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SOME REMARKS ON ALMOST KENMOTSU MANIFOLDS

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Abstract: An almost Kenmotsu manifold is an almost contact Riemannian manifold $M(\varphi, \xi, \eta, g)$ in which still $(\nabla_X \varphi)Y = -g(X, \varphi Y)\xi - \eta(X)\varphi X$ and $\nabla_X \xi = X - \eta(X)\xi$ hold. In this paper the semi-symmetry of almost Kenmotsu manifolds is investigated. It is proved that a) If an almost Kenmotsu manifold

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is semi-symmetric ($R \circ R = 0$) then it is locally symmetric ($\nabla R = 0$); b) If an almost Kenmotsu manifold is semi-Ricci-symmetric ($R \circ S = 0$) then it is locally-Ricci-symmetric ($\nabla S = 0$); c) If an almost Kenmotsu manifold is semi-conformally-symmetric ($R \circ C = 0$) then it is a conformally flat manifold ($C = 0$). Some other properties equivalent to the above are found.

1. Introduction

A Riemannian manifold (M, g) is called semi-symmetric if its curvature tensor R satisfies the condition $R \circ R = 0$. A complete instrinsic classification of these spaces was given by Z. I. Szabó [6]. However it is interesting to investigate the semi-symmetry of special Riemannian manifolds. K. Nomizu proved [4] that if M is a complete, connected semi-symmetric hypersurface of a euclidean space R^{n+1} $n > 3$, (i.e.: $R \circ R = 0$) then M is locally symmetric (i.e.: $\nabla R = 0$). For the case of a compact Kähler manifold M. Ogawa [5] proved that if it is semi-symmetric then it must be locally-symmetric. In the case of contact manifolds S. Tanno [7], showed that there exists no proper semi-symmetric (or semi-Ricci-symmetric) K-contact manifold.

Recently M. C. Chaki and M. Tarafdar [1] proved that if the curvature tensor R and the conformal curvature tensor C of a Sasakian manifold M^n ($n > 3$) satisfy the relation $R \circ C = 0$, then M^n is locally isometric with a unit sphere $S^n(1)$. Similar results were obtained by N. Guha and U. C. De [2] for the case of a K-contact manifold with characteristic vector field belonging to the K-nullity distribution.

In the present paper we consider an almost Kenmotsu manifold [3] satisfying one of the following conditions:

$$R \circ R = 0, \quad R \circ S = 0 \quad \text{or} \quad R \circ C = 0,$$

and among others we show that:

i) If an almost Kenmotsu manifold is semi-symmetric ($R \circ R = 0$) then it is locally symmetric ($\nabla R = 0$);

ii) If an almost Kenmotsu manifold is semi-Ricci-symmetric ($R \circ S = 0$) then it is locally-Ricci-symmetric ($\nabla S = 0$);

iii) If an almost Kenmotsu manifold is semi-conformally-symmetric ($R \circ C = 0$) then it is a conformally flat manifold ($C = 0$).

2. Preliminaries

Let $(M, \varphi, \xi, \eta, g)$ be an n dimensional almost contact Riemannian manifold, where φ is a $(1,1)$ tensor field, ξ is the structure vector field, η is a 1-form and g is a Riemannian metric. It is well-known that φ, ξ, η, g satisfy

$$\eta(\xi) = 1, \quad \varphi\xi = 0, \quad \eta \circ \varphi = 0 \tag{2.1}$$

$$\varphi^2 = -\operatorname{id} + \eta \otimes \xi, \quad \eta(X) = g(X, \xi) \tag{2.2}$$

$$g(\varphi x, \varphi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2.3}$$

for any vector fields X, Y on M.

If moreover

$$(\nabla_X \varphi)Y = -g(X, \varphi Y)\xi - \eta(X)\varphi X \tag{2.4}$$

$$\nabla_X \xi = X - \eta(X)\xi \tag{2.5}$$

where ∇ denotes the Riemannian connection of g, then $(M, \varphi, \xi, \eta, g)$ is called an almost Kenmotsu manifold. An almost Kenmotsu manifold is a nice example of an almost contact manifold which is not a K-contact (and hence not a Sasakian) manifold (see Kenmotsu [3]).

Now let us recall some important curvature-properties of almost Kenmotsu manifolds. (For details see [3]). We have

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X \tag{2.6}$$

$$S(X, \xi) = -2n\eta(X), \tag{2.7}$$

where S denotes the Ricci curvature tensor. From (2.6) it easily follows that

$$R(X, \xi)Y = \langle X, Y\rangle\xi - \eta(Y)X \tag{2.8}$$

$$R(X, \xi)\xi = \eta(X)\xi - X. \tag{2.9}$$

3. Almost Kenmotsu manifolds with $\mathbf{R} \circ \mathbf{R} = 0$ or $\mathbf{R} \circ \mathbf{S} = 0$

In this section we show that in the case of almost Kenmotsu manifolds we also have Tanno-type results [7]. Namely we have the following

Theorem 1. *For an almost Kenmotsu manifold the following conditions are eqivalent:*

i) *M is of constant curvature -1.*
ii) *M is locally-symmetric, i.e.: $\nabla R = 0$.*
iii) *M is semi symmetric, i.e.: $R \circ R = 0$.*
iv) *$R(X,\xi) \circ R = 0$ for any $X \in \mathfrak{X}(M)$.*

Proof. i) $\Longrightarrow$ ii) $\Longrightarrow$ iii) $\Longrightarrow$ iv) is clear.

We are going to prove iv) $\Longrightarrow$ i). Assume condition iv), which is equivalent to

$$R(X,\xi)R(U,V)W - R(R(X,\xi)U,V)W - R(U,R(X,\xi)V)W - \tag{3.1}$$
$$-R(U,V)R(X,\xi)W = 0 \quad \forall\, U,V,W \in \mathfrak{X}(M).$$

Put $U = \xi$ in (3.1). Using (2.8), (2.9) we get:

(3.2)
$$R(X,\xi)R(\xi,V)W = R(X,\xi)(\eta(W)V - \langle V,W\rangle\xi) = \eta(W)R(X,\xi)V -$$
$$-\langle V,W\rangle R(X,\xi)\xi = \eta(W)(\langle X,V\rangle\xi - \eta(V)X) - \langle V,W\rangle(\eta(X)\xi - X) =$$
$$= \eta(W)\langle X,V\rangle\xi - \eta(V)\eta(W)X - \langle V,W\rangle\eta(X)\xi + \langle V,W\rangle X.$$

$$R(R(X,\xi)\xi,V)W = R(\eta(X)\xi - X,V)W = \tag{3.3}$$
$$= \eta(X)R(\xi,V)W - R(X,V)W = \eta(X)(\eta(W)V - \langle V,W\rangle\xi) - R(X,V)W =$$
$$= \eta(X)\eta(W)V - \eta(X)\langle V,W\rangle\xi - R(X,V)W.$$

$$R(\xi,R(X,\xi)V)W = \eta(W)R(X,\xi)V - \langle R(X,\xi)V,W\rangle\xi = \tag{3.4}$$
$$= \eta(W)(\langle X,V\rangle\xi - \eta(V)X)\langle\langle X,V\rangle\xi - \eta(V)X,W\rangle\xi =$$
$$= \eta(W)\langle X,V\rangle\xi - \eta(V)\eta(W)X - \langle X,V\rangle\xi(W)\xi + \eta(V)\langle X,W\rangle\xi.$$

$$R(\xi,V)R(X,\xi)W = \eta(R(X,\xi)W)V - \langle V,R(X,\xi)W\rangle\xi = \tag{3.5}$$
$$= \langle\langle X,W\rangle\xi - \eta(W)X,\xi\rangle V - \langle V,\langle X,W\rangle\xi - \eta(W)X\rangle\xi =$$
$$= \langle X,W\rangle V - \eta(X)\eta(W)V - \langle X,W\rangle\eta(V)\xi + \eta(W)\langle X,V\rangle\xi.$$

Taking into account (3.2–5) and using (3.1) we obtain:

$$\langle V,W\rangle X + R(X,V)W - \langle X,W\rangle V = 0$$

or

$$R(X,V)W = \langle X,W\rangle V - (V,W)X,$$

that is M is of constant curvature -1. $\Diamond$

Theorem 2. *In an almost Kenmotsu manifold the following conditions are equivalent:*

i) *M is an Einstein space with $S = -2ng$*
ii) $\nabla S = 0$
iii) $R(X,Y) \circ S = 0$ *for any X, Y*
iv) $R(X,\xi) \circ S = 0$ *for any X,*

where S denotes the Ricci curvature tensor.

Proof. i) $\Longrightarrow$ ii) $\Longrightarrow$ iii) $\Longrightarrow$ iv) is clear.

Now we assume condition iv), which is equivalent to

$$S(R(X,\xi)U,V) + S(U,R(X,\xi)V) = 0, \tag{3.6}$$

and we conclude i).

From (2.8) and (2.7) it follows that

$$S(R(X,\xi)U,V) = S(\langle X,U\rangle\xi - \eta(U)X,V) = \tag{3.7}$$

$$= \langle X,U\rangle S(\xi,V) - \eta(U)S(X,V) = -2n\langle X,U\rangle\eta(V) - \eta(U)S(X,V)$$

$$S(U,R(X,\xi)V) = S(U,\langle X,V\rangle\xi - \eta(V)X) = \tag{3.8}$$

$$= \langle X,V\rangle S(U,\xi) - \eta(V)S(U,X) = -2n\langle X,V\rangle\eta(U) - \eta(V)S(U,X).$$

These yield

$$2n\langle X,U\rangle\eta(V) + \eta(U)S(X,V) + 2n\langle X,V\rangle\eta(U) + \eta(V)S(U,X) = 0.$$

Put $U = \xi$. Using (2.7) we get

$$S(X,V) + 2n\langle X,V\rangle = 0.$$

Therefore M is an Einstein space with $S = -2ng$. $\Diamond$

4. Almost Kenmotsu manifolds with $\mathbf{R} \circ \mathbf{C} = 0$

Sasakian manifolds with $R \circ C = 0$ (where C is the conformal curvature tensor) were investigated by M. Chaki and M. Tarafdar [1]. They showed that a Sasakian manifold satisfying $R \circ C = 0$ is of constant curvature 1, and hence it is locally isomorphic with $S^n(1)$.

Now we show

Theorem 3. *For an almost Kenmotsu manifold M^{2n+1} the following conditions are equivalent:*

i) *M is of constant curvature -1*
ii) *M is conformally flat i.e.: $C = 0$*
iii) *M is conformally symmetric i.e.: $\nabla C = 0$*

iv) *M is semi-conformally-symmetric i.e.:* $R(X,Y)\circ C=0$ *for any* X, Y.

v) $R(X,\xi)\circ C=0$ *for any* X.

Proof. i) $\iff$ ii) is proved by Kenmotsu ([3], Prop. 11).

i) $\implies$ ii) $\implies$ iii) $\implies$ iv) $\implies$ v) is clear.

So it is enough to see that v) $\implies$ ii).

By definition

$$C(X,Y)Z=R(X,Y)Z- \tag{4.1}$$

$$-\frac{1}{2n-1}\{g(Y,Z)QX-g(X,Z)QY+S(Y,Z)X-S(X,Z)Y\}+$$

$$+\frac{r}{2n(2n-1}\{g(Y,Z)X-g(X,Z)Y\},$$

where Q is the Ricci operator defined by $S(X,Y)=g(QX,Y)$.

Assume condition v), which is equivalent to

$$(RX,\xi)C(U,V)W-C(R(X,\xi)U,V)W- \tag{4.2}$$

$$-C(U,R(X,\xi)V)W-C(U,V)R(X,\xi W)=0.$$

According to (4.1) we have

$$\langle C(X,Y)Z,\xi\rangle=\langle R(X,Y)Z,\xi\rangle-$$

$$-\frac{1}{n-2}\{\langle Y,Z\rangle\langle QX,\xi\rangle-\langle X,Z\rangle\langle QY,\xi\rangle+$$

$$+S(Y,Z)\langle X,\xi\rangle-S(X,Z)\langle Y,\xi\rangle\}+$$

$$+\frac{r}{(n-1)(n-2)}\{\langle Y,Z\rangle\langle X,\xi\rangle-\langle X,Z\rangle\langle Y,\xi\rangle\}.$$

From (2.7) it follows that

$$\langle QX,\xi\rangle=S(X,\xi)=-2n\eta(X)\quad\text{for any } X.$$

Thus

$$\langle C(X,Y)Z,\xi\rangle=-\langle R(X,Y)\xi,Z\rangle-$$

$$\frac{1}{2n-1}\{-2n\langle Y,Z\rangle\eta(Y)+2n\langle X,Z\rangle\eta(Y)+S(Y,Z)\eta(X)-S(X,Z)\eta(Y)\}+$$

$$+\frac{r}{2n(2n-1)}\{\langle Y,Z\rangle\eta(X)-\langle X,z\rangle\eta(Y)\},$$

or

(4.3)
$$\langle C(X,Y)Z,\xi\rangle = \frac{1}{2n-1}\Big[\Big(\frac{r}{2n}+1\Big)\left(\langle Y,Z\rangle\eta(X)-\langle X,Z\rangle\eta(Y)\right)-$$
$$-\big(S(Y,Z)\eta(X)-S(X,Y)\eta(Y)\big)\Big].$$

Taking $X=\xi$ in (4.3), we get

$$(4.4)\qquad \langle C(\xi,Y)Z,\xi\rangle = \frac{1}{2n-1}\Big[(\langle Y,Z\rangle-\eta(Z)\eta(Y))\cdot\Big(\frac{r}{2n}+1\Big)-$$
$$-(S(Y,Z)+2n\eta(Z)\eta(Y)\Big].$$

Now applying (4.3), (4.4) to (4.2) we get

$$(4.5)\qquad \big\langle R(X,\xi)C(U,V)W,\xi\big\rangle = \big\langle C(U,V)W,X\big\rangle - \big\langle C(U,V)W,\xi\big\rangle =$$
$$= \big\langle C(U,V)W,X\rangle - \eta(X) - \eta(X)\frac{1}{2n-1}\Big[\Big(\frac{r}{2n}+1\Big)\big(\langle V,W\rangle\eta(U)-$$
$$-\langle U,W\rangle\eta(V)\big) - (S(V,W)\eta(U) - S(U,W)\eta(V))\Big]$$

$$(4.6)\qquad \big\langle C(R(X,\xi)U,V)W,\xi\big\rangle = \big\langle C(\langle X,Y\rangle\xi-\eta(U)X,V)W,\xi\big\rangle =$$
$$= \langle X,U\rangle\langle C(\xi,V)W,\xi\rangle - \eta(U)\langle C(X,V)W,\xi\rangle = \langle X,Y\rangle\frac{1}{2n-1}\cdot$$
$$\cdot\Big[\Big(\frac{r}{2n}+1\Big)\big(\langle V,W\rangle-\eta(V)\eta(W)\big)-(S(V,W)+2n\eta(V)\eta(W)\Big]-$$
$$-\eta(U)\cdot\frac{1}{2n-1}\Big[\Big(\frac{1}{2n}+1\Big)\left(\langle V,W\rangle\eta(X)-\langle X,W\rangle\eta(V)\right)-$$
$$-(S(V,W)\eta(X)-S(X,W)\eta(V)\Big]$$

(4.7)
$$\big\langle C(U,R(X\xi))W,\xi\big\rangle = -\big\langle C(R(X,\xi)V,U)W,\xi\rangle = -\langle X,V\rangle\frac{1}{2n-1}\cdot$$

$$\cdot\left[\left(\frac{r}{2n}+1\right)(\langle U,W\rangle-\eta(U)\eta(W))-(S(U,W)+2n\eta(U)\eta(W)\right]+$$

$$+\eta(V)\frac{1}{2n-1}\left[\left(\frac{r}{2n}+1\right)(\langle U,W\rangle\eta(X)-\langle X,W\rangle\eta(U))-\right.$$

$$\left.-(S(U,W)\eta(X)-S(X,W)\eta(U)\right]$$

$$\text{(4.8)}\quad \langle C(U,V)R(X,\xi)W,\xi\rangle=\langle C(U,V)(\langle X,W\rangle\xi-\eta(W)X),\xi\rangle=$$

$$=\langle X,W\rangle\langle C(U,V)\xi,\xi\rangle-\eta(W)\langle C(U,V)X,\xi\rangle-$$

$$-\eta(W)\cdot\frac{1}{2n-1}\left[\left(\frac{r}{2n}\right)(\langle V,X\rangle\eta(U)-\langle U,X\rangle\eta(U))-\right.$$

$$\left.-\big(S(V,X)\eta(U)-S(U,X)\eta(V)\big)\right].$$

Thus (4.5–8) yield

$$\text{(4.9)}\qquad \langle C(U,V)W,X\rangle-\eta(U)\frac{1}{2n-1}\left(\frac{r}{2n}+1\right)\langle X,W\rangle\eta(V)-$$

$$-\langle X,U\rangle\frac{1}{2n-1}\left[\left(\frac{r}{2n}+1\right)\langle V,W\rangle-S(V,W)-2n\eta(V)\eta(W)\right]+$$

$$+\langle X,V\rangle\frac{1}{2n-1}\left[\left(\frac{r}{2n}+1\right)\langle U,W\rangle-S(U,W)-2n\eta(U)\eta(W)\right]+$$

$$+\eta(V)\frac{1}{2n-1}\left[\left(\frac{r}{2n}+1\right)\langle X,W\rangle\eta(U)\right]-$$

$$-\eta(W)\cdot\frac{1}{2n-1}\big(S(V,X)\eta(U)-S(U,X)\eta(V)\big)=0.$$

Let $\{e_i,\ i=1,\dots,2n+1\}$ be an orthonormal basis of the tangent space at the points of M. Then from (4.1) it follows that

$$\text{(4.10)}\qquad \sum_{i=1}^{2n+1}\langle C(e_i,Y)Z,e_i\rangle=0.$$

Put $U=X=e_i$ in (4.9). Summarizing for $1\le i\le 2n+1$ and taking into account (4.10), we obtain

$$(4.11) \qquad S(V,W) = \left(\frac{r}{2n}+1\right) g(Y,Z) + \left(\frac{r}{2n}+1\right) \eta(V)\eta(W).$$

Finallly, using (4.11), (4.9) reduces to

$$\langle C(U,V)W, X\rangle = 0$$

i.e.: $C(U,V)W = 0$. ◊

A comparision of Theorems 1 and 3 shows that in an almost Kenmotsu manifold $R \circ R = 0$ is equivalent to $R \circ C = 0$.

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