Mathematica Pannonica
12/1 (2001), 129–132

AN INFINITE-DIMENSIONAL SUBSPACE OF C[0,1] CONSISTING OF FUNCTIONS WITH NO FINITE ONE-SIDED DERIVATIVES

Roland **Girgensohn**

GSF — National Research Center for Environment and Health, Institute of Biomathematics and Biometry, Ingolstädter Landstraße 1, D-85764 Neuherberg, Germany

Received: June 2000

MSC 2000: 26 A 27

Keywords: Nowhere differentiable functions, van der Waerden function, Faber-Schauder system.

Abstract: We give a simple construction of an infinite dimensional, closed subspace E of $C[0,1]$ such that every non-zero element of E has no finite one-sided derivative anywhere.

This note is motivated by the recent paper [2] where a closed, infinite-dimensional subspace E of $C[0,1]$ is constructed with the following properties:

(1) Every element of E except the zero function is nowhere differentiable, and

(2) there is a set A of Lebesgue measure 1 such that every non-zero element of E has no finite one-sided derivative except possibly on the set A.

E-mail address: girgen@gsf.de

This result generalized earlier constructions where only non-differentiability, resp. non-existence of one-sided derivatives almost everywhere was demanded. It was left open if a space exists where every non-zero element has no one-sided derivative anywhere.

This question can be answered using the methods which I have developed in [3] and [4], and via a construction similar to (but slightly simpler than) the one given in [2].

Theorem. *There exists a closed, infinite-dimensional subspace E of $C[0,1]$ such that every non-zero element of E has no finite one-sided derivative anywhere.*

Proof. We begin by reviewing some of the methods of [3] and [4]. The classical Faber-Schauder system is the system of continuous functions

$$\{\sigma_{i,n} : n \in \mathbb{N},\ i = 0, \dots, 2^{n-1} - 1\},$$

where the function $\sigma_{i,n}$ linearly interpolates the points

$$(0,0), \quad \left(\frac{i}{2^{n-1}}, 0\right), \quad \left(\frac{2i+1}{2^n}, 1\right), \quad \left(\frac{i+1}{2^{n-1}}, 0\right), \quad (1,0).$$

The Faber-Schauder system is a basis of the space $\{f \in C[0,1] : f(0) = f(1) = 0\}$, more precisely: Every continuous function $f : [0,1] \to \mathbb{R}$ with $f(0) = f(1) = 0$ has a unique, uniformly convergent expansion of the form

$$f(x) = \sum_{n=1}^{\infty} \sum_{i=0}^{2^{n-1}-1} \gamma_{i,n}(f)\, \sigma_{i,n}(x),$$

where the coefficients $\gamma_{i,n}(f)$ are given by

$$\gamma_{i,n}(f) \;=\; f\left(\frac{2i+1}{2^n}\right) - \frac{1}{2} f\left(\frac{i}{2^{n-1}}\right) - \frac{1}{2} f\left(\frac{i+1}{2^{n-1}}\right).$$

Now, Th. 3 in [3] says that there are simple criteria for non-differentiability of a function f in terms of its Faber-Schauder coefficients: Let f have a finite one-sided derivative at some point in $[0,1]$. Then $\delta_{i,n}(f) := 2^n\, \gamma_{i,n}(f)$ satisfies

(1) $$\lim_{n \to \infty} \min_{i=0,\dots,2^{n-1}-1} |\delta_{i,n}(f)| \;=\; 0.$$

This is a generalization of a theorem by Faber, [Fa10], about two-sided derivatives.

Now, similar to the construction in [2], define $u_1(x) := \mathrm{dist}(x, \mathbb{Z})$, and $u_m(x) := 2^{-(m-1)}\, u_1(2^{m-1}\, x)$ for $m \in \mathbb{N}$. Further, for $p \in \mathbb{N}_0$ let $\sigma_p := \{k \cdot 2^p : k \in \mathbb{N}\}$ and $\varphi_p(x) := \sum_{m \in \sigma_p} u_m(x)$, where this sum is of course uniformly convergent. Now it can be computed, directly from the definition of $\gamma_{i,n}(f)$, that

$$\gamma_{i,n}(u_m) \;=\; \begin{cases} 0, & \text{if } n \neq m, \\ \dfrac{1}{2^{m+1}}, & \text{if } n = m, \end{cases}$$

for all $i = 0, \ldots, 2^{n-1} - 1$. Because of the continuity of the Faber-Schauder coefficients it follows that

$$\gamma_{i,n}(\varphi_p) \;=\; \sum_{m \in \sigma(p)} \gamma_{i,n}(u_m) \;=\; \begin{cases} \dfrac{1}{2^{n+1}}, & \text{if } n \in \sigma_p, \\ 0, & \text{otherwise}, \end{cases}$$

again independently of i. Here the condition $n \in \sigma_p$ is equivalent to the condition $n = k \cdot 2^p$ with some $k \in \mathbb{N}$.

As in [2], it can be proved, along the lines of the proof of the corresponding statement for lacunary trigonometric series, that the sequence $(\varphi_p)_{p \in \mathbb{N}_0}$ is a basis for its closed linear span, $E := \overline{\mathrm{span}\{\varphi_p : p \in \mathbb{N}_0\}}$. Thus every element $\psi \in E$ can be written in the form

$$\psi \;=\; \sum_{p=0}^{\infty} a_p\, \varphi_p$$

with uniform convergence. Take such a $\psi \neq 0$ and assume that in this representation we have $a_0 = \cdots = a_{q-1} = 0$ and $a_q \neq 0$.

For $j \in \mathbb{N}$, let $n_j = (2j+1) \cdot 2^q$. Again using continuity of the Faber-Schauder coefficients, we get

$$\gamma_{i,n_j}(\psi) \;=\; \sum_{p=0}^{\infty} a_p \gamma_{i,n_j}(\varphi_p) \;=\; a_q \cdot \frac{1}{2^{n_j+1}},$$

again for all i, and where we have used that $\gamma_{i,n_j} \neq 0$ if and only if $n_j = (2j+1) \cdot 2^q = k \cdot 2^p$ for some $k \in \mathbb{N}$, and because of $p \geq q$ this is equivalent to $p = q$ and $k = 2j+1$.

This shows that $\delta_{i,n_j}(\psi) = a_q/2$ for all $i = 0, \ldots, 2^{n_j - 1} - 1$, so that for the subsequence n_j we have $\min_i |\delta_{i,n_j}(f)| = a_q/2$. Thus condition (1) cannot be satisfied. $\Diamond$

References

[1] FABER, G.: Über stetige Funktionen I,II, *Math. Ann.* **66** (1908), 81–93; *Math. Ann.***69** (1910), 372–443.

[2] FONF, V.P., GURARIY, V.I. and KADETS, M.I.: An infinite-dimensional subspace of $C[0,1]$ consisting of nowhere differentiable functions, *C. R. Acad. Bulgare Sci.* **52**/11–12 (1999), 13–16.

[3] GIRGENSOHN, R.: Functional equations and nowhere differentiable functions, *Aequationes Math.* **46** (1993), 243–256.

[4] GIRGENSOHN, R.: Nowhere differentiable solutions of a system of functional equations, *Aequationes Math.* **47** (1994), 89–99.