obtain

$$\frac{\sigma_S(n_k)}{n_k} = \prod_{\substack{p\le t\\ p\in P}} \left(1+\frac{1}{p}+\frac{1}{p^2}+\ldots+\frac{1}{p^{a-1}}\right)\times$$

$$\times \prod_{\substack{p\le t\\ p\notin P}} \left(1+\frac{1}{p}+\frac{1}{p^2}+\ldots+\frac{1}{p^{2s(p)-1}}\right) \prod_{t<p\le e^k} \left(1+\frac{1}{p}\right) \ge$$

$$\ge \prod_{p\le t} \left(1-\frac{1}{p^a}\right) \prod_{p\notin P} \left(1-\frac{1}{p^{2s(p)}}\right) \prod_{p>t} \left(1-\frac{1}{p^2}\right) \prod_{p\le e^k} \left(1-\frac{1}{p}\right)^{-1} \ge$$

$$\ge (1-\varepsilon)^2 \prod_{p\notin P} \left(1-\frac{1}{p^{2s(p)}}\right) e^\gamma k(1+o(1)) \qquad \text{as} \qquad k\to\infty,$$

applying Mertens' theorem again.

Furthermore, considering the Chebysev function $\theta(x) = \sum_{p\le x} \log p$ and using the elementary estimate $\theta(x) = O(x)$, we get

$$\log n_k \le O(1) + \theta(e^k) = O(e^k).$$

Hence, for sufficiently large k,

$$\log\log n_k \le O(1) + k < (1+\varepsilon)k.$$

Therefore

$$\limsup_{k\to\infty} \frac{\sigma_S(n_k)}{n_k \log\log n_k} \ge \frac{(1-\varepsilon)^2}{1+\varepsilon} e^\gamma \prod_{p\notin P} \left(1-\frac{1}{p^{2s(p)}}\right),$$

and the proof is complete. ◊

A direct consequence of Th. 5.1 is the following result.

Theorem 5.2. *Let S be an arbitarary multiplicative subset and suppose that there exists $s\in\mathbb{N}$ such that for every prime p, $p^j\in S$ for every $1\le j<s$ and $p^s\notin S$. Then*

$$\limsup_{n\to\infty} \frac{\sigma_S(n)}{n\log\log n} = \frac{e^\gamma}{\zeta(2s)}.$$

This result can be applied for $S = Q_k$ (case $s = k \ge 1$), for $S = L_k$ (case $s = 1$).

What is the maximal order of $\sigma_S(n)$ for an arbitrary subset S ?

Theorem 5.3. *Let S be an arbitrary subset such that $1\in S$. Then*

$$\limsup_{n\to\infty} \frac{\log \tau_S(n) \log\log n}{\log n} = \log 2. \tag{4.5}$$

Proof. It is well-known that this result holds for the function $\tau(n)$ (case $S = \mathbb{N}$) and that for the sequence $n_k = p_1 p_2 ... p_k$, where p_i is the i-th prime,

$$\lim_{k\to\infty} \frac{\log \tau(n_k) \log\log n_k}{\log n_k} = \log 2.$$

Taking into account that if $1 \in S$, then $\tau_S(n) = \tau(n)$ for every squarefree n and $\tau_S(n) \leq \tau(n)$ for every $n \in \mathbb{N}$, (4.5) follows at once. ◊

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