Mathematica Pannonica
13/1 (2002), 129–135

DEVELOPABLE AND METRIZABLE SPACES AND PROBLEMS OF FLETCHER AND LINDGREN AND GITTINGS

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Dedicated to Professor Hans Sachs on his 60th birthday

Received: April 2001

MSC 2000: 54 E 30, 54 E 35

Keywords: Quasi-$w\Delta$-space, quasi-developable space, quasi-γ-space, β-space, quasi-G_δ^*-diagonal, semi-stratifiable, c-semi-stratifiable, wM-space, metrizable space.

Abstract: In this paper, we answers two questions of P. Fletcher and W. Lindgren [1] and R. Gittings [4], one of which is partially answered. We prove that a space X is developable if and only if it is $w\Delta$-space with a quasi-G_δ^*-diagonal; a space X is developable if and only if it is quasi-developable, β-space; a space X is developable if and only if it is β, quasi-γ-space with a quasi-G_δ^*-diagonal; a space is metrizable if and only if it is wM-space with a quasi-G_δ^*-diagonal.

1. Introduction

In this brief note we present some conditions which imply develop-

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The author acknowledge the support of the Marsden Fund Award UOA 611, from the Royal Society of New Zealand.

ability and metrizability, and consequently we give a full positive answer to Fletcher and Lindgren's question [1] and a partial answer to R. Gittings's question [4] respectively: is every quasi-developable β-space developable? Is every wM-space with G_δ-diagonal metrizable?

In [13], the author makes it possible to factorize quasi-developability into two parts: *a space X is quasi developable if and only if it is a quasi-$w\Delta$-space with a quasi-G_δ^*-diagonal.* This result plays an important role in getting the results in this paper.

A COC-map (= countable open covering map) for a topological space X is a function from $N \times X$ into the topology of X such that for every $x \in X$ and $n \in \mathbb{N}, x \in g(n,x)$ and $g(n+1,x) \subseteq g(n,x)$. It is well known that many important classes of generalized metrizable spaces can be characterized in terms of a COC-map. In particular, X is developable [5] ($w\Delta$-space) if and only if X has a COC-map g such that if $\{p, x_n\} \subseteq g(n, x_n)$ for all n, then $\langle x_n \rangle$ converges to p (then $\langle x_n \rangle$ has a cluster point).

A space X is called quasi-γ [10] if and only if X has a COC-map g such that if $x_n \in g(n, y_n)$ for each $n \in \mathbb{N}$, and the sequence $\langle y_n \rangle$ converges in X, then the sequence $\langle x_n \rangle$ has a cluster point; a space X is called semi-stratifiable [7] (β-space [6]) if and only if X has a COC-map g such that if for each $x \in g(n, x_n)$ for each $n \in \mathbb{N}$ then x is a cluster point of $\langle x_n \rangle$ ($\langle x_n \rangle$ has a cluster point).

Let $\mathcal{G} = \{\mathcal{G}_n\}_{n \in \mathbb{N}}$ be a sequence of open families of X. Define $c(x) = \{n : x \in \mathcal{G}^*{}_n = \bigcup\{G : G \in \mathcal{G}_n\}\}$. A space X has a quasi-G_δ^*-diagonal [13] (quasi-$G_\delta^*(2)$-diagonal) if there is such a sequence $\mathcal{G}$ such that for any distinct $x, y \in X$, there exists $n \in \mathbb{N}$ such that $x \in$ $\in \overline{st(x, \mathcal{G}_n)} \subset X - \{y\}$ ($x \in \overline{st^2(x, \mathcal{G}_n)} \subset X - \{y\}$); a space X is called a quasi-$w\Delta$-space [13] if X has such a sequence $\mathcal{G}$ such that

(1) for all x, $c(x)$ is infinite,
(2) if $\langle x_n \rangle$ is a sequence with $x_n \in st(x, \mathcal{G}_n)$ for all $n \in c(x)$ then $\langle x_n \rangle$ has a cluster point.

If we take $\mathcal{G}$ as a sequence of open covers of X with the condition (2) ($\langle x_n \rangle$ is a sequence with $x_n \in st^2(x, \mathcal{G}_n)$ for all $n \in \mathbb{N}$ then $\langle x_n \rangle$ has a cluster point), then X is a $w\Delta$-space (wM-space).

A space X is called an c-semi-stratifiable [10] if there is a sequence $\langle g(n,x) \rangle$ of open neighborhoods of x such that for each compact set $K \subset X$, if $g(n, K) = \bigcup\{g(n,x) : x \in K\}$, then $\bigcap\{g(n,K) : n \geq 1\} =$ $= K$. The COC-map $g : \mathbb{N} \times X \to \tau$ is called a c-semi-stratification of X.

A space X is quasi-developable if there exists a sequence $\langle\mathcal{G}_n\rangle$ of families of open subsets of X such that for each $x \in X$, $\{st(x, \mathcal{G}_n) : n \in \in \mathbb{N}\} - \{\emptyset\}$ is a base at x.

All spaces will be regular, unless we state otherwise.

2. Main results

Lemma 2.1. *Let X be a space with a quasi-G_δ^*-diagonal sequence. Then X has a quasi-G_δ^*-diagonal sequence $\langle\mathcal{G}_n : n \in \mathbb{N}\rangle$ such that for each $x \in X$ there is an infinite subset $d(x) \subseteq c_{\mathcal{G}}(x)$ such that if $x_n \in \in st(x, \mathcal{G}_n)$ for each $n \in d(x)$ then $\langle x_n\rangle$ either clusters at x or it does not cluster at all.*

Proof. Let $\langle\mathcal{H}_n : n \in \mathbb{N}\rangle$ be a quasi-G_δ^*-diagonal sequence of X. Without loss of generality we may assume that $c_{\mathcal{H}}(x)$ is infinite for each $x \in X$ and $\mathcal{H}_1 = \{X\}$. Let $\mathcal{F}$ denote the non-empty finite subsets of $\mathbb{N}$. For each $F \in \mathcal{F}$ set

$$\mathcal{G}_F = \left\{\bigcap\nolimits_{i\in F} H_i : H_i \in \mathcal{H}_i\right\}.$$

For $n \in \mathbb{N}$ and $x \in X$, set $F_n(x) = c_{\mathcal{H}}(x) \cap \{1, 2, \ldots, n\}$. Put $d(x) = = \{F_n(x) : n \in \mathbb{N}\}$. Note that $d(x) \subseteq c_{\mathcal{G}}(x)$. Since $c_{\mathcal{H}}(x)$ is infinite, $d(x)$ is infinite. Because $F_n(x) \subseteq F_m(x)$ for $m \geq n$, $st(x, \mathcal{G}_{F_m(x)}) \subseteq \subseteq st(x, \mathcal{G}_{F_n(x)})$ for $m \geq n$.

For each $n \in \mathbb{N}$ suppose that $x_n \in st(x, \mathcal{G}_{F_n(x)})$. Then for $m \geq n$ we have

$$x_m \in st(x, \mathcal{G}_{F_m(x)}) \subset st(x, \mathcal{G}_{F_n(x)}),$$

so

$$\overline{\{x_m \;/\; m \geq n\}} \subset \overline{st(x, \mathcal{G}_{F_n(x)})}.$$

Since $\cap_{n\in\mathbb{N}}\overline{st(x, \mathcal{G}_{F_n(x)})} = \{x\}$ it follows that either $\langle x_n\rangle$ clusters at x or does not cluster at all. $\diamondsuit$

Remark 2.2. Let X be a space and $\langle\mathcal{G}_n : n \in \mathbb{N}\rangle$ a countable family of collections of open subsets of a space X, such that for all x, $c(x) = = \{n \in \mathbb{N} : x \in \mathcal{G}_n^*\}$ is infinite. Consider the following condition on $\langle\mathcal{G}_n : n \in \mathbb{N}\rangle$: *if $\langle x_n : n \in \mathbb{N}\rangle$ is a sequence with $x_n \in st(x, \mathcal{G}_n)$ for all $n \in c(x)$ then x is a cluster point of $\langle x_n : n \in \mathbb{N}\rangle$.* For all spaces, this condition is equivalent to the following condition: for each point $x \in X$ the set $st(x, \mathcal{G}_n)$ is nonempty for infinitely many n and the nonempty sets of the form $st(x, \mathcal{G}_n)$ form a local base at x for all $x \in X$. Thus *the condition above is a characterization of a quasi-developable space.*

Theorem 2.3. *Every quasi-developable space is a c-semi-stratifiable space.*

Proof. Let $\langle \mathcal{G}_n : n \in \mathbb{N} \rangle$ be a quasi-development sequence in a space X. Define

$$g(n,x) = \begin{cases} st(x, \mathcal{G}_n) & \text{if } x \in \mathcal{G}^*{}_n. \\ X & \text{if } x \notin \mathcal{G}^*{}_n. \end{cases}$$

Let $h(n,x) = \bigcap_{i=1}^{n} g(i,x)$. We prove that $h(n,x)$ is a c-semi-stratifiable-map. We claim that $C = \cap_{n \in \mathbb{N}} h(n,C)$ for any compact $C \subset X$, where $h(n,C) = \bigcup_{c \in C} h(n,c)$. As $\mathcal{G}_1 = \{X\}$ it follows readily that $C \subset \cap_{n \in \mathbb{N}} h(n,C)$ so it is appropriate concentrate on the reverse inclusion. To prove that, let $y \in \bigcap h(n,C)$, so $y \in h(n,c_n)$ for some $c_n \in C$. Then $y \in st(c_n, \mathcal{G}_n)$ for infinitely many $n \in \mathbb{N}$. It follows that $c_n \in st(y, \mathcal{G}_n)$ for infinitely many $n \in \mathbb{N}$. From Remark 2.2, $\langle c_n \rangle$ clusters at y. Hence, $y \in C$. ◊

Lemma 2.4. *A space is semi-stratifiable if and only if it is a c-semi-stratifiable β-space.*

Proof. Only if part is clear. If part: Let X be a regular c-semi-stratifiable β-space. Let f be a c-semi-stratifiable-map and g be a β-map. Define $h(n,x) = f(n,x) \cap g(n,x)$. It is clear that h is a c-semi-stratifiable, β-map. Since X is a regular and h is a c-semi-stratifiable, β-map, $\overline{h(n+1,x)} \subset h(n,x)$ for all $x \in X$ and all $n \in \mathbb{N}$ and such that if $x \in \in h(n,x_n)$ for $n \in \mathbb{N}$, then the sequence $\langle x_n \rangle$ has a cluster point. Now to prove that h is a semi-stratifiable-map, let $x \in h(n,x_n)$ for $n \in \mathbb{N}$, we must prove that the sequence $\langle x_n \rangle$ is convergent to x.

Now, the sequence $\langle x_n \rangle$ has at least one cluster point. Moreover, it is easy to show that every subsequence of $\langle x_n \rangle$ also has at least one cluster point. Suppose p is a cluster point of $\langle x_n \rangle$ and that $p \neq x$. Choose a s subsequence of $\langle x_{n_i} \rangle$ of $\langle x_n \rangle$ such that $x_{n_i} \in g(i,p)$ for $i \in \in \mathbb{N}$ and $x \neq x_{n_i}$ for all i. Since every subsequence of $\langle x_{n_i} \rangle$ has a cluster point, it follows that $\langle x_{n_i} \rangle$ converges to p. Therefore $K = \{p\} \cup \{x_{n_i}\}$ is compact. There exists m such that $x \notin h(m,K)$. Choose $k > m$ such that $x_k \in K$; then $x \notin h(m,x_k)$. But $h(k,x_k) \subset h(m,x_k)$, so $x \notin h(k,x_k)$, which is a contradiction. It follows that x is the only cluster point of $\langle x_n \rangle$. Since every subsequence of $\langle x_n \rangle$ has a cluster point, necessarily $\langle x_n \rangle$ converges to x. ◊

Theorem 2.5. *A space is developable if and only if it is quasi-develop-*

able β-space.

Proof. Only if part: clear. If part: follows from Lemma 2.4 and Th. 2.3. $\Diamond$

Corollary 2.6. *A space X is developable if and only if X is $w\Delta$-space with a quasi-G^*_δ-diagonal*

Proof. This follows from [13, Th. 3.1], Th. 2.7 and since every $w\Delta$-space is β-space. $\Diamond$

Theorem 2.7. *A space X is developable if and only if it is β, quasi-γ-space with a quasi-G^*_δ-diagonal.*

Proof. The necessity of the conditions is obvious. To prove the sufficiency of the conditions, let f be a β-map and g a quasi-γ-map of X. Define $h(n,x) = f(n,x) \cap g(n,x)$. It is clear that h is a β and quasi-γ-map of X. We prove that h is a $w\Delta$-map of X. Let $\{x, x_n\} \subset h(n, y_n)$. By the β-condition, $\langle y_n \rangle$ converges and so by the quasi-γ-condition $\langle x_n \rangle$ has a cluster point. Thus h is $w\Delta$-map of X. From Cor. 2.6, X is a developable space. $\Diamond$

Corollary 2.8. *A space is metrizable if and only if it is wM-space with a quasi-G^*_δ-diagonal.*

Proof. Let X be a regular, wM-space with a quasi-G^*_δ-diagonal. Every wM-space is a $w\Delta$-space so that (by Cor. 2.6) X is developable. Every developable, wM-space is metrizable, this completes the proof. $\Diamond$

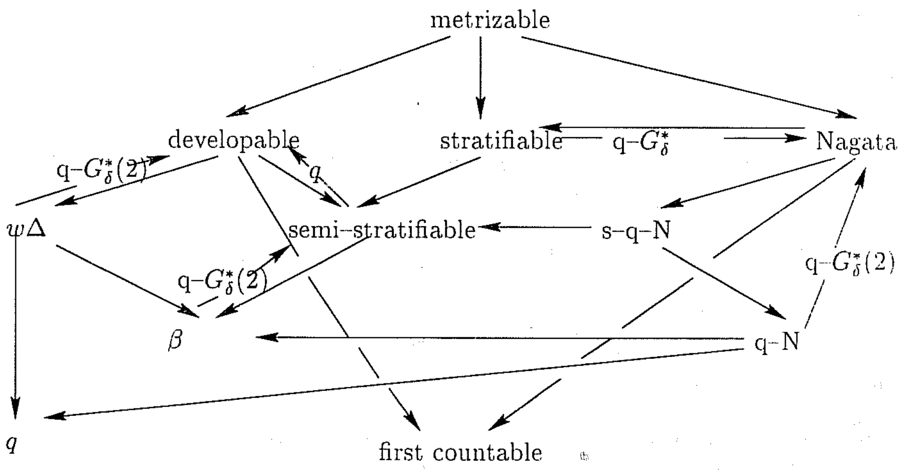
Now, it is natural to ask:

Question 2.9. Is every quasi-wΔ-space (quasi-wM-space) with G^*_δ-diagonal developable (metrizable)?

We answer this question in negative manner.

Example 2.10. *There is a p-adic analytic manifold which is separable, submetrizable, quasi-wM, quasi-developable, but not perfect (*[12, Ex. 7.4.7]*).* This example also can serve as *a quasi-semi-stratifiable space (see* [8] *for the definition) which has a G^*_δ-diagonal but which is not semi-stratifiable.* $\Diamond$

Example 2.11. *There is a quasi-developable manifold which has a G_δ-diagonal but not a G^*_δ-diagonal (see* [3, Ex. 2.2]*)* This example also can serve as a *quasi-$w\Delta$ manifold which is not $w\Delta$. (It is not even a β-manifold).* $\Diamond$



Relationships between some generalized metric spaces and quasi-G_δ^*-diagonal.

Acknowledgement. The author is grateful to Prof. David Gauld for his kind help and valuable comments and suggestions on this paper.

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