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# SOME GROUP THEORETIC PROBLEMS INSPIRED BY RING THEORETIC ANALOGIES

Péter P. **Pálfy**

*Department of Algebra and Number Theory, Eötvös University, Budapest, P.O.Box 120, H-1518 Hungary*

Ottó **Steinfeld**†

*Mathematical Institute of the Hungarian Academy of Sciences, Reáltanoda u. 13–15, 1053 Budapest, Hungary*

**Dedicated to Richárd Wiegandt on the occasion of his 70th birthday**

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**Abstract**: We characterize the solvable groups with the property that $[[A,B],C] = [A,[B,C]]$ holds for all normal subgroups $A$, $B$, $C$. We give an example that this property is not inherited by normal subgroups in general. We also construct an infinite group $G$ such that $[A,B] = A \cap B$ holds for all normal subgroups $A$, $B$ of $G$ but this is no longer true in a normal subgroup of $G$.

Large part of this paper was written in 1990 just before the untimely death of Ottó Steinfeld (1924–1990). Then the project was aban-

*E-mail address:* ppp@cs.elte.hu
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doned and it has been resuscitated only recently. With the addition of some new results it is now hopefully ripe enough for publication.

We are going to study some properties of the operation taking the mutual commutator of two subgroups. For two subgroups $A$, $B$ of a group $G$ let $[A, B]$ be the subgroup generated by the commutators $[a, b] = a^{-1}b^{-1}ab$ ($a \in A$, $b \in B$). This binary operation defined on the subgroup lattice of $G$ turns it into a so-called groupoid-lattice (see [11], [12], [15]). There is an analogous operation for subrings $A$, $B$ of a ring, namely, let $AB$ be the subring generated by the products $ab$ ($a \in A$, $b \in B$). This way the lattice of subrings becomes a groupoid-lattice as well. Since the commutator of two normal subgroups is again a normal subgroup, the lattice of normal subgroups is also a groupoid-lattice. Similarly, the ideals of a ring form a groupoid-lattice, too. There is a number of interesting analogies between the operations $[A, B]$ in groups and $AB$ in rings. An early study in this direction was written by Eugene Schenkman [10]. (We note that the example given in Remark 12 of [10] contains an error.) The analogy with primary decomposition of ideals in Noetherian rings was also investigated by Kurata [5]. Left, right, and two-sided ideals of a ring $R$ are defined by the properties $RX \subseteq X$, $XR \subseteq X$, and both of them. In groups $[X, Y] = [Y, X]$, hence we can single out only those subgroups $X$ for which $[X, G] \subseteq X$, i.e., normal subgroups. (In a groupoid-lattice such elements are called absorbents.)

## 1. Associativity

For ideals (even for quasi-ideals, see [14]) we always have $(AB)C = A(BC)$. (We have to warn the reader that this is not true for subrings in general.) The analogous result does not hold for normal subgroups, for example, we can take $G = A = B = S_3$ and $C = A_3$. This will motivate our first question.

**Definition.** We say that the groupoid-lattice of normal subgroups of $G$ is *associative* if

(A) $$[[A, B], C] = [A, [B, C]]$$

holds for arbitrary normal subgroups $A$, $B$, $C$ of $G$. We say that the groupoid-lattice of all subgroups of $G$ is associative if (A) holds for all subgroups $A$, $B$, $C$.

The problem of characterizing groups with associative groupoid-lattice of normal subgroups was formulated, e.g., in [15, Problem 2].

If we require (A) for all subgroups, the answer is known. An unpublished result of Ian Macdonald and Bernhard Neumann states the following (see [15], Prop. 2).

**Proposition 1** (Macdonald and Neumann). *Equation* (A) *holds for all subgroups $A$, $B$, $C$ of a group $G$, if and only if $G$ is a 2-Engel group, that is, $[[x, y], y] = 1$ holds for all $x, y \in G$.*

Hence the groups with associative groupoid-lattice of subgroups form a variety. The same is not true for the groups with associative groupoid-lattice of normal subgroups. Namely, every simple group trivially satisfies (A) for all normal subgroups, and every group (with a nonassociative groupoid-lattice of subgroups) can be embedded into a simple group. We will even show that this property is not inherited by normal subgroups in general, see Ex. 1 below.

Now we describe the finite solvable groups satisfying (A) for all normal subgroups, thus giving a partial solution of Problem 2 in [15].

**Theorem 1.** *Let $G$ be a finite solvable group. The following are equivalent:*

(1) $[[A, B], C] = [A, [B, C]]$ *for all normal subgroups* $A, B, C \lhd G$;
(2) $[[A, B], C] = [A, [B, C]]$ *for all subgroups* $A, B, C \leq G$;
(3) $[[x, y], y] = 1$ *for all elements* $x, y \in G$.

**Proof.** (2) $\Longrightarrow$ (1) is trivial; (3) $\Longleftrightarrow$ (2) is the already mentioned unpublished result of I. D. Macdonald and B. H. Neumann. We are going to prove that (1) implies (3).

Assume that (1) holds. First we show that $G$ is nilpotent. Let $G$ be a minimal counterexample. Then $\mathbf{Z}(G) = 1$, since the condition is inherited by quotient groups. Let $A \lhd G$ be a minimal normal subgroup in $G$. By solvability, $A$ is abelian. We have $\mathbf{C}_G(A) < G$. Let $B/\mathbf{C}_G(A) \lhd \lhd G/\mathbf{C}_G(A)$ be a minimal normal subgroup. This is also abelian, hence $[B, B] \leq \mathbf{C}_G(A)$. So $[[B, B], A] = 1$. On the other hand, $[B, A]$ is a nontrivial normal subgroup contained in $A$, so — by minimality — $[B, A] = A$. Thus $[B, [B, A]] = A$, a contradiction.

Now let the nilpotence class of $G$ be $k$. Then any commutator of length $k + 1$ is trivial, and any fixed bracketing of $[x_1, \ldots, x_k]$ gives a homomorphism of $G^k$ into an abelian group (the last nontrivial term of the lower central series). Hence only the cosets by $G' = [G, G]$ have to be considered. For $g_1, \ldots, g_k \in G$ we have

$$[\langle g_1, G' \rangle, \ldots, \langle g_k, G' \rangle] = \langle [g_1, \ldots, g_k] \rangle,$$

where the same bracketing has to be taken on both sides. However, each $\langle g_i, G' \rangle \lhd G$, so we can apply associativity on the left hand side (by induction (1) implies it for an arbitrary number of normal subgroups), so it does not matter which bracketing is used on the left hand side. Therefore, it has no effect on the right hand side either. In particular, if $g_i = g_{i+1}$, we can use a bracketing containing the subterm $[g_i, g_{i+1}]$, hence in this case the commutator is trivial under arbitrary bracketing. Now assume for contradiction that the nilpotence class $k \geq 4$. Condition (1) is inherited by quotient groups, so it holds for the largest quotient of $G$ of nilpotence class 4 as well. So this quotient group satisfies $[[[x,y],y],z] = 1$ and $[[[x,y],z],z] = 1$. By a result of Hermann Heineken (see [4], p. 293) these two laws imply that the nilpotence class of the group is at most 3, a contradiction. So the nilpotence class of $G$ is at most 3, and the previous argument yields that the Engel condition $[[x,y],y] = 1$ holds in $G$. $\Diamond$

Next we show that that the associativity of the groupoid-lattice of normal subgroups of $G$ is not inherited by normal subgroups of $G$.

**Example 1.** *There exists a group $G$ and a normal subgroup $P \lhd G$ such that the groupoid-lattice of normal subgroups of $G$ is associative, but that of $P$ is not.*

**Proof.** Let $p > 5$ be a prime and let $F$ be the free group on the generators $g_1, \ldots, g_5$ in the variety $\mathcal{V}$ of nilpotent groups of class 3 and of exponent $p$ (i.e., defined by the laws $[[[x_0,x_1],x_2],x_3] = 1$, $x^p = 1$). Let the alternating group $A_5$ act on $F$ by permuting the given generators. The kernel $K$ of the homomorphism $F \to C_p$ mapping each $a_i$ to the same generator of the cyclic group of order $p$ is obviously $A_5$-invariant. It is routine to calculate that the centralizer $C = \mathbf{C}_K(A_5)$ has order $p$ and it is contained in $\mathbf{Z}(K)$. Let $\bar{P}$ be a minimal $A_5$-invariant subgroup of $F$ such that $F'\bar{P} = K$. If $\bar{P} \geq C$, then let $P = \bar{P}/C$, otherwise put $P = \bar{P}$. Since $A_5$ acts on $P$, we can form the semidirect product $G = PA_5$.

Now we have $\mathbf{C}_P(A_5) = 1$, so $[N,G] = N$ for every $N \lhd G$, $N \leq P$. Since $A_5$ acts irreducibly on $K/F'$, the minimal choice of $\bar{P}$ and the complete reducibility of the action of $A_5$ on $\bar{P}/\bar{P}'$ yield that $\bar{P}' = \bar{P} \cap F'$ and $P/P' \cong \bar{P}/\bar{P}' \cong K/F'$ even as $A_5$-modules. So $A_5$ acts irreducibly on $P/P'$. Hence if $P' \leq N \lhd G$, then $N$ is one of $P'$, $P$, or $G$. Since $P$ has exponent $p$, its Frattini subgroup is just $P'$. If $N \lhd G$ is such that $NP' \geq P$ then $N \geq P$. Otherwise, we have $N \leq P'$.

Now we can show that the commutator is an associative operation on the set of normal subgroups of $G$. Let $X$, $Y$, and $Z \lhd G$. If one of them is $G$, then (A) obviously holds, since $[N,G] = N$ for every $N \lhd G$.

So we may assume that $X, Y, Z \leq P$. If one of them is contained in $P'$, then $[[X,Y],Z] = [X,[Y,Z]] = 1$, since the nilpotence class of $P$ is at most 3. So the only case remaining is when all of them are equal to $P$, but then the statement is trivial.

If we take four elements of $\bar{P}$ that are linearly independent in the quotient group $\bar{P}/\bar{P} \cap F'$, then we see that $\bar{P}$ contains a free group on four generators from the variety $\mathcal{V}$, therefore $P$ has nilpotence class 3. Hence $P$ does not satisfy $[[x,y],y] = 1$, so in virtue of Th. 1 the groupoid-lattice of normal subgroups of $P$ is not associative. ◊

## 2. Regularity

An important class of rings is the class of von Neumann regular rings. By definition this means (see [8]) that for every $r \in R$ there exists an $x \in R$ such that $r = rxr$. The following nice characterization is due to L. G. Kovács.

**Proposition 2** (L. G. Kovács [6]). *A ring is von Neumann regular, if and only if $RL = R \cap L$ for every right ideal $R$ and left ideal $L$ of the ring.*

This motivates our definition.

**Definition.** We call a group $G$ *regular*, if

$$[N, M] = N \cap M \tag{R}$$

for every pair of normal subgroups $N$, $M \lhd G$.

A similar notion in universal algebra has been introduced by Joachim Hagemann and Christian Herrmann [3], p. 243. They called an algebra *neutral* if the commutator of arbitrary congruences $\alpha$ and $\beta$ coincides with their intersection.

Note that in a regular group the groupoid-lattice of normal subgroups is obviously associative.

It is easy to see that a group is regular iff $[N, N] = N$ for every $N \lhd G$. For finite groups it is equivalent to the property that every chief factor of $G$ is nonabelian. Since chief factors are direct products of isomorphic simple groups, it is further equivalent to the property that every composition factor of $G$ is nonabelian. So it follows that every normal subgroup of a finite regular group is also regular. For regular rings it is very easy to see that the analogous statement is true without any finiteness requirement (see [2], Lemma 1.3). In contrast, we are going to give an example of an infinite regular group containing a non-regular normal subgroup, hereby solving Problem 4 from [15].

**Example 2.** There exists an infinite regular group $G$ containing a normal subgroup $M$ that is not regular.

**Proof.** We will use McLain's group, see [9], pp. 347–349. McLain's group is constructed in the following way. We take an arbitrary field $F$ and a countably infinite dimensional vector space over $F$ with basis vectors $b_q$ indexed by rational numbers. Then $M$ is the group of linear transformations generated by all transformations of the form $1 + ae_{qr}$ where $a \in F$, $q < r$ are rational numbers, $e_{qr}b_r = b_q$ and $e_{qr}b_t = 0$ for all $t \neq r$. This group has abelian normal subgroups and it is characteristically simple. Let $R$ be the group consisting of those order preserving permutations of the rational numbers which have bounded support (that is, they are identical outside a finite interval). One can check that $R$ is a simple group. Moreover, the proof showing that $M$ is characteristically simple may be carried out using only outer automorphisms of $M$ induced by elements of $R$. This yields that the only normal subgroups of the semidirect product $G = MR$ are 1, $M$, and $G$, so $G$ is regular. However, $M$ has abelian normal subgroups, so it is not regular. $\diamondsuit$

## 3. Subnormal subgroups

Recall that a subring $M$ of an associative ring $R$ is called accessible if there exists a chain of subrings $M = M_0 \leq M_1 \leq \cdots \leq M_{r-1} \leq M_r = R$ such that $M_i \lhd M_{i+1}$ for all $i = 0, 1, \ldots, r-1$. An important result of Reinhold Baer [1] gives that for any accessible subring (he used the term meta ideal of finite index) there exists an ideal $I \lhd R$ and a natural number $n$ such that $I^n \leq M \leq I$ holds. An analogous statement does not hold for arbitrary subnormal subgroups in groups, we can take, for example, the wreath product of any nonabelian simple group with the cyclic group of order 2. In groupoid-lattices one can introduce the notion of metaabsorbents (see [13]) as a common generalization of the concepts of accessible subrings and subnormal subgroups.

For a subgroup $H \leq G$ let $H^G$ denote the normal closure of $H$ in $G$ and $H_G$ the normal core (that is, the intersection of all conjugates) of $H$ in $G$. Let $\mathbf{X}$ denote the class of all finite groups $G$ such that for every $H \leq G$ the quotient $H^G/H_G$ is nilpotent. There is the following unpublished description of these groups.

**Theorem 2** (L. G. Kovács [7]). *A finite group $G$ belongs to $\mathbf{X}$, if and only if all nonabelian chief factors of $G$ are simple and, for each complemented abelian chief factor $H/K$ of $G$ and for each prime $p$, $\mathbf{O}_p(G/\mathbf{C}_G(H/K))$ is cyclic or (possibly generalized) quaternion.*

Here, as usual, $\mathbf{C}_G(H/K)$ denotes the centralizer of $H/K$ in $G$, and $\mathbf{O}_p(G)$ the largest normal $p$-subgroup of $G$.

From his characterization L. Kovács has derived several consequences.

Let $A$, $B$ be finite groups, $C \triangleleft B$. If $A, B \in \mathbf{X}$, then $A \times B \in \mathbf{X}$ and $B/C \in \mathbf{X}$. If $B/C \in \mathbf{X}$ and $C \leq \Phi(B)$, then $B \in \mathbf{X}$. If $A \in \mathbf{X}$ is a solvable group, then the nilpotent length (also called the Fitting height) of $A$ is at most 4, more precisely, the nilpotent length of $A''$ is at most 2. The class $\mathbf{X}$ is not closed under taking subgroups, nor for subdirect products, nor for products of normal subgroups. Hence the solvable groups in $\mathbf{X}$ form a normal subgroup closed Schunk class which is neither a formation nor a Fitting class.

We finish with a problem motivated by Ex. 1. Recall that $H \leq G$ is a subnormal subgroup of defect at most 2 iff $H \triangleleft H^G \triangleleft G$.

**Problem.** Assume that (A) holds for all subnormal subgroups of defect at most 2. Does it follow that (A) holds for all subnormal subgroups?

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