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AMICABLE PELL NUMBERS

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Abstract: In this note, we show that there are no amicable Pell numbers.

For any positive integer n let $\sigma(n)$ denote the divisor sum of n. Two positive integers m and n are called amicable if $\sigma(m) = \sigma(n) = m + n$. A positive integer n which is amicable with itself is called *perfect*.

Let $(P_n)_{n\geq 1}$ be the Pell sequence given by $P_0 = 0$, $P_1 = 1$ and

$$P_{n+2} = 2P_{n+1} + P_n \qquad \text{for } n \geq 0.$$

Various arithmetic properties of the Pell numbers have been intensively studied. For example, Cohn (see [1]) has shown that the only perfect powers in the Pell sequence are $P_1 = 1$ and $P_7 = 169 = 13^2$. In this note, we prove the following:

Theorem. *There are no amicable Pell pairs.*

Proof. Assume that P_m and P_n are amicable for some $m \leq n$. We distinguish three cases:

CASE 1. $P_n \not\equiv P_m \pmod 2$. In this case, the sum $P_n + P_m$ is odd. Let s be one of the two numbers m and n such that P_s is odd. Since

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$\sigma(P_s)$ is odd, it follows that P_s is a perfect square. Hence, $s = 1$ or $s = 7$. The case $s = 1$ gives $P_s = 1$, which is impossible because 1 is not amicable with any other number. The case $s = 7$ gives $P_s = 169 = 13^2$, $\sigma(P_s) - P_s = \sigma(169) - 169 = 14$ and 14 is not a member of the Pell sequence.

CASE 2. Both P_n and P_m are even. Notice that both n and m are even and that $m \geq 2$.

Assume first that $m < n$. In this case, $n \geq m + 2$. Since P_n is even, it follows that $P_n/2$ is a divisor of P_n. Hence,

$$P_m + P_n = \sigma(P_n) \geq P_n + \frac{P_n}{2}$$

or

$$2P_m \geq P_n. \tag{1}$$

But the inequality (1) is impossible because

$$P_n \geq P_{m+2} = 2P_{m+1} + P_m = 5P_m + 2P_{m-1} > 2P_m. \tag{2}$$

Assume now that $m = n$. In this case, P_m is an even perfect number. Hence,

$$P_m = 2^{p-1}(2^p - 1), \tag{3}$$

where both p and $2^p - 1$ are primes. One can check that P_m is not of this form for $m = 2,\ 4,\ 6$. Assume now that $m \geq 8$. In particular, $p \geq 4$. It is easy to prove that for any $k \geq 1$, $2^k \mid P_t$ if and only if $2^k \mid t$. Indeed, since $(P_t, P_s) = P_{(s,t)}$, it follows that it suffices to show that the order at which 2 divides P_{2^k} is precisely k for any $k \geq 1$. This follows easily by induction. The case $k = 1$ is obvious. For the induction step, let $(Q_n)_n$ be the companion sequence to $(P_n)_n$. This sequence is given by $Q_1 = 1$, $Q_2 = 3$ and

$$Q_{n+2} = 2Q_{n+1} + Q_n \qquad \text{for } n \geq 0. \tag{4}$$

It is well-known that the pairs $(Q_n,\ P_n)$ give all the solutions of the Pell equation

$$X^2 - 2Y^2 = \pm 1$$

and that

$$Q_n^2 - 2P_n^2 = (-1)^n. \tag{5}$$

In particular, Q_n is odd for all $n \geq 1$. The induction step follows now from the formula

$$P_{2n} = 2P_nQ_n \tag{6}$$

for $n = 2^{k-1}$.

The above arguments together with equation (3) imply $2^{p-1} \mid m$. Since

$$P_t > 2t^2 \qquad \text{for all } t \geq 7,$$

it follows that

$$2^{p-1}(2^p - 1) = P_m \geq P_{2^{p-1}} \geq 2(2^{p-1})^2 = 2^{2p-1},$$

which is an obvious contradiction.

CASE 3. Both P_m and P_n are odd. In this case, both m and n are odd. The sequence $(P_t)_t$ is periodic modulo 4 with period 4. By analyzing the first 4 terms, one concludes easily that $P_n + P_m \equiv \equiv 2 \pmod 4$, whenever both m and n are odd. In particular $2 \mid\mid \sigma(P_m)$. Now it follows easily that both P_m and P_n are of the form p_1x^2 for some prime number p_1 such that $p_1 \equiv 1 \pmod 4$. We analyze only P_m since the situation is symmetric in m and n.

We need to investigate the equation

$$P_m = p_1x^2. \tag{7}$$

Assume first that m is not prime. Let q be the largest prime number dividing m. Since $q \mid m$, it follows that $P_q \mid P_m$. Write equation (7) as

$$P_q\frac{P_m}{P_q} = p_1x^2. \tag{8}$$

It is well-known that

$$\left(P_q,\ \frac{P_m}{P_q}\right) = (P_q,\ m/q). \tag{9}$$

We use formula (9) to show that the greatest common divisor appearing in (9) is, in fact, 1. Indeed, it is well-known that if p is any prime, then $p \mid P_{p-e}$, where $e = 0$ for $p = 2$ and $e = \left(\frac{2}{p}\right)$, where $\left(\frac{a}{p}\right)$ denotes the Jacobi symbol of a with respect to p. In particular, $p \mid P_{p^2-1}$ when p is odd. Assume now that the greatest common divisor appearing in formula (9) is not 1. Pick a prime divisor p of it. Notice that p is odd because m is odd. On the one hand, it follows that $p \mid P_q$. On the other hand, by the previous remarks, it follows that $p \mid P_{p^2-1}$. Hence,

$$p \mid (P_q,\ P_{p^2-1}) = P_{(q,p^2-1)}. \tag{10}$$

Since q was the largest prime divisor of m, it follows that $q \geq p$. In particular, $(q,\ p^2-1)=1$, which contradicts formula (10). Hence, P_q and P_m/P_q are coprime. Equation (7) implies now that either

$$P_q = p_1 x_1^2 \text{ and } P_m = P_q x_2^2 \tag{11}$$

or

$$P_q = x_1^2 \text{ and } P_m = p_1 P_q x_2^2 \tag{12}$$

for some positive integers x_1 and x_2 such that $x_1 x_2 = x$.

We treat first situation (11). Combining (11) with formula (5) and with the fact that m is odd, we get the equation

$$Q_m^2 - (2P_q^2)x_2^4 = -1. \tag{13}$$

From the main result in [2], we know that the equation

$$X^2 - dY^4 = -1$$

has at most one solution when $d > 3$. Taking $d = 2P_q^2$, we get that the equation

$$X^2 - (2P_q^2)Y^4 = -1$$

has two solutions, namely $(X,\ Y) = (Q_q,\ 1)$ and $(Q_m,\ x_2)$, which is a contradiction.

We now analyze situation (12). The first equation (12) implies $q = 7$. We now show that m/q is a prime. Indeed, assume that m/q is not a prime and let q_1 be the largest prime dividing m/q. Rewrite the second equation in (12) as

$$\frac{P_m}{P_{qq_1}} \cdot \frac{P_{qq_1}}{P_q} = p_1 x_2^2. \tag{14}$$

One can employ the previous argument to show that the two factors of the product appearing on the left side of (14) are coprimes. Hence, equation (14) implies that one of the two numbers P_m/P_{qq_1} and $P_{qq_1}/$ $/P_q$ is a square. But this is again impossible by the main result in [2], so m/q is prime. Since $q = 7$ and q is the largest prime dividing m, it follows that $m \in \{21,\ 35,\ 49\}$. One can check that none of the numbers P_m for these values of m is of the form (7).

Finally, assume now that both m and n are prime. Let $m = p$. Since $n \geq p$, we get

$$\sigma(P_p) = P_p + P_n \geq 2P_p$$

or

$$2 \le \frac{\sigma(P_p)}{P_p}. \tag{15}$$

Decompose P_p in prime factors as

$$P_p = q_1 ... q_t \qquad \text{where } q_1 \le q_2 \le ... \le q_t. \tag{16}$$

It is well-known that every prime q_i is congruent to ± 1 modulo p. In particular $q_1 \ge 2p - 1$. Since

$$P_k < (1 + \sqrt{2})^k \qquad \text{for } k \ge 0,$$

we get

$$p \log(1 + \sqrt{2}) > \log P_p \ge \sum_{i=1}^{t} \log q_i \ge t \log(2p - 1),$$

or

$$t < \log(1 + \sqrt{2}) \frac{p}{\log(2p - 1)}. \tag{17}$$

On the other hand, by fomula (15), we get

$$2 \le \frac{\sigma(P_p)}{P_p} \le \frac{P_p}{\phi(P_p)} = \prod_{i=1}^{t} \left(1 + \frac{1}{q_i - 1}\right) \le \left(1 + \frac{1}{2p - 2}\right)^t. \tag{18}$$

Here ϕ is the Euler function. Inequalities (17) and (18) force
(19)

$$\log 2 \le t \log\left(1 + \frac{1}{2p - 2}\right) < \frac{t}{2p - 2} < \log(1 + \sqrt{2}) \frac{p}{(2p - 2)\log(2p - 1)}.$$

Inequality (19) implies $p \le 3$. But for $p = 3$, one gets $\sigma(P_p) = \sigma(5) = 6 = 5 + 1$, which leads to the pair $(5,\ 1)$, which is certainly not amicable.

The theorem is proved. $\Diamond$

Remark I.t is probably true that there are only finitely many Fibonacci pairs, Lucas pairs or Fibonacci-Lucas pairs of amicable numbers. In [3], we showed that there are no perfect Fibonacci or Lucas numbers and in [4] we showed that there are only finitely many multiply perfect Fibonacci or Lucas numbers. Unfortunately, the methods employed in the present paper or in [3] or [4] are not powerful enough to deal with the amicability problem for such numbers.

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References

[1] COHN, J.H.E.: Perfect Pell powers, *Glasgow. Math. J.* **38** (1996), 19–20.

[2] HUA, C.J. and VOUTIER, P.: Complete solution of the diophantine equation $x^2 + 1 = dY^2$ and a related family of quartic Thue equations, *J. Number Theory* **62** (1997), 71–99.

[3] LUCA, F.: Perfect Fibonacci and Lucas numbers, *Rend. Circ. Mat. Palermo* Ser. II **49** (2000), 651–661.

[4] LUCA, F.: Multiply perfect numbers in Lucas sequences, *Publ. Math. (Debrecen)* **58** (2001), 121–155.