

# CONVEXITY STRUCTURES IN ZERO-DIMENSIONAL COMPACT SPACES

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**Abstract:** We investigate some properties of compact zero-dimensional spaces with additional convexity structures. As a main result, we prove that every retract of a Cantor cube has a binary subbase closed under complementation.

## Introduction

We show that every retract of a Cantor cube, called a *zero-dimensional Dugundji space*, has a binary subbase closed under complementation. This strengthens the result of Heindorf [3] that zero-dimensional Dugundji spaces are supercompact and admit a binary subbase consisting of clopen sets. Introducing a suitable *convexity* (see the definitions

below) we construct a binary subbase consisting of convex sets, which is closed under complements. We also state and use some properties of inverse systems of topological convexity spaces. The proof of our main result is simpler than Heindorf's one and does not require algebraic or lattice structures.

## 1. Preliminaries

By a *geometrical space* we mean a set  $X$  together with a collection  $\mathcal{G} \subset \mathcal{P}(X)$  such that:

- (1)  $\emptyset, X \in \mathcal{G}$ ,
- (2)  $\bigcap \mathcal{A} \in \mathcal{G}$  for nonempty  $\mathcal{A} \subset \mathcal{G}$ ,
- (3) If  $A \subset X$  and for every  $a, b \in A$  there exists a  $G \in \mathcal{G}$  with  $a, b \in G \subset A$  then  $A \in \mathcal{G}$ .

A collection  $\mathcal{G}$  satisfying these conditions is called an *interval convexity*, see Calder [1]. Elements of  $\mathcal{G}$  will be called *convex sets*. A *halfspace* is a convex set with convex complement. The *segment joining*  $a, b \in X$  is defined by  $[a, b] = \bigcap \{G \in \mathcal{G} : a, b \in G\}$ . Note that  $G \in \mathcal{G}$  iff for every  $a, b \in G$  it holds that  $[a, b] \subset G$ . Consequently, an interval convexity is determined by its segments. If  $(X, \mathcal{G})$  is a geometrical space and  $M \subset X$  then  $\mathcal{G}_M = \{A \subset M : \forall a, b \in A, [a, b] \cap M \subset A\}$  is easily seen to be an interval convexity in  $M$ , called the *subspace convexity*. Clearly,  $G \cap M \in \mathcal{G}_M$  whenever  $G \in \mathcal{G}$  and the segment joining points  $a, b$  in  $M$  equals  $[a, b] \cap M$ . If  $(X, \mathcal{G})$  and  $(Y, \mathcal{H})$  are two geometrical spaces then the product  $X \times Y$  is a geometrical space with the interval convexity consisting of all sets  $G \times H$  where  $G \in \mathcal{G}$ ,  $H \in \mathcal{H}$  (see [7, p. 14]). Note that  $[\langle a_1, a_2 \rangle, \langle b_1, b_2 \rangle] = [a_1, b_1] \times [a_2, b_2]$  for each  $\langle a_1, a_2 \rangle, \langle b_1, b_2 \rangle \in X \times Y$ . Let  $X$  and  $Y$  be two geometrical spaces. A map  $f: X \rightarrow Y$  is called *convexity preserving*, *cp-map* for short, provided  $f^{-1}(G)$  is convex in  $X$  for each convex set  $G \subset Y$ . This is equivalent to the condition  $f([a, b]) \subset [f(a), f(b)]$  for each  $a, b \in X$ , see [7, p. 15]. For the study of convexity theory we refer to van de Vel's monograph [7].

A *Boolean median space* is a triple  $(X, \mathcal{T}, \mathcal{G})$  where  $(X, \mathcal{G})$  is a geometrical space,  $(X, \mathcal{T})$  is a compact topological space and the following conditions are satisfied:

- (M1) For each two distinct points  $a, b \in X$  there exists a clopen half-space  $H \subset X$  such that  $a \in H$  and  $b \notin H$ .
- (M2) For each  $a, b, c \in X$  the set  $[a, b] \cap [a, c] \cap [b, c]$  is nonempty.

Condition (M1) implies that the intersection  $[a, b] \cap [a, c] \cap [b, c]$  consists of a single point, called the *median of  $a, b, c$*  and denoted by  $m(a, b, c)$ . Let  $X$  be a Boolean median space. A subset  $M \subset X$  is *median-stable* [7, p. 121] provided  $m(a, b, c) \in M$  whenever  $a, b, c \in M$ . Every closed median-stable subset of  $X$  with the subspace topology and the subspace convexity is again a Boolean median space. Observe that the product of two Boolean median spaces is a Boolean median space. Every Boolean median space is a topological median space in the sense of van de Vel [7, p. 269] and it is a zero-dimensional compact Hausdorff topological space.

The following fact will be essential for us. Namely, the collection of all clopen halfspaces in a Boolean median space forms a binary closed subbase for its topology (cf. van Mill [5, Th. 1.3.3] or van de Vel [7, Th. II.1.7]). Recall that a nonempty family of sets is *binary* if each its subcollection with empty intersection contains two disjoint sets.

Let  $\Sigma$  be a directed partially ordered set and let  $S = \{X_\sigma, p_\sigma^\tau, \Sigma\}$  be an inverse system of sets such that each  $X_\sigma$  is a geometrical space and  $p_\sigma^\tau$ 's are cp-maps, i.e.  $p_\sigma^\tau: X_\tau \rightarrow X_\sigma$  and  $p_\sigma^\tau p_\tau^\mu = p_\sigma^\mu$  whenever  $\sigma \leq \tau \leq \mu$ . The system  $S$  will be called an *inverse system of geometrical spaces*. Let  $\varprojlim S$  be the inverse limit of  $S$  in the category of sets, i.e. the set consisting of all points  $x \in \prod_{\sigma \in \Sigma} X_\sigma$  such that  $p_\sigma^\tau(x_\tau) = x_\sigma$  for all  $\sigma \leq \tau$ . Denote by  $p_\sigma$  the projection of  $\varprojlim S$  into  $X_\sigma$ .

The proof of the following proposition is straightforward and therefore will be omitted.

**Proposition 1.1.** *Let  $S = \{X_\sigma, p_\sigma^\tau, \Sigma\}$  be an inverse system of geometrical spaces. There exists an interval convexity  $\mathcal{G}$  in  $\varprojlim S$  with the following properties:*

- (a) *The geometrical space  $(\varprojlim S, \mathcal{G})$  is the inverse limit in the category of geometrical spaces. In other words, if  $Y$  is a geometrical space and  $\{f_\sigma: Y \rightarrow X_\sigma\}_{\sigma \in \Sigma}$  is a collection of cp-maps such that  $p_\sigma^\tau f_\tau = f_\sigma$  for  $\sigma \leq \tau$  then there exists a unique cp-map  $h: Y \rightarrow \varprojlim S$  with the property  $p_\sigma h = f_\sigma$  for all  $\sigma \in \Sigma$ .*
- (b)  *$\mathcal{G}$  is the least convexity in  $\varprojlim S$  such that all the projections  $p_\sigma$  are cp-maps.*
- (c) *For every  $a, b \in \varprojlim S$  it holds that  $[a, b]_{\mathcal{G}} = X \cap \prod_{\sigma \in \Sigma} [a_\sigma, b_\sigma]$ .*

By an *inverse system of Boolean median spaces* we mean an inverse system  $S = \{X_\sigma, p_\sigma^\tau, \Sigma\}$  where  $X_\sigma$ 's are Boolean median spaces and each  $p_\sigma^\tau$  is cp and continuous.

**Proposition 1.2.** *The inverse limit of a system of Boolean median spaces is a Boolean median space.*

**Proof.** Let  $S = \{X_\sigma, p_\sigma^\tau, \Sigma\}$  be an inverse system of Boolean median spaces. Let  $p_\sigma: \varprojlim S \rightarrow X_\sigma$  be the projection. The collection

$$\{p_\sigma^{-1}(H) : \sigma \in \Sigma, H \text{ is a clopen halfspace in } X_\sigma\}$$

is point-separating and consists of clopen halfspaces in  $\varprojlim S$ . This shows (M1).

Fix  $a, b, c \in \varprojlim S$ . For each  $\sigma \in \Sigma$  let  $x_\sigma = m(a_\sigma, b_\sigma, c_\sigma)$ . If  $\tau \leq \sigma$  then  $p_\tau^\sigma(x_\sigma) = m(a_\tau, b_\tau, c_\tau)$  since  $p_\tau^\sigma$  is cp. Hence  $p_\tau^\sigma(x_\sigma) = x_\tau$ . It follows that  $x = \{x_\sigma\}_{\sigma \in \Sigma} \in \varprojlim S$  and  $x \in [a, b] \cap [a, c] \cap [b, c]$ . This shows (M2).  $\diamond$

## 2. Some properties of Boolean median spaces

Fundamental examples of Boolean median spaces are Cantor cubes. Specifically, for a Cantor cube  $\{0, 1\}^\kappa$  we call its subset  $M$  *convex* iff

$$\{x \in \{0, 1\}^\kappa : a^{-1}(1) \cap b^{-1}(1) \subset x^{-1}(1) \subset a^{-1}(1) \cup b^{-1}(1)\} \subset M,$$

for every  $a, b \in M$  (cf. [7, p. 60]). The set  $H_\alpha = \{x \in \{0, 1\}^\kappa : x(\alpha) = 1\}$  is a clopen halfspace in  $\{0, 1\}^\kappa$  and the collection  $\{H_\alpha\}_{\alpha < \kappa}$  is point-separating. Finally, the median of  $a, b, c \in \{0, 1\}^\kappa$  is the characteristic function of  $(a^{-1}(1) \cap b^{-1}(1)) \cup (a^{-1}(1) \cap c^{-1}(1)) \cup (b^{-1}(1) \cap c^{-1}(1))$ .

The following lemma is an analogue of the result of van Mill and Wattel [6].

**Lemma 2.1** cf. [7, Lemma I.3.16]. *Every Boolean median space of weight  $\leq \kappa$  is isomorphic to a closed median-stable subset of a Cantor cube  $\{0, 1\}^\kappa$ .*

**Proof.** Let  $X$  be a Boolean median space of weight  $\leq \kappa$ . Then there exists a subbase of the topology  $\{H_\alpha\}_{\alpha < \kappa}$  consisting of clopen halfspaces in  $X$ . Put  $j(x)(\alpha) = 1$  iff  $x \in H_\alpha$ . This defines a map  $j: X \rightarrow \{0, 1\}^\kappa$ , which is a topological embedding. By [7, Prop. I.1.12] it remains to show that  $j([a, b]) = [j(a), j(b)] \cap j(X)$  for each  $a, b \in X$ . If  $x \in [a, b]$  then  $j(a)(\alpha) = j(b)(\alpha) = 1$  implies  $j(x)(\alpha) = 1$  and  $j(x)(\alpha) = 1$  implies  $j(a)(\alpha) = 1$  or  $j(b)(\alpha) = 1$ . This means that  $j(x) \in [j(a), j(b)]$ . If  $x \notin [a, b]$  then taking  $y = m(a, b, x)$  we get  $x \neq y$  and hence there exists an  $\alpha < \kappa$  with  $y \in H_\alpha$  and  $x \notin H_\alpha$ . Now  $a, b \in H_\alpha$  and therefore  $j(a)(\alpha) = j(b)(\alpha) = 1$  while  $j(x)(\alpha) = 0$ . Hence  $j(x) \notin [j(a), j(b)]$ .  $\diamond$

**Theorem 2.2.** *Let  $X$  be a Boolean median space of weight  $\kappa \geq \omega$ . There exists an inverse system of Boolean median spaces  $S = \{X_\alpha, p_\alpha^\beta, \alpha < \beta < \kappa\}$  such that  $X = \varprojlim S$  and*

- (1)  $|X_0| = 1$ .
- (2)  $X_\lambda = \varprojlim\{X_\alpha : \alpha < \lambda\}$ , for a limit ordinal  $\lambda < \kappa$ .
- (3) For each  $\alpha < \kappa$  there exist closed convex sets  $A_\alpha, B_\alpha \subset X_\alpha$  such that  $A_\alpha \cup B_\alpha = X_\alpha$ ,  $X_{\alpha+1} = (A_\alpha \times \{0\}) \cup (B_\alpha \times \{1\})$ ,  $p_\alpha^{\alpha+1}$  is the projection and the convexity of  $X_{\alpha+1}$  is inherited from the product  $X_\alpha \times \{0, 1\}$ .

**Proof.** In view of Lemma 2.1 we may assume that  $X$  is a closed median-stable subspace of  $\{0, 1\}^\kappa$ . Set  $X_\alpha = \{x|\alpha : x \in X\}$ . Let  $p_\alpha^\beta : X_\beta \rightarrow X_\alpha$  be the projection. Clearly, each  $p_\alpha^\beta$  is continuous cp and  $p_\alpha^\beta p_\beta^\gamma = p_\alpha^\gamma$ . Let

$$A_\alpha = \{x|\alpha : x(\alpha) = 0, x \in X\},$$

$$B_\alpha = \{x|\alpha : x(\alpha) = 1, x \in X\}.$$

We have  $A_\alpha \cup B_\alpha = X_\alpha$  and  $A_\alpha, B_\alpha$  are closed. One can easily check that  $X = \varprojlim S$  and conditions (1), (2) are satisfied. Clearly  $X_{\alpha+1} = (A_\alpha \times \{0\}) \cup (B_\alpha \times \{1\})$ . It remains to show that  $A_\alpha, B_\alpha$  are convex.

Let  $x|\alpha, y|\alpha \in A_\alpha$  and  $z|\alpha \in [x|\alpha, y|\alpha]$ , where  $x, y, z \in X$ . Setting  $v = m(x, y, z)$  we see that  $v|\alpha = z|\alpha$  and  $v(\alpha) = m(x(\alpha), y(\alpha), z(\alpha)) = 0$ ; hence  $z|\alpha \in A_\alpha$ .  $\diamond$

### 3. Main result

The following lemma is known, we present a proof for the sake of completeness.

**Lemma 3.1.** *Let  $\mathcal{P}$  be a closed under complements subbase of a zero-dimensional compact topological space  $X$  and let  $\mathcal{B}$  be the collection of all finite intersections of sets from  $\mathcal{P}$ . Then every clopen subset of  $X$  can be partitioned into a finite number of members of  $\mathcal{B}$ .*

**Proof.** Set  $\mathfrak{M} = \{M_1 \cup \dots \cup M_n : M_i \in \mathcal{B}, M_i \cap M_j = \emptyset \text{ for } i \neq j, n \in \mathbb{N}\}$ . Since  $\mathcal{B} \subset \mathfrak{M}$  and  $\mathcal{B}$  is an open base of  $X$ , it is enough to show that  $\mathfrak{M}$  is an algebra of sets. Fix  $B, C \in \mathfrak{M}$  and let  $B$  and  $C$  have partitions  $M_1 \cup \dots \cup M_n$  and  $N_1 \cup \dots \cup N_k$  respectively, where  $M_i, N_j \in \mathcal{B}$ . We have  $B \cap C = \bigcup_{i,j} M_i \cap N_j$  and  $M_i \cap N_j$ 's are pairwise disjoint. It follows that  $B \cap C \in \mathfrak{M}$ . Now observe that  $X \setminus M \in \mathfrak{M}$  for  $M \in \mathcal{B}$ . Indeed, if  $M = H_1 \cap \dots \cap H_n$  where  $H_i \in \mathcal{P}$  for  $i \leq n$ ,

then  $X \setminus M$  has a partition into sets of the form  $H_1^{\varepsilon(1)} \cap \dots \cap H_n^{\varepsilon(n)}$  where  $\varepsilon: \{1, \dots, n\} \rightarrow \{-1, 1\}$  is a function not equal constantly to 1 and  $H_i^1 = H_i, H_i^{-1} = X \setminus H_i$ . Finally, if  $B = M_1 \cup \dots \cup M_n \in \mathfrak{M}$  where  $M_i \in \mathcal{B}$  then the set  $X \setminus B = \bigcap_{i \leq n} (X \setminus M_i)$  does belong to  $\mathfrak{M}$ . This completes the proof.  $\diamond$

**Theorem 3.2.** *If  $X$  is a retract of a Cantor cube then there exists a convexity in  $X$  such that  $X$  is a Boolean median space.*

**Proof.** According to [2] and [4, Th. 2.7] we can represent  $X$  as the limit of an inverse system  $S = \{X_\alpha, p_\alpha^\beta, \alpha < \beta < \tau\}$  with the following properties:

- (1)  $|X_0| = 1,$
- (2)  $X_\gamma = \varprojlim \{X_\alpha, \alpha < \gamma\}$  for limit ordinals  $\gamma < \tau,$
- (3)  $X_{\alpha+1} = (X_\alpha \times \{0\}) \cup (U_\alpha \times \{1\})$  where  $U_\alpha$  is clopen in  $X_\alpha$  and  $p_\alpha^{\alpha+1}: X_{\alpha+1} \rightarrow X_\alpha$  is the projection.

We define inductively suitable convexities in  $X_\alpha$ 's in such a way that each  $X_\alpha$  becomes a Boolean median space and each  $p_\alpha^{\alpha+1}$  becomes cp. Suppose that this is already done for all  $\xi < \gamma$  and assume that  $\gamma = \alpha + 1$ .

By Lemma 3.1,  $U_\alpha = G_1 \cup \dots \cup G_n$  where  $G_i$ 's are pairwise disjoint clopen and convex. Hence  $X_{\alpha+1} = X_\alpha \oplus G_1 \oplus \dots \oplus G_n$  and  $p_\alpha^{\alpha+1}$  is the superposition of  $n$  projections of the form  $X_\alpha \oplus G_1 \oplus \dots \oplus G_{i+1} \rightarrow X_\alpha \oplus G_1 \oplus \dots \oplus G_i$ . Thus we may assume that  $U_\alpha$  is convex. Now  $X_{\alpha+1}$  is a median-stable subset (the union of two convex sets) of the product  $X_\alpha \times \{0, 1\}$ . It follows that  $X_{\alpha+1}$  with the subspace convexity is a Boolean median space. Clearly  $p_\alpha^{\alpha+1}$  is cp.

If  $\gamma$  is a limit ordinal and convexities  $\mathcal{G}_\alpha$  are already defined for  $\alpha < \gamma$  then, by Prop. 1.2,  $X_\gamma$  with the convexity of the limit is a Boolean median space. This completes the proof.  $\diamond$

**Remark 3.3.** Actually, we have proved that if  $X$  is a zero-dimensional Dugundji space then there exists a convexity  $\mathcal{G}$  in  $X$  and an inverse system of Boolean median spaces  $S = \{(X_\alpha, \mathcal{G}_\alpha), p_\alpha^\beta, \alpha < \beta < \tau\}$  such that  $(X, \mathcal{G}) = \varprojlim S$  and  $S$  has properties (1)–(3) above, with  $X_{\alpha+1} = X_\alpha \oplus U_\alpha$ , where  $U_\alpha$  is clopen and convex in  $X_\alpha$ . On the other hand, by Haydon's Theorem [2], the inverse limit of such a system  $S$  is a topological zero-dimensional Dugundji space.

**Corollary 3.4.** *Every retract of a Cantor cube has a binary subbase closed under complementation.*

**Proof.** The desired subbase consists of the all clopen halfspaces, with

respect to the convexity given by Th. 3.2.  $\diamond$

The example below shows that the converse does not hold.

**Example 3.5.** Consider the one-point compactification  $\alpha\kappa$  of the discrete space of cardinality  $\kappa$ . Let  $\mathcal{P}$  be the collection of all one-element subsets of  $\kappa$  and all their complements. One can check that  $\mathcal{P}$  is a binary subbase. On the other hand, if  $\kappa > \omega$  then  $\alpha\kappa$  is not dyadic and therefore cannot be a Dugundji space.

**Question.** Does there exist a space with a binary subbase consisting of clopen sets, which does not have a binary subbase closed under complementation?

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