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ON A WEAKLY BERWALD FINSLER SPACE OF KROPINA TYPE

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Abstract: In this paper we investigate a Finsler space, where the (hv)-Ricci tensor G_{ij} vanishes, but the (hv)-curvature tensor G^h_{ijk} is not necessarily equal to zero. The aim of this paper to give an example for the so-called weakly Berwald Finsler space (WBFS), and a sufficient condition for the existence of a WBFS of Kropina type is determined also.

1. Introduction

Let $F^n(M^n, L)$ be a Finsler space of dimension n, where M^n is a connected n-dimensional differentiable manifold and the domain of the metric fundamental function $L(x, y)$ is the set $TM\backslash 0$ of the non zero tangent vectors. We will assumed that L is positive and the met-

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ric fundamental tensor $g_{ij}(x,y) = \frac{1}{2}L^2_{(i)(j)}$ (where $(i) := \frac{\partial}{\partial y^i}$) is not necessarily positive definite.

The equation of the (canonically parametrized) geodetics of F^n is given by

$$\frac{d^2x^i}{dt^2} = -2G^i(x,y)\ , \qquad \left(\frac{dx^i}{dt} = y^i\right)\ ,$$

where

$$G^i = \frac{1}{2}g^{ir}\left(y^s\frac{\partial L^2_{(r)}}{\partial x^s} - \frac{\partial L^2}{\partial x^r}\right). \tag{1.1}$$

The Berwald connection of the space is defined by its connection coefficients $G^i_{jk}(x,y)$ which can be computed from G^i according to the following formulae:

$$G^i_j(x,y) = G^i_{(j)}; \quad G^i_{jk}(x,y) = G^i_{j(k)}. \tag{1.2}$$

Definition 1.1. [3] A metric $L(x,y) = \alpha^2(x,y)/\beta(x,y)$ and the corresponding Finsler space will be called a *Kropina metric* and a *Kropina space*, respectively, where $\alpha^2(x,y) = a_{ij}(x)y^iy^j$ is a Riemannian metric and $\beta(x,y) = b_i(x)y^i$ is a one-form.

Definition 1.2. [4] *A* spray is called a *weakly affine spray* if the (hv)-Ricci curvature tensor $G_{kl} = 0$ where $G_{kl} = G^r_{rkl}$, and the (hv)-curvature tensor is given by $G^i_{jkl} = G^i_{jk(l)}$.

Definition 1.3. *A* Finsler space is called a *weakly Berwald space* if the (hv)-Ricci curvature tensor $G_{kl} = 0$.

We will use the following notations

$$E_{ij} = (\nabla_j b_i + \nabla_i b_j)/2, \tag{1.3}$$

$$F_{ij} = (\nabla_j b_i - \nabla_i b_j)/2, \tag{1.4}$$

$$F^i_j = a^{ir}F_{rj}; \quad F_i = b_rF^r_i; \quad F^i = a^{ir}F_r, \tag{1.5}$$

$$b^i = a^{ir}b_r; \quad b^2 = b^rb_r, \tag{1.6}$$

$$E_{0i} = E_{ri}y^r; \quad E_{00} = E_{rs}y^ry^s, \tag{1.7}$$

$$F_0 = F^ry_r; \quad F^i_0 = F^i_ry^r, \tag{1.8}$$

where ∇ denotes the covariant derivation of the Levi-Civita connection of $\alpha(x,y)$, and (a^{ij}) is the inverse matrix of matrix a_{ij} and the transvection by y^i is denoted by subscript 0.

Z. Shen has studied the weakly affine spray in Randers space [4], in this paper we investigate weakly Berwald Finsler space of Kropina type. Finally we construct an example for this space.

2. Weakly Berwald Finsler spaces of Kropina type

It is well-known that in a Kropina space the functions $G^i(x,y)$ are given by

$$(2.1)\quad 2G^i = \Gamma^i_{00}(x) - 2\left(F_0 + \frac{\beta E_{00}}{\alpha^2}\right)\frac{y^i}{b^2} - \frac{\alpha^2 F^i_0}{\beta} + \left(\frac{\alpha^2 F_0}{\beta} + E_{00}\right)\frac{b^i}{b^2}$$

where $\Gamma^i_{jk}(x)$ denotes the Christoffel symbols of the Riemannian metric $\alpha(x,y)$ [5], [3].

Using of (1.2) we obtain

$$(2.2)\quad \begin{aligned} 2G^i_j =& 2\Gamma^i_{j0} - 2\left[F_j + \frac{b_j E_{00} + 2\beta E_{j0}}{\alpha^2} - \frac{2E_{00}a_{j0}}{\alpha^4}\right]\frac{y^i}{b^2} - \\ & - (2F^i_0 a_{j0} + \alpha^2 F^i_j)\beta + \frac{\alpha^2 F^i_0 b_j}{\beta^2} + \\ & + \left[(2F_0 a_{j0} + \alpha^2 F_j)\beta - \frac{\alpha^2 F_0 b_j}{\beta^2} + 2E_{j0}\right]\frac{b^i}{b^2}. \end{aligned}$$

After contracting (2.2) by the indices i, j and differentiating this equation by y^k, y^l we get the (hv)-Ricci tensor in the following form

$$(2.3)\quad \begin{aligned} G_{kl} =& -\frac{2(n+1)}{\alpha^2 b^2}\left[-\left(\frac{b_k a_{l0} + b_l a_{k0} + \beta a_{kl}}{\alpha^2} - \frac{4\beta a_{k0}a_{l0}}{\alpha^4}\right)E_{00} + \right. \\ & \left. + \frac{b_k - 2\beta a_{k0}}{\alpha^2}E_{l0} + \frac{b_l - 2\beta a_{l0}}{\alpha^2}E_{k0} - \beta E_{kl}\right]. \end{aligned}$$

If we differentiate the connection coeffitients G^i_{jk} by y^l we have the (hv)-curvature tensor of Kropina type:

$$
\begin{aligned}
G^i_{jkl} =& U^i_l E_{jk} + U^i_j E_{kl} + U^i_k E_{lj} + V^i_{kl} E_{j0} + V^i_{lj} E_{k0} + \\
&+ V^i_{jk} E_{l0} + Z^i_{jkl} E_{00} + A_{jk}\big(- F^i_l + F_l \frac{b^i}{b^2}\big) + \\
&+ A_{kl}\big(- F^i_j + F_j \frac{b^i}{b^2}\big) + A_{lj}\big(- F^i_k + F_k \frac{b^i}{b^2}\big) + \\
&+ B_{jkl}\big(- F^i_0 + F_0 \frac{b^i}{b^2}\big)
\end{aligned}
\tag{2.4}
$$

Consequently G^i_{jkl} of (2.4) is written as

$$
\begin{aligned}
G^i_{jkl} =& U^i_l E_{jk} + V^i_{kl} E_{j0} + \frac{1}{3} Z^i_{jkl} E_{00} + A_{jk}\big(- F^i_l + F_l \frac{b^i}{b^2}\big) + \\
&+ \frac{1}{3} B_{jkl}\big(- F^i_0 + F_0 \frac{b^i}{b^2}\big) + (j,k,l)
\end{aligned}
\tag{2.5}
$$

where (j,k,l) denotes the cyclic permutation of the indices (j,k,l) and

$$
\begin{aligned}
K_j &= -b_j + 2\frac{\beta a_{j0}}{\alpha^2}, \\
U^i_j &= \frac{2}{\alpha^2 b^2}(K_j y^i - 2\beta \delta^i_j), \\
V^i_{jk} &= \frac{4}{\alpha^2 b^2}\Bigg[\bigg((b_j a_{k0} + b_k a j_0 + \beta a_{jk})\alpha^2 - \frac{4 a_{j0} a_{k0}}{\alpha^4}\bigg) y^i + \\
&\quad + K_j \delta^i_k + K_k \delta^i_j\Bigg], \\
Z^i_{jkl} &= \frac{2}{\alpha^4 b^2}\Bigg\{\Big[b_j a_{kl} - \frac{4}{\alpha^2}(b_j a_{k0} + \beta a_{jk}) a_{l0} + \frac{8\beta a_{j0} a_{k0} a_{l0}}{\alpha^4} + \\
&\quad + (j,k,l)\Big] y^i + \Big[2(b_j a_{k0} b_k a_{j0} + \beta a_{jk} - \frac{4\beta a_{j0} a_{k0}}{\alpha^2})\delta^i_l + (j,k,l)\Big]\Bigg\}, \\
A_{jk} &= \frac{2}{\beta}\left[a_{jk} - \frac{a_{j0} b_k + a_{k0} b_j}{\beta} + \frac{\alpha^2 b_j b_k}{\beta^2}\right], \\
B_{jkl} &= \frac{2}{\beta^2}\left(\frac{2 a_{j0} b_k b_l}{\beta} - a_{jk} b_l - \frac{\alpha^2 b_j b_k b_l}{\beta^2} + (j,k,l)\right).
\end{aligned}
$$

Therefore from the structure of the equations (2.3) and (2.5) we have the following two theorems:

Theorem 1. *In a n-dimensional Kropina space if $E_{kl} = 0$ then $G_{kl} = 0$ holds good.*

Theorem 2. *In a n-dimensional Kropina space if F_{kl} is not equal* *to zero, then the (hv)-curvature tensor G^i_{jkl} is not necessarily equal to zero.*

3. An example for the weakly-Berwald space

We give a covariant vector field $(b_i(x))$ in an odd-dimensional Euclidean space so that $E_{kl} = 0$ and $F_{kl} \neq 0$ hold good.

Let Ω_{ij} be an $n \times n$ type quadratic skew symmetric matrix, and x^i denote coordinates of a point.

We consider the following vector field $b_i = \Omega_{ij}x^j + c_i$, where c_i are constants. Easy to see, that in this special case $\nabla_j b_i + \nabla_i b_j = 0$, and $\nabla_j b_i - \nabla_i b_j \neq 0$. So a Kropina space, which is generated by $b_i = \Omega_{ij}x^j + c_i$, is a weakly-Berwald space, and it is not Berwald space.

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