Mathematica Pannonica
13/1 (2002), 79–84

CHAIN OF DENDRITES WITHOUT RETRACT INFIMUM

Pavel **Podbrdský**

Department of Mathematical Analysis, Charles University, Sokolovská 83, CZ-186 75 Prague 8, Czech Republic

Pavel **Pyrih**

Department of Mathematical Analysis, Charles University, Sokolovská 83, CZ-186 75 Prague 8, Czech Republic

Dedicated to Professor Hans Sachs on his 60th birthday

Received: January 2002

MSC 2000: 54 C 10, 54 F 50

Keywords: Dendrite, retraction, chain.

Abstract: We consider an ordering with respect to retractions on the class of all dendrites and construct a bounded chain of dendrites which does not have an infimum answering a question of its existence. A simple example answers a similar question for a supremum.

An ordering

All spaces considered in this paper are assumed to be metric. A continuum means a nonempty compact connected space. A simple closed curve is any space which is homeomorphic to the unit circle.

E-mail addresses: podbrdsk@karlin.mff.cuni.cz, pyrih@karlin.mff.cuni.cz
Joint research of the Open Problem Seminar, Charles University, Prague. Partially supported by grants GAUK 165/1999 and GAUK 252/2000 from the Grant Agency of Charles University and partially supported by grant number MSM 113200007 from the Czech Ministry of Education.

A dendrite means a locally connected continuum containing no simple closed curve.

A mapping means a continuous function. A surjective mapping $f: X \to Y \subset X$ is said to be a retraction provided $f = f^2$.

In [1, p. 7] J. J. Charatonik, W. J. Charatonik and J. R. Prajs introduced a quasiordering $\leq_R$ on the class of all dendrites. We recall the definition. If X, Y are dendrites, then $Y \leq_R X$ if for some $Z \subset \subset X$ homeomorphic to Y there is a retraction $f: X \to Z$. This quasiordering is not an ordering (i.e. $X \leq_R Y$ & $Y \leq_R X$ does not imply X is homeomorphic to Y). To guarantee this they said that X and Y are $\mathbb{R}$-equivalent if $X \leq_R Y$ and $Y \leq_R X$ and then considered the quotient of the class of all dendrites. The quasiordering $\leq_R$ induces the ordering $\leq_{\mathbb{R}}$ on the quotient. We will use for short $\leq$ instead of $\leq_{\mathbb{R}}$ in this paper.

J. J. Charatonik, W. J. Charatonik and J. R. Prajs posed a question if every chain bounded with respect to this ordering has an infimum or a supremum (see [1, §7, Q4(b)$\mathbb{R}$, p. 51]). In Theorem we give a negative answer to this question for an infimum by constructing a bounded sequence of dendrites with no retraction infimum. In Example we give a negative answer to this question for a supremum by constructing a bounded sequence of dendrites with no retraction supremum. We will use for short 'chain', 'supremum' etc. without explicitly noticing the respect to the ordering $\leq$ in this paper.

We shall use the notion of order of a point in the sense of Menger-Urysohn (see e.g. [2, §51, I, p. 274], and we denote by $\operatorname{ord}(p, X)$ the order of the continuum X at a point $p \in X$ or just $\operatorname{ord}(p)$ if there is no risk of confusion. Points of order 1 in a continuum X are called end points of X. Points of order 2 are called ordinary points of X. Points of order at least 3 are called ramification points of X.

An arc is any space which is homeomorphic to $I = [0, 1]$. We denote by pq an arc with end points p and q. An arc pq in a space X is said to be free if $pq \setminus \{p, q\}$ is an open subset of the space X. Denote by S_n the n-od, i.e. the dendrite where the only ramification point is of order $n \in \mathbb{N}$. Similarly denote by S_ω the *star* , i.e. the dendrite where the only ramification point is of order ω.

First we will prove an easy lemma, about the ordering $\leq$ on dendrites.

Lemma. *Let A, B be dendrites. Then $A \leq B$ if and only if A is homeomorphic to some subset of B.*

Proof. Clearly by definition, if $A \le B$, then A is homeomorphic to some subset of B. For the other implication we can assume that $A \subset \subset B$. We have to find a retraction $f : B \to A$. We define $f \equiv \mathrm{Id}$ on A. Let $x \in B \setminus A$. We take C that component of $B \setminus A$ which contains x. There is a unique point $y \in \mathrm{cl}\,(C) \setminus C$; we define $f(x) = y$. It is easy to check that f is a retraction. ◇

An infimum free chain

K. Sieklucki in [3, §1, p. 331–333] constructed a decreasing sequence of dendrites $\{B_n\}$; in [3, §2, p. 333–334] he used the sequence $\{B_n\}$ to obtain a sequence $\{C_n\}$ of uncomparable dendrites; in [3, §3, p. 334–335] he created an uncountable family of uncomparable dendrites. We modify the figure of such a family (see [3, Fig. 4, p. 334] to obtain a sequence of dendrites without an infimum. We recall from [3, p. 334–335] an important information.

Proposition. *There exists a sequence $\{C_n\}_{n=1}^{\infty}$ of dendrites such that the following conditions hold:*

(1) *for each $n \in \mathbb{N}$ the dendrite C_n has only ramification points of finite order;*
(2) *$C_n \le C_m$ for some $m, n \in \mathbb{N}$ if and only if $n = m$.*

Fix an end point e_n in C_n, $n \in \mathbb{N}$. Put $a = (0,0)$, $b = (2,0)$ and for each $n \in \mathbb{N}$ denote $a_n = (-1/n, 1/n^2)$, $b_n = (2 + 1/n, 1/n^2)$, $c_n = (2 - 1/n, 0)$, $d_n = (c_n + c_{n+1})/2$. For each $n \in \mathbb{N}$, attach a sufficiently small copy $\widehat{C_n}$ of C_n in the point c_n (identifying e_n and c_n) and a sufficiently small copy $\widehat{D_n}$ of an arc in the point d_n (identifying d_n and some ordinary point of an arc) such that

$$X_m = ab \cup \bigcup_{n=1}^{\infty} aa_n \cup \bigcup_{n=1}^{\infty} bb_n \cup \bigcup_{n=1}^{\infty} \widehat{C_n} \cup \bigcup_{n=m}^{\infty} \widehat{D_n}$$

is a dendrite for each $m \in \mathbb{N}$. See Fig. 1.

Similarly we denote

$$C = ab \cup \bigcup_{n=1}^{\infty} aa_n \cup \bigcup_{n=1}^{\infty} bb_n \cup \bigcup_{n=1}^{\infty} \widehat{C_n}$$

and

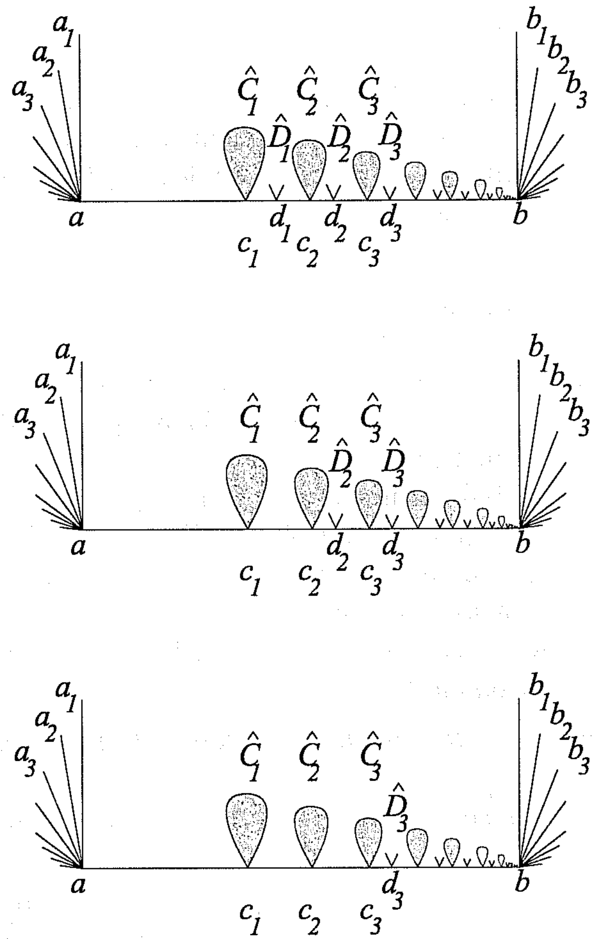

Fig. 1 – dendrites X_1, X_2, X_3, $\cdots$

$$D = ab \cup \bigcup_{n=1}^{\infty} aa_n \cup \bigcup_{n=1}^{\infty} bb_n \cup \bigcup_{n=1}^{\infty} \widehat{D_n}.$$

Theorem. *The chain* $X_1 \geq X_2 \geq X_3 \geq \cdots \geq C$ *is bounded and has no infimum.*

Proof. Clearly, $\{X_n\}$ is a decreasing sequence. Suppose that Y is an infimum of the sequence $\{X_n\}$. We conclude a contradiction.

Observe that $X_m \geq C$ and $X_m \geq D$ for each $m \in \mathbb{N}$. We conclude that $Y \geq C$ and $Y \geq D$. We may assume $C \subset Y$.

Fix $m \in \mathbb{N}$. We have $X_m \geq Y$ and $Y \geq D$. By Lemma there are homomorphisms $f_m : Y \leftrightarrow Y_m \subset X_m$ and homomorphism $g : D \leftrightarrow \leftrightarrow Z \subset Y$.

Notice that the dendrites X_m, Y and D have exactly two points of order ω (the points a, b), only one of which (b) is an accumulating point of ramification points. We conclude $f_m(a) = a$, $f_m(b) = b$, $f_m(ab) = ab$, $g(a) = a$, $g(b) = b$ and $g(ab) = ab$. For each point $c_k \in Y$ (with $k \in \in \mathbb{N}$), $f(c_k)$ is a ramification point of $ab \subset X_m$. Because the dendrites $\widehat{C}_k, \widehat{C}_l$ are $\leq$-uncomparable for $k \neq l$ (with $k, l \in \mathbb{N}$, see Proposition (2)) and also $\widehat{C}_k$ is not a retract of an arc (the 'v-sets' $\widehat{D_n}$) we conclude $f_m(c_k) = c_k$.

Finally we have $f_m(ac_m) = ac_m \subset X_m$ and we conclude that Y has no point of order larger than 3 on the arc ac_m for each $m \in \mathbb{N}$, and consequently Y has no point of order larger than 3 on the arc ab.

But we see that each point $g(d_k) \in ab \subset Y$ (with $k \in \mathbb{N}$) has order at least 4. This gives a contradiction. ◊

A supremum free chain

Example. *The chain $S_3 \leq S_4 \leq S_5 \leq \cdots \leq S_\omega$ is bounded and has no supremum.*

Proof. Let Y be a supremum of the sequence $\{S_n\}$. We conclude a contradiction.

(i) Clearly $Y \leq S_\omega$. This gives that Y has at most one ramification point.

(ii) Denote $a = (0, 0)$, $b = (1, 0)$ and for each $n \in \mathbb{N}$ put $s_n = (1/ /n, 0)$. Let $L \supset ab$ be a dendrite where the only ramification points are the points s_n, $3 \leq n \in \mathbb{N}$, and the order of s_n is n for each $3 \leq n \in \mathbb{N}$. See Fig. 2.

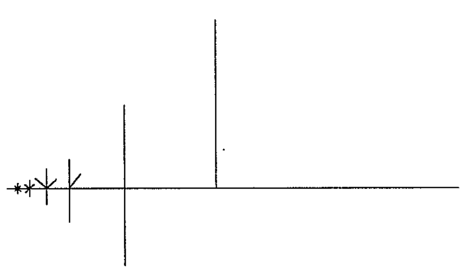

Fig. 2 – a dendrite L

Obviously, $S_n \leq L$ for each $n \in \mathbb{N}$. This gives $Y \leq L$. We conclude that Y has ramification points of finite order only.

(iii) From (i) and (ii) we see that Y is a n-*od* for some $n \in \mathbb{N}$, which gives a contradiction. $\Diamond$

Question. Are infinite decreasing chains of dendrites only of K. Sieklucki ([3]) type?

References

[1] CHARATONIK, J. J., CHARATONIK, W. J. and PRAJS, J. R.: Mapping hierarchy for dendrites, *Dissertationes Math.* (Rozprawy Mat.) **333** (1994), 1–52.

[2] KURATOWSKI, K.: Topology, Vol. II, Academic Press, New York, 1968

[3] SIEKLUCKI, K.: On a family of power c consisting of $\mathfrak{R}$-uncomparable dendrites, *Fund. Math.* **46** (1959), 331–335