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AN ARCWISE CONNECTED CONTINUUM WITHOUT NON-BOUNDARY PROPER ARCWISE CONNECTED SUBCONTINUA

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Abstract: We construct an arcwise connected continuum such that each proper arcwise connected subcontinuum has the empty interior.

A continuum means a nonempty compact connected metric space.

In a private communication to the last named author in 1998, W. Makuchowski and J.R. Prajs posed a question of whether there exists a nondegenerate arcwise connected continuum such that each of its proper arcwise connected subcontinua has the empty interior.

A positive answer was found during the Continuum Theory Prague 2001, Open Problem Workshop.

A construction of the example is inspired by the example of J. B. Fugate and L. Mohler of a continuum with exactly two dense arc components (see [1, p. 397]).

Example. Denote by C the Cantor ternary set. Set $A = [0,3] \times C$. Choose a dense in C sequence $\{r_n\}_{n=1}^{\infty}$ of distinct points of C.

Denote

$$a_n = \frac{1}{n} - \frac{1}{77n^2}, \qquad b_n = \frac{1}{n}, \qquad c_n = \frac{1}{n} + \frac{1}{77n^2},$$

$$d_n = 3 - \frac{1}{n} - \frac{1}{77n^2}, \qquad e_n = 3 - \frac{1}{n}, \qquad f_n = 3 - \frac{1}{n} + \frac{1}{77n^2}$$

and

$$L_n = [a_n, c_n] \times \left[r_n - \frac{1}{777n^3}, r_n + \frac{1}{777n^3}\right],$$

$$R_n = [d_n, f_n] \times \left[r_n - \frac{1}{777n^3}, r_n + \frac{1}{777n^3}\right]$$

for $n \in \mathbb{N}$.

We modify A in such a way that, for each $n \in \mathbb{N}$,

(1) we replace the arc $[a_n, c_n] \times \{r_n\}$ by the union

$$\mathcal{L}_n = \left\{ (x,y) \in L_n : x \in [a_n, b_n),\ y = r_n + \frac{1}{7777n^4} \sin\left(\frac{1}{x - b_n}\right)\right\} \cup$$
$$\cup \left\{ (b_n, y) \in L_n : y \in \left[r_n - \frac{1}{7777n^4}, r_n + \frac{1}{7777n^4}\right]\right\} \cup$$
$$\cup \left\{ (x, r_n) \in L_n : x \in (b_n, c_n]\right\}$$

and the arc $[d_n, f_n] \times \{r_n\}$ by the copy $\mathcal{R}_n \subset R_n$ of $\mathcal{L}_n$ symmetric to

$\mathcal{L}_n$ with respect to point (e_n, r_n).

(2) other arcs in $A \cap L_n$ and $A \cap R_n$ bend a little bit so that they approximate the inserted $\sin(\frac{1}{x})$ curves (in fact the whole $\mathcal{L}_n$ or $\mathcal{R}_n$) but they remain mutually disjoint, disjoint with $\mathcal{L}_n$ and $\mathcal{R}_n$, and must stay in L_n or R_n, respectively (see Fig. 1).

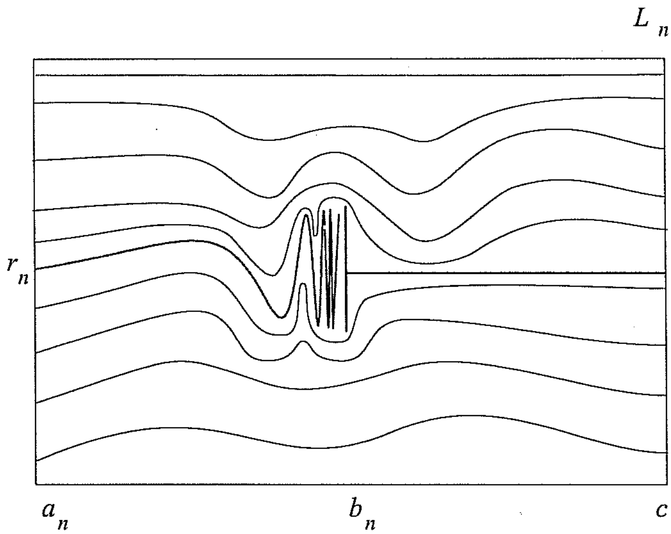

Fig. 1 - a replacement in L_n

Denote the modified compact space by B.

Next, we will do some identifications in B to get a continuum with exactly two dense arc components.

(i) For each odd number $n \in \mathbb{N}$, identify points (e_n, r_n) and (e_n, r_{n+1}).

(ii) For each even number $n \in \mathbb{N}$, identify points (b_n, r_n) and (b_n, r_{n+1}) (see Fig. 2).

(iii) Identify all points in $\{0\} \times C$.

(iv) Identify all points in $\{3\} \times C$.

One can easily check that the decomposition of B described by (i)–(iv) is upper semi-continuous. Therefore, if $q : B \to X' = q(B)$ is the quotient map of the decomposition, then X' is compact and connected because it has exactly two dense arc components: the one containing points $q(0,0)$, $q(3,0)$ together with all $\sin(\frac{1}{x})$ curves, and the other that is uniquely arcwise connected and contains the dense in X' ray $W = \bigcup_{n=1}^{\infty} W_n$, where

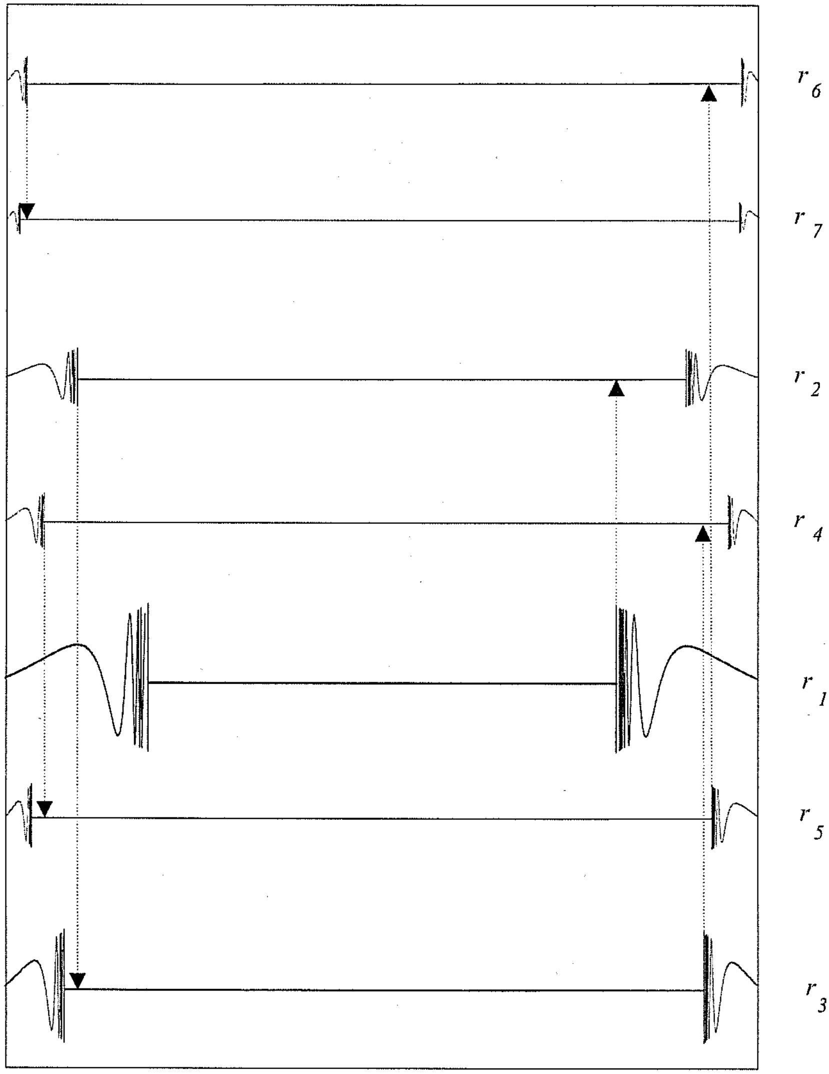

Fig. 2 - an identification in B

$$W_1 = q(\{r_1\} \times [b_1, e_1])$$

and

$$W_n = \begin{cases} q(\{r_n\} \times [b_{n-1}, e_n]) & \text{if } n > 1 \text{ is odd,} \\ q(\{r_n\} \times [b_n, e_{n-1}]) & \text{if } n \text{ is even.} \end{cases}$$

Finally, we obtain our example X as the quotient of X' by identifying two points $q(0,0)$ and $q(b_1,r_1)$. Let $q' : X' \to X$ be the quotient map.

Properties of X.

(a) *X is an arcwise connected continuum*, since points $q(0,0)$ and $q(b_1,r_1)$ belong to the only two different arc components of continuum X'.

(b) *Each arcwise connected proper subcontinuum $Y \subset X$ has the empty interior in X.* Indeed, observe that, for each k, the ray $\bigcup_{n=k}^{\infty} W_n$ is dense in X' and q' maps W homeomorphically onto $q'(W)$, hence the set $q'(\bigcup_{n=k}^{\infty} W_n)$ is a dense in X ray, for each k. If $\operatorname{int} Y \neq \emptyset$, then Y contains a sequence of points $w_1, w_2, \ldots \in$ $\in q'(W) \cap Y$ such that $w_n \in q'(W_{k_n})$ for some subsequence $1 \leq k_1 < k_2 < \ldots$. By definition of q' and the arcwise connectedness of Y, the ray $q'(\bigcup_{n=k_2}^{\infty} W_n)$ is contained in Y, so $Y = X$.

(c) *X is non-planar.* In fact, suppose X is embedded in the plane. Choose numbers r_n and $s_1, s_2 \in C \setminus \{r_1, r_2, \ldots\}$ such that s_1 is between r_1 and r_n and r_n is between s_1 and s_2. Then point $p = q'q(3/2, r_1)$ lies outside the simple closed curve $S = q'q(([0,3] \times$ $\times \{s_1\}) \cup ([0,3] \times \{s_2\}))$ and point $p' = q'q(3/2, r_2)$ is inside S. By construction, the unique in X arc pp' is contained in the ray $q'(W)$, so it is disjoint with S, a contradiction with the Jordan curve theorem.

Question. Does there exist an arcwise connected continuum Z in the plane such that each arcwise connected proper subcontinuum of Z is boundary?

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