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ON THE TWO-PARAMETER VILENKIN DERIVATIVE

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Abstract: We consider the two-parameter generalization of the dyadic derivative with respect to the Vilenkin system. In the dyadic case it is known that the so-called integral function is a.e. differentiable in this sense. This is a simple consequence of a weak type inequality, i.e. that the maximal operator of the derivative of the integral is of weak type (1,1). Moreover, the maximal operator is bounded from certain Hardy-Lorentz spaces $H^{p,q}$ into the Lorentz space $L^{p,q}$. To this end it is enough to show that this operator is p-quasi-local. The aim of this paper is to give the generalization of these results for the Vilenkin system. For simplicity we formulate all theorems in the two-dimensional case, only. Of course, they can be extended to higher dimensions in a natural way.

1. Introduction

The concept of the so-called dyadic derivative and integral is due to Butzer and Wagner [1]. Later the generalization of this concept with respect to the Vilenkin system was introduced by Onneweer [6] (see also Pál and Simon [7], [8]). In the Walsh case Schipp [9] proved the boundedness of the maximal operator of the dyadic integral from $L^p[0,1)$ to $L^p[0,1)$ $(1 < p \leq \infty)$. Moreover, he shown in his work that this maximal operator is of weak type (1,1). The dyadic analogue of the classical differentiation theorem of the integral function follows from this in a simple way. The Vilenkin analogue of the Schipp's results was proved by Pál and Simon [7], assumed the bounded structure of the Vilenkin group in the question.

For the Walsh system some results of more general characters were given by Weisz [15], [16], [17], in the two-parameter case, too. Namely, he considered the maximal operator mentioned above as mapping from the Hardy-Lorentz space $H^{p,q}$ into the Lorentz space $L^{p,q}$ and proved its boundedness under the assumption $p_0 < p < \infty, 0 < q \leq \infty$ with $p_0 = = 1/2$ in the one dimensional case and $p_0 = 2/3$ for the two-parameter system, respectively. Note that in the case $p = q$ the usual definition of Hardy spaces $H^{p,p} = H^p$ is obtained. In particular, the Weisz's boundedness theorem implies the weak type (1,1) of the maximal operator, from which the a.e. (dyadic) differentiability of the integral function follows in the usual way (see also Gát [4]). The a.e. differentiability for two-dimensional Vilenkin systems (in the so-called bounded case) can be found in Nagy and Gát [5].

The Weisz's results were extended to the one-parameter Vilenkin system in Simon and Weisz [13]. In the present work we consider two-parameter Vilenkin systems of bounded structure and give the generalizations of the corresponding results of Weisz [15]. As in the one-parameter case the atomic structure of the Hardy spaces plays an important role. By well-known results on interpolation it is enough to prove only the p-quasi-locality of the maximal operator. To this end we need some estimations and also their modifications from the one-parameter case.

First we establish the results that will be used later. In this connection we refer to the books written by Schipp, Wade, Simon and Pál [10] and Weisz [14]. The main results are formulated in Section 3.

2. Preliminaries and notations

In this section the most important definitions and notations will be introduced. Furthermore, we formulate some known results with respect to the Vilenkin system used in this note, too.

To the definition of the so-called Vilenkin group G_m let $m = (m_0, m_1, \dots, m_k, \dots)$ be a sequence of natural numbers such that $m_k \geq 2$ $(k \in \mathbb{N} := \{0, 1, \dots\})$. For all $k \in \mathbb{N}$ we denote by Z_{m_k} the m_k th discrete cyclic group, where Z_{m_k} is represented by $\{0, 1, \dots, m_k - 1\}$. Then G_m is the complete direct product of Z_{m_k}'s, which is a compact Abelian group with Haar measure 1. The elements of G_m are the sequences of the form $(x_0, x_1, \dots, x_k, \dots)$, where $x_k \in Z_{m_k}$ $(k \in \mathbb{N})$. The group operation $\dotplus$ in G_m is the $\operatorname{mod} m_k$ $(k \in \mathbb{N})$ addition of the corresponding coordinates. Let the inverse of $\dotplus$ be denoted by $\dot-$. It is clear that the topology of the group G_m is completely determined by the sets

$$I_n := I_n(0) := \Big\{(x_0, x_1, \dots, x_k, \dots) \in G_m : x_j = 0 \quad (j = 0, \dots, n-1)\Big\}$$

$(0 \neq n \in \mathbb{N}, I_0 := I_0(0) := G_m)$. If $I_n(x) := x \dotplus I_n(0)$ $(n \in \mathbb{N})$ is the coset of $I_n(0)$ by a given $x \in G_m$ then $I_n(x)$'s will be called intervals. The Haar measure $|I_n(x)|$ of $I_n(x)$ is M_n, where the generalized powers M_n $(n \in \mathbb{N})$ are defined in the following way: $M_0 := 1$, $M_n := \prod_{j=0}^{n-1} m_j$ $(0 < n \in \mathbb{N})$.

The characters of G_m form a complete orthonormal system $\widehat{G}_m$ in $L^1(G_m)$. To the description of $\widehat{G}_m$ let

$$r_n(x) := \exp \frac{2\pi i x_n}{m_n}$$

$(n \in \mathbb{N}, x = (x_0, x_1, \dots) \in G_m, i := \sqrt{-1})$ and

$$\Psi_n := \prod_{k=0}^{\infty} r_k^{n_k},$$

where $n = \sum_{k=0}^{\infty} n_k M_k$ $(n_k \in Z_{m_k}$ $(k \in \mathbb{N}))$. It is not hard to see that $\widehat{G}_m = \{\Psi_n : n \in \mathbb{N}\}$. This system of the functions Ψ_n $(n \in \mathbb{N})$ is the so-called *Vilenkin system.*

The kernels of Dirichlet type will be denoted by

$$D_n := \sum_{k=0}^{n-1} \Psi_k \qquad (n \in \mathbb{N}).$$

The most important property of D_n's is the equality

(1) $$D_{M_n}(x) = \begin{cases} M_n & (x \in I_n) \\ 0 & (x \in G_m \setminus I_n) \end{cases} \qquad (n \in \mathbb{N})$$

The *two-parameter Vilenkin system* is defined on G_m^2 as the Kronecker products of the functions Ψ_j and Ψ_k, i.e. for $(j,k) \in \mathbb{N}^2$ let

$$\Psi_{j,k}(x,y) := \Psi_j(x)\Psi_k(y) \qquad ((x,y) \in G_m^2).$$

The symbol $L^p(G_m^2)$ $(p > 0)$ stands for the usual Lebesgue space, i.e. for the set of all complex valuable functions f defined on G_m^2 such that f is measurable with respect to the product measure generated by the Haar measure of G_m and the norm (or quasi-norm) $\|f\|_p :=$ $:= \left(\int\int |f|^p\right)^{1/p}$ $(p < \infty)$ or $\|f\|_\infty := \inf\{\alpha : |f| \leq \alpha \text{ a.e.}\}$ is finite, respectively.

If $f \in L^1(G_m^2)$ then the maximal function $f^\star$ is defined by

$$f^\star(x,y) = \sup_{I,J}(|I||J|)^{-1}\left|\int_{I\times J} f\right|,$$

where $(x,y) \in G_m^2$ and the supremum is taken over all intervals $I, J \subset$ $\subset G_m$ such that $(x,y) \in I \times J$ and $|I| = |J|$. Then the set $I \times J$ will be called *m-adic square.*

Let $p > 0$ and denote by $H^p(G_m^2)$ the (Hardy) space of f's for which

$$\|f\|_{H^p} := \|f^\star\|_p < \infty.$$

The Lorentz- and Hardy-Lorentz spaces $L^{p,q}(G_m^2)$ and $H^{p,q}(G_m^2)$ $(0 < p < \infty, 0 < q \leq \infty)$ with the norm (or quasi-norm) $\|f\|_{p,q} < \infty$ and $\|f\|_{H^{p,q}} < \infty$, respectively, will be taken in the usual way. That is, if f is a complex valued measurable function given on G_m^2 and $\tilde{f}$ is the non-increasing rearrangement of its distribution function then for $0 < p < \infty, 0 < q < \infty$ let the Lorentz norm (or quasi-norm) $\|f\|_{p,q}$ be defined by

$$\|f\|_{p,q} := \left(\int_0^\infty \tilde{f}(t)^q t^{q/p}\frac{dt}{t}\right)^{1/q}.$$

If $0 < p < \infty, q = \infty$ then let $\|f\|_{p,\infty} := \sup_{t>0} t^{1/p}\tilde{f}(t)$ (for details see e.g. Weisz [17] or Simon [12]). As special case we get $H^{p,p}(G_m^2) =$ $= H^p(G_m^2), L^{p,p}(G_m^2) = L^p(G_m^2)$. We remark that the extensions of these spaces in view of martingales can be accomplished in a usual way (see e.g. Weisz [14]).

The following result on interpolation plays an important part in the investigations concerning mappings between the above mentioned spaces.

Theorem A [14, Weisz]. *Suppose that T is a sublinear operator which is bounded from $H^{p_0}(G_m^2)$ to $L^{p_0}(G_m^2)$ for certain $0 < p_0 < \infty$ and from $L^\infty(G_m^2)$ to $L^\infty(G_m^2)$. Then T is also bounded from $H^{p,q}(G_m^2)$ to $L^{p,q}(G_m^2)$ if $p_0 < p < \infty$ and $0 < q \le \infty$. In particular, if $p_0 < 1$ then T is of weak type* (1,1).

A function $a \in L^2(G_m^2)$ will be called *p-atom* if

(i) $\operatorname{supp} a \subset I \times J$ for an m-adic square $I \times J$;

(ii) $\|a\|_\infty \le |I \times J|^{-1/p} = |I|^{-2/p}$;

(iii) $\int\int a = 0$.

To the application of Th. A we need the next statement on the so-called p-quasi-local operators (for details see the above items).

Theorem B. *Let T be a sublinear operator and assume that for each $p_0 < p \le 1$ it is p-quasi-local, i.e. there exists a constant C_p such that*

$$\sup_a \int_{G_m \setminus I} \int_{G_m \setminus J} |Ta|^p < \infty.$$

(Here $0 < p_0 < 1$ is fixed, the supremum is taken over all p-atoms a and $\operatorname{supp} a \subset I \times J$ (see (i)*). If T is bounded from $L^\infty(G_m^2)$ to $L^\infty(G_m^2)$ then it is also bounded from $H^p(G_m^2)$ to $L^p(G_m^2)$.*

Throughout this paper the symbols C, C_p and $C_{p,q}$ will denote positive constants depending at most only on p, q, α, m, which are not always the same in different occurences.

3. The derivative

In Butzer and Wagner [1] a concept of the so-called dyadic derivative was introduced. Its generalization for the Vilenkin analysis is due to Onneweer [6]. To the definition let f be a real or complex valued function given on G_m and for all $n \in \mathbb{N}$ introduce the function $d_n f$ as follows:

$$d_n f(x) := \sum_{j=0}^{n-1} M_j \sum_{k=0}^{m_j-1} k m_j^{-1} \sum_{l=0}^{m_j-1} r_j(le_j)^{m_j-k} f(x \dotplus le_j) \qquad (x \in G_m).$$

If there exists $\lim_{n\to\infty} d_n f(x)$ at some point $x \in G_m$ then this limes is called the (pointwise) derivative of f at x and is denoted by $df(x)$. It is not hard to see that $d_n \Psi_l(x) = \Psi_l(x) \sum_{j=0}^{n-1} l_j M_j$ $(n, l \in \mathbb{N}, x \in G_m)$. Moreover, $d\Psi_n(x) = n\Psi_n(x)$ $(n \in \mathbb{N}, x \in G_m)$.

Furthermore, Butzer and Engels [3] introduced the two-dimensional dyadic derivative, too. This concept can be extended to the two-parameter Vilenkin system in a simple way. Namely, let $(x,y) \in G_m^2, n, s \in \mathbb{N}$ and

$$d_{n,s}f(x,y) := \sum_{j=0}^{n-1}\sum_{l=0}^{s-1} M_j M_l \sum_{k=0}^{m_j-1}\sum_{t=0}^{m_l-1} \frac{kt}{m_j m_l} \cdot$$

$$\cdot \sum_{u=0}^{m_j-1}\sum_{v=0}^{m_l-1} r_j(ue_j)^{m_j-k} r_l(ve_l)^{m_l-1} f(x+ue_j, y+ve_l),$$

where f is an arbitrary real or complex valued function defined on G_m^2. The function f will be said *differentiable at* (x,y) if the limit $df(x,y)$ of $d_{n,s}f(x,y)$ exists as $\min(n,s) \to \infty$. Especially, $d\Psi_{k,l} = kl\Psi_{k,l}$ $(k,l \in \mathbb{N})$.

We remark that the definition of df can be extended to higher dimension in analogous way, but for the sake of simplicity we deal only with the two-dimensional case.

Furthermore, let $W, W_K \in L^2(G_m)$ $(0 < K \in \mathbb{N})$ denote the functions

$$W_0 := W := \Psi_0 + \sum_{k=1}^{\infty} \frac{\Psi_k}{k} \qquad W_K := \sum_{k=M_K}^{\infty} \frac{\Psi_k}{k},$$

respectively. The system $\widehat{G}_m$ is complete, therefore the functions W, W_K are (in L^2-sense) uniquely determined. For $f \in L^1(G_m^2)$ define the integral $\mathbf{I}f := f * (W \times W)$ of f as

$$\mathbf{I}f(x,y) := \int\int f(t,u) W(x \dot{-} t) W(y \dot{-} u)\, dt\, du \qquad ((x,y) \in G_m^2).$$

Then $d_{n,s}(\mathbf{I}f) = f * (d_n W \times d_s W)$. Let us define the two-parameter maximal function $\mathbf{I}^*{}_\alpha f$ in the following way: for $\alpha \ge 0$ let

$$\mathbf{I}^*{}_\alpha f := \sup_{|n-s|\le\alpha} |d_{n,s}(\mathbf{I}f)| \qquad (f \in L^1(G_m^2)).$$

We introduce the notations

$$A_{n,l}(x) := \sum_{j=0}^{n-1} M_j \sum_{q=0}^{m_j-1} D_{M_l}(x + qe_j) \qquad (n, l \in \mathbb{N}, x \in G_m),$$

$$B_n(x) := \sum_{k=0}^{n-1}(n-k)M_k D_{M_k}(x) \qquad (n \in \mathbb{N}, x \in G_m),$$

$$F_n(x) := \sum_{i=1}^{n-1}(n-i)\sum_{k=0}^{i-1} M_k \sum_{l=1}^{m_k-1} D_{M_i}(x \dotplus le_k) \qquad (n \in \mathbb{N}, x \in G_m).$$

Then we get the following estimations for $d_n W_K$ $(n, K \in \mathbb{N})$.

Lemma 1 [13, Simon and Weisz]. *Assume that m is bounded, $n, K \in \in \mathbb{N}$. Then in case $n \leq K$*

$$|d_n W_K| \leq C\Big(\frac{1}{M_K}\sum_{l=n}^{K} A_{n,l} + \sum_{l=K+1}^{\infty} \frac{1}{M_l}A_{n,l} + 1M_K B_n + \frac{1}{M_K}F_n\Big),$$

while for $n > K$

$$|d_n W_K| \leq D_{M_K} + C\Big(\sum_{i=n}^{\infty}\frac{1}{M_i}A_{n,i+1} + D_{M_n} + \frac{1}{M_n}B_n + \frac{1}{M_n}F_n\Big).$$

In Simon and Weisz [13] we proved the next estimations. Namely, let $K \in \mathbb{N}$ and for $\mathbb{N} \ni n \leq K$ we choose an arbitrary function

$$\Phi_{n,K} \in \Big\{\frac{1}{M_K}\sum_{l=n}^{K} A_{n,l}, \sum_{l=K+1}^{\infty}\frac{1}{M_l}A_{n,l}, \frac{1}{M_K}B_n, \frac{1}{M_K}F_n\Big\}.$$

Analogously let $K < j \in \mathbb{N}$ and

$$\Psi_{j,K} \in \Big\{\sum_{i=j}^{\infty}\frac{1}{M_i}A_{j,i+1}, D_{M_j}, \frac{1}{M_j}B_j, \frac{1}{M_j}F_j\Big\}.$$

Then for all $1/2 < p \leq 1$

$$M_K \int_{G_m \setminus I_K} \big(\sup_{n \leq K}\int_{I_K} \big|\Phi_{n,K}(x \dot{-} t)\big|\, dt\big)^p dx \leq C_p \tag{2}$$

and

$$M_K \int_{G_m \setminus I_K} \big(\sup_{j > K}\int_{I_K} \big|\Psi_{j,K}(x \dot{-} t)\big|\, dt\big)^p dx \leq C_p. \tag{3}$$

(Here we recall the definition of I_K, that is, I_K is the set of all $(x_0, x_1, \dots) \in\in G_m$ such that $x_0 = x_1 = \dots = x_{K-1} = 0$.)

Of course, the inequalities (2), (3) and Lemma 1 imply that

$$M_K \int_{G_m \setminus I_K} \Big(\sup_{n}\int_{I_K} \big|d_n W_K(x \dot{-} t)\big|\, dt\Big)^p dx \leq C_p. \tag{4}$$

As a special case we get from Lemma 1 the next estimation for

$|d_nW+1|$:

$$|d_nW+1| \le C\Big(\sum_{i=n}^{\infty}\frac{1}{M_i}A_{n,i+1}+D_{M_n}+\frac{1}{M_n}B_n+\frac{1}{M_n}F_n\Big).$$

Taking into account the previous inequalities it follows immediately for all $K\in\mathbb{N}$ and $1/2<p\le 1$ that

$$(5)\qquad M_K\int_{G_m\setminus I_K}\Big(\sup_{n\le K}\frac{M_n}{M_K}\int_{I_K}\Big|d_nW(x\dot{-}t)+1\Big|\,dt\Big)^p dx\le C_p$$

and

$$(6)\qquad M_K\int_{G_m\setminus I_K}\big(\sup_{n>K}\int_{I_K}\big|d_nW(x\dot{-}t)+1\big|\,dt\big)^p dx\le C_p.$$

Furthermore, we get

$$\|d_nW+1\|_1\le C\Big(\sum_{i=n}^{\infty}\frac{1}{M_i}\|A_{n,i+1}\|_1+1+\frac{1}{M_n}\|B_n\|_1+\frac{1}{M_n}\|F_n\|_1\Big),$$

where

$$\|A_{n,i+1}\|_1=\int_{G_m}\Big|\sum_{j=0}^{n-1}M_j\sum_{q=0}^{m_j-1}D_{M_{i+1}}(x\dot{+}qe_j)\Big|\,dx\le$$

$$\le\sum_{j=0}^{n-1}M_j\sum_{q=0}^{m_j-1}1\le CM_n,$$

that is, $\sum_{i=n}^{\infty}M_i^{-1}\|A_{n,i+1}\|_1\le C$ and

$$\|B_n\|_1=\|\sum_{k=0}^{n-1}(n-k)M_kD_{M_k}\|_1\le\sum_{k=0}^{n-1}(n-k)M_k\le$$

$$\le M_n\sum_{k=0}^{n-1}\frac{n-k}{2^{n-k}}\le CM_n,$$

$$\|F_n\|_1\le\sum_{i=1}^{n-1}(n-i)\sum_{k=0}^{i-1}M_k\sum_{l=1}^{m_k-1}1\le C\sum_{i=1}^{n-1}(n-i)M_i\le CM_n.$$

In other words we have

$$(7)\qquad \sup_n\|d_nW+1\|_1<\infty.$$

Some estimations for the integrals $\int_{I_K}\big|d_nW_K(x\dot{-}t)\big|\,dt$ $(x\in I_K, n,K\in\in\mathbb{N})$ will be needed, too. To this end let $n\le K$. Then for $l=n,\dots,K$

$$\int_{I_K} A_{n,l}(x \dot{-} t)\, dt = \sum_{j=0}^{n-1} M_j \sum_{q=0}^{m_j-1} \int_{I_K} D_{M_l}(x \dot{-} t \dot{+} qe_j)\, dt =$$

$$= \sum_{j=0}^{n-1} M_j \int_{I_K} D_{M_l}(x \dot{-} t)\, dt = \frac{M_l}{M_K} \sum_{j=0}^{n-1} M_j \le C \frac{M_l M_n}{M_K},$$

while for $l = K+1, K+2, \ldots$

$$\int_{I_K} A_{n,l}(x \dot{-} t)\, dt = \sum_{j=0}^{n-1} M_j \int_{I_K} D_{M_l}(x \dot{-} t)\, dt = \sum_{j=0}^{n-1} M_j \le C M_n.$$

Furthermore, by (1) we get for all $i = 1, \ldots, n-1$ and $k = 0, \ldots, i-1$ and $l = 1, \ldots, m_k - 1$ that $\int_{I_K} D_{M_i}(x \dot{-} t \dot{+} le_k)\, dt = 0$ and so $\int_{I_K} F_n(x \dot{-} t)\, dt = 0$. Finally

$$\int_{I_K} B_n(x \dot{-} t)\, dt = \sum_{k=0}^{n-1} (n-k) M_k \int_{I_K} D_{M_k}(x \dot{-} t)\, dt =$$

$$= \frac{1}{M_K} \sum_{k=0}^{n-1} (n-k) M_k^2 \le C \frac{M_n^2}{M_K}.$$

Therefore in the case $n \le K$ the next estimation is true:

$$\int_{I_K} \left| d_n W_K(x \dot{-} t) \right| dt \le C \int_{I_K} \Big(\frac{1}{M_K} \sum_{l=n}^{K} A_{n,l}(x \dot{-} t) +$$

$$+ \sum_{l=K+1}^{\infty} \frac{1}{M_l} A_{n,l}(x \dot{-} t) + \frac{1}{M_K} B_n(x \dot{-} t) + \frac{1}{M_K} F_n(x \dot{-} t) \Big)\, dt \le$$

$$\le C \Big(\frac{M_n}{M_K^2} \sum_{l=n}^{K} M_l + M_n \sum_{l=K+1}^{\infty} \frac{1}{M_l} + \frac{M_n^2}{M_K^2} \Big) \le C \frac{M_n}{M_K},$$

i.e. if $n, K \in \mathbb{N}, n \le K$ and $x \in I_K$ then

$$\int_{I_K} \left| d_n W_K(x \dot{-} t) \right| dt \le C \frac{M_n}{M_K}. \tag{8}$$

On the other hand, for the above K, x but for $n = K+1, K+2, \ldots$ we get $\int_{I_K} D_{M_n}(x \dot{-} t)\, dt = 1$ and

$$\int_{I_K} A_{n,i+1}(x \dot{-} t)\,dt = \sum_{j=0}^{n-1} M_j \sum_{q=0}^{m_j-1} \int_{I_K} D_{M_{i+1}}(x \dot{-} t \dot{+} qe_j) \le$$
$$\le C \sum_{j=0}^{n-1} M_j \le CM_n,$$

$$\int_{I_K} B_n(x \dot{-} t)\,dt = \sum_{k=0}^{n-1} (n-k) M_k \int_{I_K} D_{M_k}(x \dot{-} t)\,dt \le$$
$$\le \sum_{k=0}^{n-1} (n-k) M_k \le CM_n,$$

$$\int_{I_K} F_n(x \dot{-} t)\,dt = \sum_{i=1}^{n-1} (n-i) \sum_{k=0}^{i-1} M_k \sum_{l=1}^{m_k-1} \int_{I_K} D_{M_i}(x \dot{-} t \dot{+} le_k)\,dt \le$$
$$\le C \sum_{i=1}^{n-1} (n-i) \sum_{k=0}^{i-1} M_k \le C \sum_{i=1}^{n-1} (n-i) M_i \le CM_n.$$

Thus it follows obviously that

$$\int_{I_K} \left| d_n W_K(x \dot{-} t) \right| dt \le \int_{I_K} D_{M_K}(x \dot{-} t)\,dt +$$
$$+ C \int_{I_K} \Big(\sum_{i=n}^{\infty} \frac{1}{M_i} A_{n,i+1}(x \dot{-} t) +$$
$$+ 1 + \frac{1}{M_n} B_n(x \dot{-} t) + \frac{1}{M_n} F_n(x \dot{-} t) \Big)\,dt,$$

i.e. if $n, K \in \mathbb{N}, n > K$ and $x \in I_K$ then

$$\text{(9)} \qquad \int_{I_K} \left| d_n W_K(x \dot{-} t) \right| dt \le C \Big(1 + M_n \sum_{i=n}^{\infty} \frac{1}{M_i} \Big) \le C.$$

Taking into account the inequality (8) with respect to the case $n \le K$ it follows also

$$\text{(10)} \qquad \int_{I_K} \left| d_n W_K(x \dot{-} t) \right| dt \le C \qquad (n, K \in \mathbb{N}, x \in I_K).$$

Now we show some estimations analogous to (2) and (3) for the slightly modified functions

$$\widetilde{A}_{n,l}(x) := \sum_{j=0}^{n-1} \sqrt{M_j} \sum_{q=0}^{m_j-1} D_{M_l}(x \dot{+} qe_j) \qquad (n, l \in \mathbb{N}, x \in G_m),$$

$$\widetilde{B}_n := \frac{1}{\sqrt{M_n}} B_n, \qquad \widetilde{F}_n := \frac{1}{\sqrt{M_n}} F_n \qquad (n \in \mathbb{N}),$$
$$A^*_{n,l} := \sqrt{M_n} A_{n,l}, \qquad B^*_n := \sqrt{M_n} B_n \qquad (n, l \in \mathbb{N}),$$
$$F^*_n := \sqrt{M_n} F_n, \qquad {D^*}_{M_n} := \sqrt{M_n} D_{M_n} \qquad (n \in \mathbb{N}).$$

To this end let $K, n \in \mathbb{N}, n \leq K$ and

$$\widetilde{\Phi}_{n,K} \in \Big\{ \frac{1}{M_K} \sum_{l=n}^{K} \widetilde{A}_{n,l}, \sum_{l=K+1}^{\infty} \frac{1}{M_l} \widetilde{A}_{n,l}, \frac{1}{M_K} \widetilde{B}_n, \frac{1}{M_K} \widetilde{F}_n \Big\}.$$

Furthermore, for $K, n \in \mathbb{N}, n \leq K$ let

$$\Phi^*_{n,K} \in \Big\{ \sum_{i=n}^{\infty} \frac{1}{M_i} {A^*}_{n,i+1}, \frac{1}{M_n} {F^*}_n, {D^*}_{M_n} \Big\}.$$

Then for these functions the next assertions are true.

Lemma 2. *If m is bounded then for all $2/3 < p \leq 1$ and for an arbitrary $K \in \mathbb{N}$ we have*

$$\text{(11)} \qquad M_K \int_{G_m \setminus I_K} \big(\sup_{n \leq K} \int_{I_K} \widetilde{\Phi}_{n,K}(x \dot{-} t)\, dt \big)^p dx \leq C_p M_K^{-p/2}$$

and

$$\text{(12)} \qquad M_K \int_{G_m \setminus I_K} \big(\sup_{n \leq K} \int_{I_K} \Phi^*_{n,K}(x \dot{-} t)\, dt \big)^p dx \leq C_p M_K^{p/2}.$$

Furthermore, the inequality

$$\text{(13)} \qquad M_K \int_{G_m \setminus I_K} \big(\sup_{n \leq K} \int_{I_K} \frac{1}{M_n} B^*_n(x \dot{-} t)\, dt \big)^p dx \leq C_p M_K^{1-p}$$

holds, too.

Proof. First we prove (11) in the case $\widetilde{\Phi}_{n,K} := \frac{1}{M_K} \sum_{l=n}^{K} \widetilde{A}_{n,l}$, i.e. when

$$\int_{I_K} \widetilde{\Phi}_{n,K}(x \dot{-} t)\, dt =$$

$$= \frac{1}{M_K} \sum_{l=n}^{K} \sum_{j=0}^{n-1} \sqrt{M_j} \sum_{q=0}^{m_j-1} \int_{I_K} D_{M_l}(x \dot{+} qe_j \dot{-} t)\, dt \qquad (x \in G_m \setminus I_K).$$

Let $k = 0, \dots, K-1$, $J_{k,K-1} := \{x \in I_k \setminus I_{k+1} : x_{k+1} = \dots = x_{K-1} = 0\}$ and

$$J_{k,s}:=\Big\{x\in I_k\setminus I_{k+1}: x_{k+1}=\ldots=x_s=0, x_{s+1}\neq 0\Big\}\quad (s=k,\ldots,K-2),$$

then

$$\int_{G_m\setminus I_K}\big(\sup_{n\le K}\int_{I_K}\widetilde{\Phi}_{n,K}(x\dot{-}t)\,dt\big)^p\,dx=$$

$$=\sum_{k=0}^{K-1}\int_{I_k\setminus I_{k+1}}\big(\sup_{n\le K}\int_{I_K}\widetilde{\Phi}_{n,K}(x\dot{-}t)\,dt\big)^p\,dx\le$$

$$\le\sum_{k=0}^{K-1}\Big(\int_{I_k\setminus I_{k+1}}\big(\sup_{k<n\le K}\int_{I_K}\widetilde{\Phi}_{n,K}(x\dot{-}t)\,dt\big)^p\,dx+$$

$$+\int_{I_k\setminus I_{k+1}}\big(\sup_{n\le k}\int_{I_K}\widetilde{\Phi}_{n,K}(x\dot{-}t)\,dt\big)^p\,dx\Big),$$

where

$$\sum_{k=0}^{K-1}\int_{I_k\setminus I_{k+1}}\big(\sup_{k<n\le K}\int_{I_K}\widetilde{\Phi}_{n,K}(x\dot{-}t)\,dt\big)^p\,dx=$$

$$=\sum_{k=0}^{K-1}\sum_{s=k}^{K-1}\int_{J_{k,s}}\big(\sup_{k<n\le K}\int_{I_K}\widetilde{\Phi}_{n,K}(x\dot{-}t)\,dt\big)^p\,dx.$$

If $x\in J_{k,s}, n>k$ and $j\neq k$ or $l>s+1$ then $x\dot{+}qe_j\dot{-}t\notin I_l$ for all $t\in I_K, q=0,\ldots,m_j-1$, i.e. by (1) $\int_{I_K}D_{M_l}(x\dot{+}qe_j\dot{-}t)\,dt=0$. Therefore

$$\Big(\sup_{k<n\le K}\int_{I_K}\widetilde{\Phi}_{n,K}(x\dot{-}t)\,dt\Big)^p\le C\frac{1}{M_K^p}\sum_{l=n}^{s+1}M_k^{p/2}\Big(\frac{M_l}{M_K}\Big)^p\le$$

$$\le CM_K^{-2p}M_k^{p/2}M_s^p.$$

This implies the next estimation:

$$\sum_{k=0}^{K-1}\sum_{s=k}^{K-1}\int_{J_{k,s}}\Big(\sup_{k<n\le K}\int_{I_K}\widetilde{\Phi}^p_{n,K}(x\dot{-}t)\,dt\Big)^p\,dx\le$$

$$\le CM_K^{-2p}\sum_{k=0}^{K-1}M_k^{p/2}\sum_{s=k}^{K-1}M_s^{p-1}\le$$

$$\le\begin{cases} C_1M_K^{-2}\sum_{k=0}^{K-1}(K-k)\sqrt{M_k}\le C_1M_K^{-3/2} & (p=1)\\ C_pM_K^{-2p}\sum_{k=0}^{K-1}M_k^{p/2}M_k^{p-1}\le C_pM_K^{-2p}M_K^{3p/2-1}= & \\ \qquad\qquad =C_pM_K^{-p/2-1} & (2/3<p<1).\end{cases}$$

Let $k=0,\dots,K-1, x\in I_k\setminus I_{k+1}$ and $n<k$. If $l\ge k$ then $x\dot{+}qe_j\dot{-}t\notin$ $\notin I_l$ for all $t\in I_K, j=0,\dots,n-1$ and for all $q=0,\dots,m_j-1$. Thus by (1) we get $D_{M_l}(x\dot{+}qe_j\dot{-}t)=0$, i.e.

$$\int_{I_K}\widetilde{\Phi}_{n,K}(x\dot{-}t)\,dt\le\frac{1}{M_K}\sum_{l=n}^{k-1}\sum_{j=0}^{n-1}\sqrt{M_j}\frac{M_l}{M_K}\le C\frac{M_k^{3/2}}{M_K^2}.$$

It is not hard to see that the above inequality holds for $n=k$, too, so

$$\sum_{k=0}^{K-1}\int_{I_k\setminus I_{k+1}}\Big(\sup_{n\le k}\int_{I_K}\widetilde{\Phi}_{n,K}(x\dot{-}t)\,dt\Big)^p\,dx\le C_pM_K^{-2p}\sum_{k=0}^{K-1}M_k^{3p/2-1}\le$$

$$\le C_pM_K^{-p/2-1}.$$

This proves (11) in the first case.

Now, we investigate (11) for $\widetilde{\Phi}_{n,K}=\sum_{l=K+1}^{\infty}\frac{1}{M_l}\widetilde{A}_{n,l}$. In this case

$$\int_{I_K}\widetilde{\Phi}_{n,K}(x\dot{-}t)\,dt=$$

$$=\sum_{l=K+1}^{\infty}\frac{1}{M_l}\sum_{j=0}^{n-1}\sqrt{M_j}\sum_{q=0}^{m_j-1}\int_{I_K}D_{M_l}(x\dot{+}qe_j\dot{-}t)\,dt\qquad(x\in G_m\setminus I_K).$$

If $t\in I_K, k=0,\dots,K-1$ and $x\in I_k\setminus I_{k+1}, l\ge K+1$ then for all $n\le k$ and $j=0,\dots,n-1$ we have $x\dot{+}qe_j\dot{-}t\notin I_l$ $(q=0,\dots,m_j-1)$, i.e. by (1) $D_{M_l}(x\dot{+}qe_j\dot{-}t)=0$. The same conclusion holds if $n>k$ and $j\ne k$ or $n>k$ and $j=k$ but $x\notin J_{k,K-1}$, resp. Therefore it can be assumed that $x\in J_{k,K-1}, n>k$ and $j=k$ when for $q=m_k-x_k$ by (1)

$$\int_{I_K} D_{M_l}(x \dotplus qe_k \dot- t)\,dt = \int_{I_l(x\dotplus qe_k)} D_{M_l}(x \dotplus qe_k \dot- t)\,dt = 1.$$

This means that $\int_{I_K} \widetilde{\Phi}_{n,K}(x \dot- t)\,dt = \sum_{l=K+1}^{\infty} \frac{1}{M_l}\sqrt{M_k} \le C\frac{\sqrt{M_k}}{M_K}$ from which

$$\begin{aligned}
&\int_{G_m \setminus I_K} \Big(\sup_{n\le K} \int_{I_K} \widetilde{\Phi}_{n,K}(x \dot- t)\,dt\Big)^p\,dx = \\
&= \sum_{k=0}^{K-1} \int_{I_k \setminus I_{k+1}} \Big(\sup_{k<n\le K} \int_{I_K} \widetilde{\Phi}_{n,K}(x \dot- t)\,dt\Big)^p\,dx = \\
&= \sum_{k=0}^{K-1} \int_{J_{k,K-1}} \Big(\sup_{k<n\le K} \int_{I_K} \widetilde{\Phi}_{n,K}(x \dot- t)\,dt\Big)\,dx \le \\
&\le C_p \sum_{k=0}^{K-1} \frac{M_k^{p/2}}{M_K^{p+1}} \le C_p M_K^{-p/2-1}.
\end{aligned}$$

Now let $\widetilde{\Phi}_{n,K}=\frac{1}{M_K}\widetilde{B}_n$. Then it follows analogously for $\int_{I_K}\widetilde{\Phi}_{n,K}(x \dot- t)dt$ $(x \in I_k \setminus I_{k+1}\ (k=0,\dots,K-1),\ n \le K)$ that

$$\int_{I_K} \widetilde{\Phi}_{n,K}(x \dot- t)\,dt = \frac{1}{M_K\sqrt{M_n}} \sum_{l=0}^{n-1} (n-l)M_l \int_{I_K} D_{M_l}(x \dot- t)\,dt =$$

$$= \frac{1}{M_K\sqrt{M_n}} \sum_{l=0}^{n-1} (n-l)M_l \begin{cases} 0 & (l > k) \\ \dfrac{M_l}{M_K\sqrt{M_n}} & (l \le k). \end{cases}$$

In other words we get

$$\int_{I_K} \widetilde{\Phi}_{n,K}(x \dot- t)\,dt \le$$

$$\le \begin{cases} \dfrac{1}{M_K^2\sqrt{M_n}} \displaystyle\sum_{l=0}^{n-1} (n-l)M_l^2 \le C\frac{M_k^{3/2}}{M_K^2} & (n \le k) \\ \dfrac{1}{M_K^2\sqrt{M_n}} \displaystyle\sum_{l=0}^{k} (n-l)M_l^2 \le C(n-k)\frac{M_k^2}{M_K^2\sqrt{M_n}} \le & \\ \qquad\qquad \le C(K-k)\dfrac{M_k^{3/2}}{M_K^2} & (n > k). \end{cases}$$

The last inequality implies

$$\int_{G_m\setminus I_K}\big(\sup_{n\le K}\int_{I_K}\widetilde{\Phi}_{n,K}(x\dot{-}t)\,dt\big)^p\,dx=$$

$$=\sum_{k=0}^{K-1}\int_{I_k\setminus I_{k+1}}\Big(\sup_{n\le K}\int_{I_K}\widetilde{\Phi}_{n,K}(x\dot{-}t)\,dt\Big)^p\,dx\le$$

$$\le C_pM_K^{-2p}\sum_{k=0}^{K-1}(K-k)^pM_k^{3p/2-1}\le$$

$$\le C_pM_K^{-p/2-1}\sum_{k=0}^{K-1}(K-k)^p\Big(\frac{1}{2^{K-k}}\Big)^{3p/2-1}\le C_pM_K^{-p/2-1}.$$

As above it follows for $\widetilde{\Phi}_{n,K}=\frac{1}{M_K}\widetilde{F}_n$ that for $x\in I_k\setminus I_{k+1}$ $(k==0,\dots,K-1)$ and for $n\le K$

$$\int_{I_K}\widetilde{\Phi}_{n,K}(x\dot{-}t)\,dt=$$

$$=\frac{1}{M_K\sqrt{M_n}}\sum_{i=1}^{n-1}(n-i)\sum_{j=0}^{i-1}M_j\sum_{l=1}^{m_j-1}\int_{I_K}D_{M_i}(x\dot{+}le_j\dot{-}t)\,dt=$$

$$=\frac{1}{M_K\sqrt{M_n}}\sum_{i=k+1}^{n-1}(n-i)M_k\int_{I_K}D_{M_i}(x\dot{+}(m_k-x_k)e_k\dot{-}t)\,dt.$$

Furthermore, it can be assumed that $n>k+1$. If $x\in J_{k,s}$ $(s==k,\dots,K-1)$ then

$$\int_{I_K}\widetilde{\Phi}_{n,K}(x\dot{-}t)\,dt=\frac{1}{M_K\sqrt{M_n}}\sum_{i=k+1}^{\min(n-1,s+1)}(n-i)M_k\frac{M_i}{M_K}\le$$

$$\le\frac{\sqrt{M_k}}{M_K^2}\sum_{i=k+1}^{s}(s-i)M_i\le\frac{\sqrt{M_k}}{M_K^2}M_s.$$

Thus we can say that

$$\int_{G_m\setminus I_K}\big(\sup_{n\le K}\int_{I_K}\widetilde{\Phi}_{n,K}(x\dot{-}t)\,dt\big)^p\,dx=$$

$$=M_K\sum_{k=0}^{K-1}\int_{I_k\setminus I_{k+1}}\big(\sup_{n\le K}\int_{I_K}\widetilde{\Phi}_{n,K}(x\dot{-}t)\,dt\big)^p\,dx\le$$

$$\leq \sum_{k=0}^{K-1} \sum_{s=k}^{K-1} \int_{J_{k,s}} \big(\sup_{n \leq K} \int_{I_K} \widetilde{\Phi}_{n,K}(x \dot{-} t)\, dt \big)^p \, dx \leq$$

$$\leq C_p \sum_{k=0}^{K-1} \sum_{s=k}^{K-1} \frac{M_k^{p/2}}{M_K^{2p}} M_s^{p-1} \leq C_p M_K^{-p/2-1}.$$

Now we prove inequality (12) for $\Phi^*_{n,K} = D^*{}_{M_n}$:

$$\int_{G_m \setminus I_K} \big(\sup_{n \leq K} \int_{I_K} \sqrt{m_N} D_{m_n}(x \dot{-} t)\, dt \big)^p \, dx =$$

$$= \sum_{l=0}^{K-1} \int_{I_l \setminus I_{l+1}} \big(\sup_{n \leq K} \int_{I_K} \sqrt{M_n} D_{M_n}(x \dot{-} t)\, dt \big)^p \, dx \leq$$

$$\leq C_p \sum_{l=0}^{K-1} \frac{1}{M_l} (M_l^{3/2} \frac{1}{M_K})^p \leq C_p M_K^{p/2-1}.$$

Furthermore, if

$$\Phi^*{}_{n,K} = \sum_{i=n}^{\infty} \frac{1}{M_i} A^*{}_{n,i+1} \qquad (K, n \in \mathbb{N}, n \leq K),$$

then for $x \in G_m \setminus I_K$

$$\int_{I_K} \Phi^*{}_{n,K}(x \dot{-} t)\, dt = \sqrt{M_n} \sum_{i=n}^{\infty} M_i^{-1} \sum_{j=0}^{n-1} M_j \sum_{q=0}^{m_j-1} \int_{I_K} D_{M_{i+1}}(x \dot{+} qe_j \dot{-} t)\, dt.$$

As above $x \in J_{k,K-1}, (k = 0, \ldots, n-1), j = k$ and $q = m_k - x_k$ can be assumed in which case $\int_{I_K} D_{M_{i+1}}(x \dot{+} qe_j \dot{-} t)\, dt = 1$. This leads to the next estimation: $\int_{I_K} \Phi^*{}_{n,K}(x \dot{-} t)\, dt = \sqrt{M_n} \sum_{i=n}^{\infty} M_i^{-1} M_k \leq$ $\leq C M_k M_n^{-1/2} \leq C\sqrt{M_k}$. Therefore we get

$$\int_{G_m \setminus I_K} \big(\sup_{n \leq K} \int_{I_K} \Phi^*{}_{n,K}(x \dot{-} t)\, dt \big)^p \, dx =$$

$$= \sum_{k=0}^{K-1} \int_{J_{k,K-1}} \big(\sup_{k < n \leq K} \int_{I_K} \Phi^*{}_{n,K}(x \dot{-} t)\, dt \big)^p \, dx \leq$$

$$\leq C_p \sum_{k=0}^{K-1} M_k^{p/2} M_K^{-1} \leq C_p M_K^{p/2-1}.$$

Let us investigate the estimations (13). If $x \in G_m \setminus I_K, n \leq K$, then

$$\int_{I_K} \frac{1}{M_n} B^*{}_n(x \dot{-} t)\, dt = \frac{1}{\sqrt{M_n}} \sum_{l=0}^{n-1} (n-l) M_l \int_{I_K} D_{M_l}(x \dot{-} t)\, dt.$$

First we note that, by (1), $D_{M_l}(x \dot{-} t) = 0$ $(t \in I_K)$ if $x \in I_k \setminus I_{k+1}$ $(k = 0, \ldots, K-1)$ and $l > k$. Furthermore, if $l \le k$ then $D_{M_l}(x \dot{-} t) = M_l$ $(t \in I_K)$. This yields the equation $\int_{I_K} \Phi^*{}_{n,K}(x \dot{-} t)\, dt = M_n^{-1/2} \sum_{l=0}^{\min\{n-1,k\}} (n-l) M_l^2 M_K^{-1}$. Therefore

$$\int_{I_K} \frac{1}{M_n} B^*{}_n(x \dot{-} t)\, dt =$$

$$= \frac{1}{\sqrt{M_n}} \left(n \sum_{l=0}^{\min\{n-1,k\}} \frac{1}{M_l^2} - \sum_{l=0}^{\min\{n-1,k\}} \frac{l}{M_l^2} \right) \frac{1}{M_K} \le C \frac{1}{M_K},$$

which implies

$$\int_{G_m \setminus I_K} \big(\sup_{n \le K} \int_{I_K} \frac{1}{M_n} B^*{}_n(x \dot{-} t)\, dt \big)^p\, dx \le C_p \frac{1}{M_K^p}.$$

Finally, we need to show (12) in the case $\Phi^*{}_{n,K} = M_n^{-1} F_n^*$. Let $x \in G_m \setminus I_K, n \le K$, which implies

$$\int_{I_K} \Phi^*{}_{n,K}(x \dot{-} t)\, dt = \frac{1}{\sqrt{M_n}} \sum_{i=1}^{n-1} (n-i) \sum_{l=0}^{i-1} M_l \sum_{j=1}^{m_l - 1} \int_{I_K} D_{M_i}(x \dot{+} j e_l \dot{-} t)\, dt.$$

Assume $x \in I_k \setminus I_{k+1}$ $(k = 0, \ldots, K-1)$. If $i \le k$ then, by (1), $D_{M_i}(x \dot{+} j e_l \dot{-} t) = 0$ for all $l \le i-1, t \in I_K, j = 1, \ldots, m_l - 1$. The same conclusion follows when $k < i$ but $l \ne k$ or $x \notin J_{k,K-1}$. Thus it can be assumed that $k < i \le n-1, l = k, x \in J_{k,K-1}, j = m_k - x_k$, in which case we have $\int_{I_K} D_{M_i}(x \dot{+} j e_l \dot{-} t)\, dt = M_i M_K^{-1}$ $(i \le K)$ and $\int_{I_K} D_{M_i}(x \dot{+} j e_l \dot{-} t)\, dt = 1$ $(i > K)$. Therefore we get for $n \ge k+1$

$$\int_{I_K} \Phi^*{}_{n,K}(x \dot{-} t)\, dt = \frac{1}{\sqrt{M_n}} \sum_{i=k+1}^{n-1} (n-i) M_k M_i M_K^{-1} =$$

$$= \frac{\sqrt{M_n} M_k}{M_K} \sum_{i=k+1}^{n-1} \frac{(n-i) M_i}{M_n} \le C \frac{M_k}{\sqrt{M_K}}.$$

The last inequalities implies

$$\int_{G_m\setminus I_K}\big(\sup_{n\le K}\int_{I_K}\Phi^*{}_{n,K}(x\dot- t)\,dt\big)^p\,dx=$$
$$=\sum_{k=0}^{K-1}\int_{J_{k,K-1}}\big(\sup_{n\le K}\int_{I_K}\Phi^*{}_{n,K}(x\dot- t)\,dt\big)^p\,dx\le$$
$$\le C_p\frac{1}{M_K^{p/2}}\sum_{k=0}^{K-1}\frac{M_k^p}{M_K}\le C_pM_K^{p/2-1}.$$

This completes the proof of the inequalities in Lemma 2. ◊

Let $f\in L^1(G_m^2), n,s\in\mathbb{N}$ and

$$\Delta_{n,s}f(x,y):=$$
$$:=\int\int f(t,u)\big(d_nW(x\dot- t)+1\big)\big(d_sW(y\dot- u)+1\big)\,dt\,du\quad\big((x,y)\in G_m^2\big).$$

For all $\alpha\ge 0$ we introduce the maximal maximal operator $J^*{}_\alpha$ defined by

$$J^*{}_\alpha f:=\sup_{|n-s|\le\alpha}\big|\Delta_{n,s}f\big|\qquad\big(f\in L^1(G_m^2)\big).$$

If $0<p\le 1$ and a is a p-atom supported on the square $I\times\times J, |I|=|J|=M_N^{-1}$ with a suitable $N\in\mathbb{N}$ then $\int a(t,u)\Psi_k(x\dot-\dot- t)\Psi_l(y\dot- u)\,dt\,du=0$ for all $k,l=0,\dots,M_N-1$ and $x,y\in G_m$. Furthermore, for $n\in\mathbb{N}$ let d_nW be written in the form

$$d_nW=d_n\big(\Psi_0+\sum_{k=1}^{M_N-1}\frac{1}{k}\Psi_k+W_N\big)=:P_{n,N}+d_nW_N,$$

where $P_{n,N}=\sum_{j=0}^{M_N-1}\beta_j\Psi_j$ with some complex coefficients β_j's. By these observations we get for all $n,s\in\mathbb{N}$ and $(x,y)\in G_m^2$ that

$$\Delta_{n,s}a(x,y)=\int\int a(t,u)\big(d_nW_N(x\dot- t)+P_{n,N}(x\dot- t)+1\big)\times$$
$$\times\big(d_sW_N(y\dot- u)+P_{s,N}(y\dot- u)+1\big)\,dt\,du=$$
$$=\int\int a(t,u)d_nW_N(x\dot- t)\big(d_sW(y\dot- u)+1\big)\,dt\,du+$$
$$+\int\int a(t,u)\big(P_{n,N}(x\dot- t)+1\big)\big(d_sW_N(y\dot- u)+P_{s,N}(y\dot- u)+1\big)\,dt\,du=$$
$$=\int\int a(t,u)d_nW_N(x\dot- t)\big(d_sW(y\dot- u)+1\big)\,dt\,du+$$

$$+\int\int a(t,u)\big(P_{n,N}(x\dot{-}t)+1\big)d_sW_N(y\dot{-}u)\,dt\,du=$$

$$=\int\int a(t,u)d_nW_N(x\dot{-}t)\big(d_sW(y\dot{-}u)+1\big)\,dt\,du+$$

$$+\int\int a(t,u)\big(d_nW(x\dot{-}t)+1\big)d_sW_N(y\dot{-}u)\,dt\,du-$$

$$-\int\int a(t,u)d_nW_N(x\dot{-}t)d_sW_N(y\dot{-}u)\,dt\,du=:$$

$$=:\Delta^{(1)}_{n,s}a(x,y)+\Delta^{(2)}_{n,s}a(x,y)-\Delta^{(3)}_{n,s}a(x,y).$$

This decomposition implies the next estimation

$$J^*{}_\alpha a\le\sup_{|n-s|\le\alpha}\left|\Delta^{(1)}_{n,s}a\right|+\sup_{|n-s|\le\alpha}\left|\Delta^{(2)}_{n,s}a\right|+\sup_{|n-s|\le\alpha}\left|\Delta^{(3)}_{n,s}a\right|=:$$

$$=:J^*{}_{\alpha,1}a+J^*{}_{\alpha,2}a+J^*{}_{\alpha,3}a.$$

Applying these observations we will prove the next lemma.

Lemma 3. *Let m be bounded, $2/3<p\le 1$ and $\alpha\ge 0$ be arbitrarily given. Then the maximal operator $J^*{}_\alpha$ is p-quasi-local, i.e. for every p-atom a supported on the square $I\times J$ the estimation*

$$\int\int_{G_m^2\setminus(I\times J)}\left(J^*{}_\alpha a\right)^p\le C_p$$

is true.

Proof. It can be assumed that $I=J=I_N$ for some $N\in\mathbb{N}$. Then we shall investigate first the integral of $\left(J^*{}_{\alpha,1}a\right)^p$ on the set $G_m^2\setminus(I\times J)$. To this end let us decomposed this integral as follows:

$$\int\int_{G_m^2\setminus(I\times J)}\left(J^*{}_{\alpha,1}a\right)^p=\int_{G_m\setminus I_N}\int_{I_N}\left(J^*{}_{\alpha,1}a\right)^p+$$

$$+\int_{I_N}\int_{G_m\setminus I_N}\left(J^*{}_{\alpha,1}a\right)^p+\int_{G_m\setminus I_N}\int_{G_m\setminus I_N}\left(J^*{}_{\alpha,1}a\right)^p,$$

where

$$\int_{G_m\setminus I_N}\mathrm{int}_{I_N}\left(J^*{}_{\alpha,1}a\right)^p\le$$

$$\le\int_{G_m\setminus I_N}\int_{I_N}(\sup_{n,s}\int_{I_N}\int_{I_N}|a(t,u)||d_nW_N(x\dot{-}t)||d_sW(y\dot{-}u)+1|\,dt\,du)^p dx\,dy\le$$

$$\le M_N^2 \int_{G_m\setminus I_N}\int_{I_N} (\sup_{n,s}\int_{I_N} |d_nW_N(x\dot{-}t)|\,dt \int_{I_N} |d_sW(y\dot{-}u)+1|\,du)^p dx\,dy,$$

from which it follows by (4), (7) that

$$\int_{G_m\setminus I_N}\int_{I_N} (J^*{}_{\alpha,1}a)^p \le C_pM_N \int_{G_m\setminus I_N} (\sup_n \int_{I_N} |d_nW_N(x\dot{-}t)|\,dt)^p dx \le C_p.$$

(We remark that for this estimation it is enough to assume $1/2 < p \le$ ≤ 1.)

Now we deal with $\int_{I_N}\int_{G_m\setminus I_N} (J^*{}_{\alpha,1}a)^p =$

$$= \int_{I_N}\int_{G_m\setminus I_N} (\sup_{|n-s|\le\alpha} \int_{I_N}\int_{I_N} |a(t,u)|\times$$

$$\times |d_nW_N(x\dot{-}t)||d_sW(y\dot{-}u)+1|\,dt\,du)^p dx\,dy \le$$

$$\le M_N^2 \int_{I_N}\int_{G_m\setminus I_N} (\sup_{|n-s|\le\alpha}\int_{I_N} |d_nW_N(x\dot{-}t)|\,dt \int_{I_N} |d_sW(y\dot{-}u)+1|\,du)^p dx\,dy.$$

Applying the estimations (5), (6), (8), (9) we can easily deduce the inequality

$$\int_{I_N}\int_{G_m\setminus I_N} (J^*{}_{\alpha,1}a)^p \le C_p.$$

To the estimation of $\int_{G_m\setminus I_N}\int_{G_m\setminus I_N} (J^*{}_{\alpha,1}a)^p$ we write it as

$$\int_{G_m\setminus I_N}\int_{G_m\setminus I_N} (J^*{}_{\alpha,1}a)^p \le$$

$$\le \int_{G_m\setminus I_N}\int_{G_m\setminus I_N} (\sup_{|n-s|\le\alpha,s>N} |\Delta_{n,s}^{(1)}a(x,y)|)^p dx\,dy +$$

$$+ \int_{G_m\setminus I_N}\int_{G_m\setminus I_N} (\sup_{|n-s|\le\alpha,s\le N} |\Delta_{n,s}^{(1)}a(x,y)|)^p dx\,dy.$$

Here the first integral in the sum can be estimated by (4) and (6) as follows:

$$\int_{G_m\setminus I_N}\int_{G_m\setminus I_N} (\sup_{|n-s|\le\alpha,s>N} |\Delta_{n,s}^{(1)}a(x,y)|)^p dx\,dy \le$$

$$\le C_pM_N^2\Big(\int_{G_m\setminus I_N} (\sup_n \int_{I_N} |d_nW_N(x\dot{-}t)|\,dt)^p dx\Big)\times$$

$$\times\Big(\int_{G_m\setminus I_N} (\sup_{s>N}\int_{I_N} |d_sW(y\dot{-}t)+1|\,dt)^p dy\Big) \le C_p$$

(again for all $1/2 < p \le 1$.) We get analogously that

$$\int_{G_m\setminus I_N}\int_{G_m\setminus I_N}\Big(\sup_{|n-s|\le\alpha,s\le N}\big|\Delta^{(1)}_{n,s}a(x,y)\big|\Big)^p dx\,dy\le$$

$$\le M_N^2\int_{G_m\setminus I_N}\int_{G_m\setminus I_N}\Big(\sup_{|n-s|\le\alpha,s\le N}\int_{I_N}\int_{I_N}\big|d_nW_N(x\dot-t)\big|\cdot$$

$$\cdot\big|d_sW(y\dot-u)+1\big|\,dt\,du\Big)^p dx\,dy\le$$

$$\le M_N^2\Big(\int_{G_m\setminus I_N}\Big(\sup_{n\le N}\int_{I_N}\frac{1}{\sqrt{M_n}}\big|d_nW_N(x\dot-t)\big|\,dt\Big)^p dx\Big)\times$$

$$\times\Big(\int_{G_m\setminus I_N}\Big(\sup_{s\le N+\alpha}\int_{I_N}\sqrt{M_s}\big|d_sW(y\dot-u)+1\big|\,du\Big)^p dy\Big).$$

It is clear that, for $n=0,\dots,N$,

$$\frac{1}{\sqrt{M_n}}\big|d_nW_N(x\dot-t)\big|\le$$

$$\le C\Big(\frac{1}{M_N}\sum_{l=n}^{N}\widetilde{A}_{n,l}+\sum_{l=N+1}^{\infty}\frac{1}{M_l}\widetilde{A}_{n,l}+\frac{1}{M_N}\widetilde{B}_n+\frac{1}{M_N}\widetilde{F}_n\Big)$$

and for $\mathbb{N}\ni s\le N+\alpha$

$$\sqrt{M_s}\big|d_sW+1\big|\le C\Big(\sum_{i=n}^{\infty}\frac{1}{M_i}A^*{}_{n,i+1}+\frac{1}{M_n}B^*_n+\frac{1}{M_n}F^*_n\Big).$$

Taking into consideration our previous estimations (11), (12) and (13) we get the following inequality for all $2/3<p\le 1$:

$$M_N^2\int_{G_m\setminus I_N}\int_{G_m\setminus I_N}\Big(\sup_{|n-s|\le\alpha,s\le N}\big|\Delta^{(1)}_{n,s}a(x,y)\big|\Big)^p dx\,dy\le$$

$$\le C_pM_N^2\big(M_N^{-p/2-1}(M_N^{p/2-1}+M_N^{-p}\big)\le C_p.$$

At the same time this means that $\int\int_{G_m^2\setminus(I\times J)}\big(J^*{}_{\alpha,1}a\big)^p\le C_p$ if $2/3<p\le 1$. It follows in analogous way the same estimation for $J^*{}_{\alpha,2}a$ instead of $J^*{}_{\alpha,1}a$.

It remains to prove the inequality $\int\int_{G_m^2\setminus(I\times J)}\big(J^*{}_{\alpha,3}a\big)^p\le C_p$ for $2/3<p\le 1$. We write as above

$$\int_{G_m^2\setminus(I\times J)}\big(J^*{}_{\alpha,3}a\big)^p=\int_{G_m\setminus I_N}\int_{I_N}\big(J^*{}_{\alpha,3}a\big)^p+$$

$$+\int_{I_N}\int_{G_m\setminus I_N}\left(J^*{}_{\alpha,3}a\right)^p+\int_{G_m\setminus I_N}\int_{G_m\setminus I_N}\left(J^*{}_{\alpha,3}a\right)^p.$$

Then

$$\int_{G_m\setminus I_N}\int_{I_N}(J^*{}_{\alpha,3}a)^p\le$$

$$\le\int_{G_m\setminus I_N}\int_{I_N}\Big(\sup_{n,s}\int_{I_N}\int_{I_N}|a(t,u)||d_nW_N(x\dot{-}t)||d_sW_N(y\dot{-}u)|\,dt\,du\Big)^p dx\,dy\le$$

$$\le M_N^2\int_{G_m\setminus I_N}\int_{I_N}\Big(\sup_{n,s}\int_{I_N}|d_nW_N(x\dot{-}t)|\,dt\int_{I_N}|d_sW_N(y\dot{-}u)|\,du\Big)^p dx\,dy,$$

from which it follows by (4) and (10) that

$$\int_{G_m\setminus I_N}\int_{I_N}(J^*{}_{\alpha,3}a)^p\le C_pM_N\int_{G_m\setminus I_N}\Big(\sup_n\int_{I_N}|d_nW_N(x\dot{-}t)|\,dt\Big)^p dx\le C_p.$$

Of course, by symmetry we get the analogous inequality for the integral on the set $I_N\times(G_m\setminus I_N)$:

$$\int_{I_N}\int_{G_m\setminus I_N}\left(J^*{}_{\alpha,3}a\right)^p\le C_p.$$

Finally, to the estimation of $\int_{G_m\setminus I_N}\int_{G_m\setminus I_N}\left(J^*{}_{\alpha,3}a\right)^p$ we apply the inequality (4) twice, namely

$$\int_{G_m\setminus I_N}\int_{G_m\setminus I_N}(J^*{}_{\alpha,3}a)^p=$$

$$\int_{G_m\setminus I_N}\int_{G_m\setminus I_N}\Big(\sup_{n,s}\int_{I_N}\int_{I_N}|a(t,u)||d_nW_N(x\dot{-}t)||d_sW_N(y\dot{-}u)|\,dt\,du\Big)^p dx\,dy\le$$

$$\le M_N^2\Big(\int_{G_m\setminus I_N}\Big(\sup_n\int_{I_N}|d_nW_N(x\dot{-}t)|\,dt\Big)^p dx\Big)\times$$

$$\times\Big(\int_{G_m\setminus I_N}\Big(\sup_s\int_{I_N}|d_sW_N(y\dot{-}u)|\,du\Big)^p dy\Big)\le C_p.$$

This completes the proof of Lemma 3. ◊

Ths. A and B imply by Lemma 3 the next statement for $J^*{}_\alpha$.

Theorem 1. *Let m be bounded and $\alpha\ge 0$. Then for all $2/3<p<\infty$ and $0<q\le\infty$ there exists a constant $C_{p,q}$ such that*

$$\|J^*{}_\alpha f\|_{p,q}\le C_{p,q}\|f\|_{H^{p,q}}\qquad (f\in H^{p,q}(G_m^2)).$$

In particular, $J^*{}_\alpha$ is of weak type $(1,1)$.

We remark that $J^*{}_\alpha f$ can be defined also for martingales f belonging to $H^{p,q}(G_m^2)$ by density argument. (For the analogous (dyadic) situation see Weisz [17].)

Now let $f\in L^1(G_m^2)$ and assume for almost all $x,y\in G_m$ that

(14) $$\int_{G_m} f(x,u)\,du = \int_{G_m} f(v,y)\,dv = 0.$$

Then it is clear that $d_{n,s}(If) = \Delta_{n,s}f$ $(n, s \in \mathbb{N})$ a.e. which leads to $I^*{}_\alpha f = J^*{}_\alpha f$ a.e. Therefore Th. 1 implies

Theorem 2. *Suppose that m is bounded and $\alpha \geq 0$. If $2/3 < p < \infty$ and $0 < q \leq \infty$ then there exists a constant $C_{p,q}$ such that*

$$\|I^*{}_\alpha f\|_{p,q} \leq C_{p,q}\|f\|_{H^{p,q}}$$

holds for all $f \in H^{p,q}(G_m^2)$ satisfying the assumption (14). *Moreover, I^* is of restricted weak type (1,1), i.e. if $f \in L^1(G_m^2)$ such that* (14) *is true then*

$$\operatorname{mes}(I^*{}_\alpha f > \lambda) \leq C\frac{\|f\|_1}{\lambda} \qquad (\lambda > 0).$$

Finally, we note that the weak type part of Th. 2 implies by standard density argument

Theorem 3. *If m is bounded, $f \in L^1(G_m^2)$ having the property* (14) *and $\alpha \geq 0$ then $d_{n,s}(If) \to f$ a.e. as $n, s \to \infty$ and $|n - s| \leq \alpha$.*

References

[1] BUTZER, P. L. and WAGNER, H. J.: Walsh-Fourier series and the concept of a derivative, *Appl. Anal.* **3** (1973), 29–46.

[2] BUTZER, P. L. and WAGNER, H. J.: On dyadic analysis based on the pointwise dyadic derivative, *Analysis Math.* **1** (1975), 171–196.

[3] BUTZER, P. L. and ENGELS, W.: Dyadic calculus and sampling theorems for functions with multidimensional domain, *Inform. and Controll* **52** (1982), 333–351.

[4] GÁT, GY.: On the two-dimensional pointwise dyadic calculus, *Journal of Approx.* **92** (1998), 191–215.

[5] NAGY, K. and GÁT, GY.: The fundamental theorem of two-parameter pointwise derivate on Vilenkin groups, *Analysis Math.* **25** (1999), 33–55.

[6] ONNEWEER, C. W.: Differentiability for Rademacher series on groups, *Acta Sci. Math. (Szeged)* **39** (1977), 121–128.

[7] PÁL J. and SIMON, P.: On a generalization of the concept of derivative, *Acta Math. Acad. Sci. Hung.* **29** (1977), 155–164.

[8] PÁL, J. and SIMON, P.: On the generalized Butzer-Wagner type a.e. differentiability of integral function, *Annales Univ. Sci. Budapest, Sectio Math.* **20** (1977), 157–165.

[9] SCHIPP, F.: Über einen Ableitungsbegriff von P. L. Butzer und H. J. Wagner, *Mathematica Balkanica* **4** (1974), 541–546.

[10] SCHIPP, F., WADE, W. R., SIMON, P. and PÁL, J.: Walsh series: An introduction to dyadic harmonic analysis, Adam Hilger, Bristol-New York, 1990.

[11] SCHIPP, F. and WADE, W. R.: A fundamental theorem of dydadic calculus for the unit square, *Appl. Anal.* **34** (1989), 203–218.

[12] SIMON, P.: Remarks on Walsh-Fourier multipliers, *Publicationes Math. Debrecen* **52**/3-4 (1998), 635–657.

[13] SIMON, P. and WEISZ, F.: Hardy spaces and the generalization of the dyadic derivative, *Functions, Series, Operators–Alexits Memorial Conference, Budapest, 1999.* (to appear)

[14] WEISZ, F.: Martingale Hardy spaces and their applications in Fourier analysis, Lecture Notes in Math. **1568**, Springer-Verlag, Berlin-Heidelberg-New York, 1994.

[15] WEISZ, F.: Martingale Hardy spaces and the dyadic derivative, *Analysis Math.* **24** (1998), 59–77.

[16] WEISZ, F.: Some maximal inequalities with respect to two-parameter dyadic derivative and Cesaro summability, *Appl. Anal.* **62** (1996), 223–238.

[17] WEISZ, F.: (H_p, L_p)-type inequalities for the two-dimensional dyadic derivative, *Studia Math.* **120**/3 (1996), 271–288.