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POSNER'S SECOND THEOREM AND AN ANNIHILATOR CONDITION

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Abstract: Let R be a prime algebra over a commutative ring K of characteristic different from 2, $d \neq 0$ a non-zero derivation of R, $f(x_1, \ldots, x_n)$ a non-central multilinear polynomial over K in n non-commuting variables, $a \in R$ such that $a[d(f(r_1, \ldots, r_n)), f(r_1, \ldots, r_n)] = 0$, for any $r_1, \ldots, r_n \in R$. Then $a = 0$.

In [13] Posner proved that if R is a prime ring and d a non-zero derivation of R such that $[d(x), x] \in Z(R)$, the center of R, for all $x \in \in R$, then R must be commutative. Many related generalizations have been obtained in the literature, by considering the k-th commutator $[d(x), x]_k$ which, for $k > 1$, is defined by $[d(x), x]_k = [[d(x), x]_{k-1}, x]$. In [7] Lanski showed that if $[d(x), x]_k = 0$, for all x in a Lie ideal of R, then either L is central in R or $\mathrm{char}(R) = 2$ and R satisfies

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$S_4(x_1, \ldots, x_4)$. Since a non-central Lie ideal of a prime ring R contains all the commutators $[x, y]$ for x, y in some non-zero ideal except when $\text{char}(R) = 2$ and R satisfies $S_4(x_1, \ldots, x_4)$, it is natural to investigate the situation when $f(x_1, \ldots, x_n)$ is a (multilinear) polynomial and $[d(f(x_1, \ldots, x_n)), f(x_1, \ldots, x_n)]_k$ is a differential identity for some ideal of R. The result obtained by P.H. Lee and T.K. Lee in [8] and [9] show that in the multilinear case the polynomial $f(x_1, \ldots, x_n)$ must be central-valued unless $\text{char}(R) = 2$ and R satisfies $S_4(x_1, \ldots, x_4)$. In our recent paper we considered an other related generalization; more precisely in [3] we describe the structure of a semiprime algebra R such that any non-zero valuation of $[d(f(x_1, \ldots, x_n)), f(x_1, \ldots, x_n)]$ is an invertible element of R. Here we will continue the study of the set

$$S = \{[d(f(x_1, \ldots, x_n)), f(x_1, \ldots, x_n)],\ x_1, \ldots, x_n \in R\},$$

by proving the following result

Theorem 1. *Let R be a prime algebra over a commutative ring K of characteristic different from 2, d a non-zero derivation of R, $f(x_1, \ldots \ldots, x_n)$ a non-central multilinear polynomial over K in n non-commuting indeterminates, $a \in R$. If $a[d(f(r_1, \ldots, r_n)), f(r_1, \ldots, r_n)] = 0$, for any $r_1, \ldots, r_n \in R$, then $a = 0$ that is $\text{Ann}_R(S) = 0$.*

Remark 1. Our assumption on the characteristic of R is needed as the following example shows:

Let F be a field of characteristic 2, $R = M_2(F)$, the ring of 2×2 matrices over F and $f(x_1, x_2) = [x_1, x_2]$ the commutator polynomial. Let d be the inner derivation induced by a non-central element $q \in M_2(F)$, that is $d(x) = [q, x]$, for all $x \in M_2(F)$; thus, for any $r_1, r_2 \in M_2(F)$,

$$[d(f(r_1, r_2)), f(r_1, r_2)] = [q, [r_1, r_2]]_2 = [q, [r_1, r_2]^2] = 0$$

because $[r_1, r_2]^2 \in Z(M_2(F))$. This implies $S = 0$ and so $\text{Ann}_R(S) = R$.

Of course we do not consider the case when R is a domain; in fact, in this case, either $\text{Ann}_R(S) = 0$ or $[d(f(r_1, \ldots, r_n)), f(r_1, \ldots, r_n)] = 0$, for any $r_1, \ldots, r_n \in R$. In this condition, by [8], $f(x_1, \ldots, x_n)$ must be central in R.

In all that follows let Q be the Martindale quotient ring of R and $C = Z(Q)$ the center of Q, $T = Q *_C C\{X\}$ the free product over C of the C-algebra Q and the free C-algebra $C\{X\}$, with X the countable set consisting of non-commuting indeterminates $x_1, x_2, \ldots, x_n, \ldots$. We refer the reader to [1] for the definitions and the related properties of these objects.

We recall that every derivation of R can be uniquely extended to a derivation of Q. Moreover, since R is a prime ring, we may assume $K \subseteq C$ and so for any $\alpha \in K$ one has $d(\alpha) \in C$.

We will use the following notation:

$$f(x_1,\ldots,x_n) = x_1x_2\ldots x_n + \sum_{\sigma\in S_n} \alpha_\sigma x_{\sigma(1)}x_{\sigma(2)}\ldots x_{\sigma(n)}$$

for some $\alpha_\sigma \in C$ and moreover we denote by $f^d(x_1,\ldots,x_n)$ the polynomial obtained from $f(x_1,\ldots,x_n)$ by replacing each coefficient α_σ with $d(\alpha_\sigma)$. Thus we write

$$d(f(r_1,\ldots,r_n)) = f^d(r_1,\ldots,r_n) + \sum_i f(r_1,\ldots,d(r_i),\ldots,r_n),$$

for all $r_1,r_2,\ldots,r_n$ in R. Hence if

$$a \in \mathrm{Ann}_R(\{[d(f(r_1,\ldots,r_n)), f(r_1,\ldots,r_n)],\ r_i \in R\})$$

then R satisfies the generalized differential identity

$$a[d(f(x_1,\ldots,x_n)), f(x_1,\ldots,x_n)] =$$

$$= a\Big([f^d(x_1,\ldots,x_n) + \sum_i f(x_1,\ldots,d(x_i),\ldots,x_n), f(x_1,\ldots,x_n)]\Big)$$

Since by [10] R and Q satisfy the same differential identities, then

$$a[d(f(r_1,\ldots,r_n)), f(r_1,\ldots,r_n)] = 0, \quad \text{for all} \quad r_1,\ldots,r_n \in Q.$$

Of course Q is a prime ring and, by replacing R by Q, we may assume, without loss of generality, $R = Q$, $C = Z(R)$ and R is a C-algebra centrally closed. We also assume $\mathrm{char}(R) \neq 2$ and $f(x_1,\ldots,x_n)$ non-central valued.

We begin with the following:

Lemma 1. *If d is an outer derivation of R then $a = 0$.*

Proof. Suppose by contradiction that $a \neq 0$. Since R satisfies the generalized differential identity

$$a[d(f(x_1,\ldots,x_n)), f(x_1,\ldots,x_n)] =$$

$$= a\Big([f^d(x_1,\ldots,x_n) + \sum_i f(x_1,\ldots,d(x_i),\ldots,x_n), f(x_1,\ldots,x_n)]\Big)$$

and d is an outer derivation, then, by Kharchenko's theorem (see [6] and [10]), R satisfies the generalized polynomial identity

$$a[f^d(x_1,\dots,x_n)+\sum_i f(x_1,\dots,y_i,\dots,x_n),f(x_1,\dots,x_n)]$$

and in particular $a[f(x_1,x_2,\dots,y_i,\dots,x_n),f(x_1,x_2,\dots,x_n)]$ for $i = 1,\dots,n$. This means that R satisfies a non-trivial generalized polynomial identity. Since, as we said above, R is a C-algebra centrally closed, then by [12] R is a dense ring of linear transformations of a vector space V over a division ring D. We will prove that R must satisfy the ordinary polynomial identities $[f(x_1,\dots,y_i,\dots,x_n),f(x_1,\dots,x_n)]$, for any $i=1,\dots,n$. By first suppose that $\dim_D V=\infty$. Let v any vector of V. If v and av are linearly independent over D then there exist vectors $v_2,\dots,v_{n-1}$, $w_1,\dots,w_n$ such that $v, av = v_1, v_2,\dots,v_{n-1},w_1,\dots,w_n$ are linearly independent. By density of R there exist $r_1,\dots,r_n,s_n\in R$ such that

$$r_iv=0\quad\forall i;\qquad r_iv_i=v_{i-1},\quad i=2,\dots,n-1;$$
$$r_iw_i=w_{i-1},\quad i=2,\dots,n;\qquad r_1v_1=w_n;\qquad r_1w_1=v;$$
$$r_iv_j=r_iw_j=0\quad\text{for all other possible choices}$$
$$s_nv=v_{n-1};\qquad s_nv_j=s_nw_j=0\quad\forall j.$$

Thus we have:

$$f(r_1,\dots,r_n)v=0,\quad f(r_1,\dots,r_{n-1},s_n)v=w_n,\quad f(r_1,\dots,r_n)w_n=v.$$

Therefore we get the contradiction

$$0=a[f(r_1,\dots r_{n-1},s_n),f(r_1,\dots,r_n)]v=-av\neq 0.$$

This implies that, for any $v\in V$, v and av are linearly D-dependent, and by standard arguments it follows that $a\in Z(R)$. Since $a\neq 0$,we get that $[f(x_1,\dots,x_{n-1},y_n),f(x_1,\dots,x_n)]$ is an identity in R.

More generally, in the same way we can prove that, for any $i = 1,\dots,n$, $[f(x_1,\dots,y_i,\dots,x_n),f(x_1,\dots,x_n)]$ is a polynomial identity for R.

Let now $\dim_D V=k$ finite and for any $i=1,\dots,n$ let

$$S_i=\{[f(r_1,\dots,s_i,\dots,r_n),f(r_1,\dots,r_n)],\quad r_1,\dots,r_n,s_i\in R\}.$$

Consider the following subring of R : $A=\cap_{i=1}^n \mathrm{Ann}_R(S_i)$. Since R is not a domain then $k\geq 2$ and R contains some non-trivial idempotent element. Moreover A is invariant under the action of all special automorphisms of R, in the sense of [4] and so one of the following holds: either $A=R$ or $A\subseteq Z(R)$, that is $a\in Z(R)$. In both cases, since $a \neq 0$ and R is prime, as above we have that, for any i, $[f(x_1,\ldots,y_i,\ldots,x_n), f(x_1,\ldots,x_n)]$ is a polynomial identity for R.

Now we will prove that R satisfies the polynomial identity

$$[f(y_1,\ldots,y_n), f(x_1,\ldots,x_n)]_n = 0.$$

By first, applying d to $[f(s_1,r_2,\ldots,r_n), f(r_1,\ldots,r_n)] = 0$, we have

$$0 = [f^d(s_1,r_2,\ldots,r_n) + f(d(s_1),r_2,\ldots,r_n)+ \\ +\sum_{i\geq 2} f(s_1,r_2,\ldots,d(r_i),\ldots,r_n), f(r_1,\ldots,r_n)]+ \\ +[f(s_1,r_2,\ldots,r_n), f^d(r_1,\ldots,r_n) + \sum_i f(r_1,\ldots,d(r_i),\ldots,r_n)].$$

As above, since d is not an inner derivation, R must satisfy the polynomial identity

$$[f^d(y_1,x_2,\ldots,x_n) + f(z_1,x_2,\ldots,x_n)+ \\ +\sum_{i\geq 2} f(y_1,x_2,\ldots,y_i,\ldots,x_n), f(x_1,\ldots,x_n)]+ \\ +[f(y_1,x_2,\ldots,x_n), f^d(x_1,\ldots,x_n) + \sum_i f(x_1,\ldots,y_i,\ldots,x_n)].$$

Of course the blended component of this identity in the $n+2$ variables $y_1,y_2,x_1,x_2,\ldots,x_n$ is the following polynomial

$$\Big[f(y_1,y_2,x_3,\ldots,x_n), f(x_1,\ldots,x_n)\Big]+ \\ +\Big[f(y_1,x_2,\ldots,x_n), f(x_1,y_2,\ldots,x_n)\Big]$$

and, by [5, Lemma 1 pag. 15], it is a polynomial identity for R too. Therefore, by commuting this last identity with $f(x_1,.,x_n)$, we obtain the following polynomial identity

$$[f(y_1,y_2,\ldots,x_n), f(x_1,\ldots,x_n)]_2.$$

Now apply d to $[f(s_1,s_2,r_3,\ldots,r_n), f(r_1,\ldots,r_n)]_2 = 0$. We have

$$\big[f^d(s_1,s_2,r_3,\ldots,r_n), f(r_1,\ldots,r_n)\big]_2 + \\ [f(d(s_1),s_2,r_3,\ldots,r_n) + f(s_1,d(s_2),r_3,\ldots,r_n)+ \\ +\sum_{i\geq 3} f(s_1,s_2,r_3,\ldots,d(r_i),\ldots,r_n), f(r_1,\ldots,r_n)]_2+$$

$$+\Big[[f(s_1,s_2,r_3,\dots,r_n),f^d(r_1,\dots,r_n)+$$
$$+\sum_{i=1}^{n} f(r_1,\dots,d(r_i),\dots,r_n)],f(r_1,\dots,r_n)\Big]+$$

$$+\Big[[f(s_1,s_2,r_3,\dots,r_n),f(r_1,\dots,r_n)],f^d(r_1,\dots,r_n)+$$
$$+\sum_{i=1}^{n} f(r_1,\dots,d(r_i),\dots,r_n)\Big]=0.$$

As above, since d is not an inner derivation, R must satisfy the polynomial identity

$$\left[f^d(y_1,y_2,x_3,\dots,x_n),f(x_1,\dots,x_n)\right]_2+$$
$$+[f(z_1,y_2,x_3,\dots,x_n)+f(y_1,z_2,x_3,\dots,x_n)+$$
$$+\sum_{i\geq 3} f(y_1,y_2,x_3,\dots,y_i,\dots,x_n),f(x_1,\dots,x_n)]_2+$$
$$+\Big[[f(y_1,y_2,x_3,\dots,x_n),f^d(x_1,\dots,x_n)+$$
$$+\sum_{i=1}^{n} f(x_1,\dots,y_i,\dots,x_n)],f(x_1,\dots,x_n)\Big]+$$
$$+\Big[[f(y_1,y_2,x_3,\dots,x_n),f(x_1,\dots,x_n)],f^d(x_1,\dots,x_n)+$$
$$+\sum_{i=1}^{n} f(x_1,\dots,y_i,\dots,x_n)\Big].$$

The blended component of this identity in the $n+3$ variables

$$y_1,y_2,y_3,x_1,x_2,\dots,x_n$$

is the following

$$(1)\qquad [f(y_1,y_2,y_3,x_4,\dots,x_n),f(x_1,\dots,x_n)]_2+$$
$$+[[f(y_1,y_2,x_3,\dots,x_n),f(x_1,x_2,y_3,x_4,\dots,x_n)],f(x_1,\dots,x_n)]+$$
$$+[[f(y_1,y_2,x_3,\dots,x_n),f(x_1,\dots,x_n)],f(x_1,x_2,y_3,x_4,\dots,x_n)]$$

Hence R must satisfy this last identity ([5]). Since

$$[f(y_1,y_2,x_3,\dots,x_n),f(x_1,\dots,x_n)]_2$$

and

$$[f(x_1, x_2, y_3, x_4, \ldots, x_n), f(x_1, \ldots, x_n)]$$

are identities for R, by commuting the (1) with $f(x_1, \ldots, x_n)$ we get that

$$[f(y_1, y_2, y_3, x_4, \ldots, x_n), f(x_1, \ldots, x_n)]_3$$

is an identity for R.

Continuing this process we will finally get

$$[f(s_1, \ldots, s_n), f(r_1, \ldots, r_n)]_n = 0, \text{ for all } s_1, \ldots, s_n, r_1, \ldots, r_n \in R.$$

By main theorem in [8] $f(x_1, \ldots, x_n)$ is a central polynomial for R. In light of this contradiction, a must be zero and this conclude the proof. ◊

Remark 2. In all that follows we will consider the only case when d is an inner derivation induced by a non-central element q of Q.

Remark 3. Recall that if B is a basis of Q over C, then any element of $T = Q *_C C\{x_1, \ldots, x_n\}$ can be written in the form $g = \sum_i \alpha_i m_i$, where $\alpha_i \in C$ and m_i are B-monomials, that is $m_i = q_0 y_1 \cdots y_n q_n$, with $q_i \in B$ and $y_i \in \{x_1, \ldots, x_n\}$. In [2] it is showed that a generalized polynomial $g = \sum_i \alpha_i m_i$ is the zero element of T if and only if any α_i is zero. As a consequence, if $a_1, a_2 \in Q$ are linearly independent over C and $a_1 g_1(x_1, \ldots, x_n) + a_2 g_2(x_1, \ldots, x_n) = 0 \in T$, for some $g_1, g_2 \in T$, then both $g_1(x_1, \ldots, x_n)$ and $g_2(x_1, \ldots, x_n)$ are the zero element of T.

Lemma 2. *If R does not satisfy any non-trivial generalized polynomial identity, then $a = 0$.*

Proof. Since R does not satisfy any non-trivial generalized polynomial identity, we have that

$$a[q, f(x_1, \ldots, x_n)]_2$$

is the zero element in the free product $T = Q *_C C\{x_1, \ldots, x_n\}$, that is

$$a\Big(qf(x_1, \ldots, x_n)^2 + f(x_1, \ldots, x_n)^2 q -$$
$$-2f(x_1, \ldots, x_n) q f(x_1, \ldots, x_n)\Big) = 0 \in T.$$

Suppose $aq \neq 0$ and a, aq linearly independent over C. We have

$$aqf(x_1, \ldots, x_n)^2 + a(f(x_1, \ldots, x_n)^2 q -$$
$$-2f(x_1, \ldots, x_n) q f(x_1, \ldots, x_n)) = 0 \in T.$$

By Remark 3, $aqf(x_1, \ldots, x_n)^2 = 0 \in T$. Since R does not satisfy any non-trivial generalized polynomial identity, this forces $aq = 0$, which is a contradiction.

Thus we assume a, aq linearly C-dependent, that is

$$aq = \beta a, \beta \in C \quad \text{and also} \quad a(\beta - q) = 0.$$

Moreover q and $q - \beta$ induce the same inner derivation in R. Hence, without loss of generality, we may analyze the case $aq = 0$. In this case we have

$$g(x_1, \dots, x_n) = af^2(x_1, \dots, x_n)q -$$
$$-2af(x_1, \dots, x_n)qf(x_1, \dots, x_n) = 0 \in T.$$

If a and q are linearly independent over C, we can consider a representation of g in terms of B-monomials, for some basis B which contains a and q. In this representation occour two kind of B-monomials; more precisely they are:

$a \cdot x_{\pi(1)}x_{\pi(2)} \cdots x_{\pi(n)} \cdot x_{\varrho(1)}x_{\varrho(2)} \cdots x_{\varrho(n)} \cdot q$ which come from the addend $af^2(x_1, \dots, x_n)q$;

$a \cdot x_{\sigma(1)}x_{\sigma(2)} \cdots x_{\sigma(n)} \cdot q \cdot x_{\tau(1)}x_{\tau(2)} \cdots x_{\tau(n)}$ which come from the addend $af(x_1, \dots, x_n)qf(x_1, \dots, x_n)$.

By Remark 3 we obtain that both

$$af^2(x_1, \dots, x_n)q \quad \text{and} \quad af(x_1, \dots, x_n)qf(x_1, \dots, x_n)$$

are the zero element in T. Since $q \neq 0$, we get the required conclusion $a = 0$.

Finally, if a and q are linearly dependent over C then, for some $\gamma \in C$, we have

$$g(x_1, \dots, x_n) = \gamma qf^2(x_1, \dots, x_n)q -$$
$$-2\gamma qf(x_1, \dots, x_n)qf(x_1, \dots, x_n) = 0 \in T.$$

In this case, for B containing q, the B-monomials which occour are the following:

$$q \cdot x_{\pi(1)}x_{\pi(2)} \cdots x_{\pi(n)} \cdot x_{\varrho(1)}x_{\varrho(2)} \cdots x_{\varrho(n)} \cdot q;$$

$$q \cdot x_{\sigma(1)}x_{\sigma(2)} \cdots x_{\sigma(n)} \cdot q \cdot x_{\tau(1)}x_{\tau(2)} \cdots x_{\tau(n)}.$$

Since $q \notin C$ then, as above we obtain

$$qf^2(x_1, \dots, x_n)q = 0 \quad \text{and} \quad qf(x_1, \dots, x_n)qf(x_1, \dots, x_n) = 0 \quad \text{in} \quad T.$$

In any case we must have $q = 0$, a contradiction. ◊

Lemma 3. *Let R be a dense ring of linear transformations over an infinite dimensional right vector space V over a division ring D. Then $a = 0$.*

Proof. Suppose that a is non-zero.

Our first aim is to show that for any $v \in V$ then v, qv are linearly D-dependent.

By contradiction let v, qv be D-independent. There exists $w, w_1, \dots, w_{n-1}, v_1, \dots, v_{n-1} \in V$ such that $v, qv = u, w, w_1 \dots, w_{n-1}, v_1, \dots, v_{n-1}$ are linearly independent. By the density of R, there exist $r_1, \dots, r_n \in R$ such that

$$r_i v = 0, \quad \forall i; \quad r_i u = r_i w = 0, \quad \forall i \neq n; \quad r_n u = w_{n-1}, \quad r_n w = v_{n-1};$$
$$r_i w_i = w_{i-1}, \quad r_i v_i = v_{i-1}, \quad i = 2, \dots, n-1; \quad r_1 w_1 = w, \quad r_1 v_1 = v;$$
$$r_i v_j = 0 \quad r_i w_j = 0, \quad \text{for all other possible choices.}$$

By calculation we obtain:

$$f(r_1, \dots, r_n)v = 0, \quad f(r_1, \dots, r_n)u = w, \quad f(r_1, \dots, r_n)w = v.$$

Hence, if av is non-zero, then we get the contradiction

$$0 = a[q, f(r_1, \dots, r_n)]_2 v = a f(r_1, \dots, r_n)^2 u = av \neq 0.$$

Now suppose $av = 0$.

Since $a \neq 0$, there exists $w \in V$ such that $aw \neq 0$. Hence $a(w - v) = aw \neq 0$. By the previous argument we have that w, qw are linearly D-dependent and $(w - v)$, $q(w - v)$ too.

Thus there exist $c, d \in D$ such that $qw = wc$ and $q(w - v) = (w - v)d$. Moreover v, w are linearly independent and so there exist $w_3, \dots, w_{n-1} \in V$ such that $v, w, w_3, \dots, w_{n-1}$ are linearly independent and $r_1, \dots, r_n \in R$ such that

$$r_i v = 0 \quad \forall i; \quad r_n w = w_{n-1}$$
$$r_i w = 0 \quad i = 1, \dots, n-1; \quad r_i w_i = w_{i-1}, \quad i = 2, \dots, n-1$$
$$r_1 w_1 = w - v, \quad r_i w_j = 0 \quad \text{for all other possible choices.}$$

This implies that

$$f(r_1, \dots, r_n)v = 0, \quad f(r_1, \dots, r_n)w = w - v, \quad f(r_1, \dots, r_n)^2 w = w - v$$

and

$$0 = a(f(r_1,\ldots,r_n)^2q+$$
$$+qf(r_1,\ldots,r_n)^2 - 2f(r_1,\ldots,r_n)qf(r_1,\ldots,r_n))w =$$
$$= a((w-v)c + (w-v)d - 2(w-v)d) = awc - awd = aw(c-d).$$

Because $aw \neq 0$ then $c = d$ and $qv = vd$, that is v, qv are linearly D-dependent in any case. Standard arguments prove that there exists $\beta \in D$ such that $qv = v\beta$, for all $v \in V$ and also, by using this fact, that $q \in Z(R)$, which contradicts our hypotesis. ◊

Proof of Theorem 1. By the previous results, we assume that d is the inner derivation induced by $q \in R$, moreover $C = Z(R)$ and R is a C-algebra centrally closed, that is $R = RC$. If R does not satisfy any non-trivial generalized polynomial identity then, by Lemma 2, $a = 0$. Thus we may suppose that R satisfies a non-trivial generalized polynomial identity. By Martindale's theorem in [12], R is a primitive ring which is isomorphic to a dense ring of linear transformations of a vector space V over a division ring D. If $\dim_D V = \infty$, then, by Lemma 3, we get the required conclusion.

Therefore we consider the case $\dim_D(V) = k$, with k finite positive integer. Of course $k \geq 2$, because R is not a domain. In this condition R is a simple ring which satisfies a non-trivial generalized polynomial identity. By [7, Lemma 2; 14, Th. 2.3, 29] $R \subseteq M_t(F)$, for a suitable field F and $t \geq 2$, moreover $M_t(F)$ satisfies the same generalized identity of R, hence

$$a[q, f(r_1,\ldots,r_n)]_2 = 0, \text{ for all } r_1,\ldots,r_n \in M_t(F)$$

and moreover $f(x_1,\ldots,x_n)$ is a non-central polynomial for $M_t(F)$. Since $f(x_1,\ldots,x_n)$ is not central then, by [11], there exist $u_1,\ldots,u_n \in M_t(F)$ and $b \in F - \{0\}$, such that $f(u_1,\ldots,u_n) = be_{kl}$, with $k \neq l$. Here e_{kl} denotes the usual matrix unit with 1 in (k,l)-entry and zero elsewhere. Moreover, since the set $\{f(v_1,\ldots,v_n) : v_1,\ldots,v_n \in M_t(F)\}$ is invariant under the action of all F-automorphisms of $M_t(F)$, then for any $i \neq j$ there exist $r_1,\ldots,r_n \in M_t(F)$ such that $f(r_1,\ldots,r_n) = be_{ij}$. Hence, for all $i \neq j$,

$$0 = a[q, f(r_1,\ldots,r_n)]_2 = -2b^2ae_{ij}qe_{ij}.$$

In other words, since $\mathrm{char}(R) \neq 2$ and $b \neq 0$, either the i-th column of the matrix a is zero or, for all j different from i, the (j,i)-entry q_{ji} of q is zero.

Case 1: $t = 2$. Suppose that q is not a diagonal matrix, say $q_{12} \neq$

$\neq 0$. In this case, as we said above, the 2-nd column of a is zero. Of course we may assume $q_{21} = 0$, otherwise the first column of a is zero too, and we are done. In other words we are in the following situation:

$$q = \begin{bmatrix} q_{11} & q_{12} \\ 0 & q_{22} \end{bmatrix}, \quad q_{12} \neq 0; \qquad a = \begin{bmatrix} a_{11} & 0 \\ a_{21} & 0 \end{bmatrix}.$$

Now, since $f(x_1, \ldots, x_n)$ is not central for $M_(F)$, by [11, Lemmas 2 and 9], there exists a sequence of matrices $\underline{r} = (r_1, \ldots, r_n)$ such that $f(\underline{r}) = \alpha e_{11} + \beta e_{22}$ is not central, that is $\alpha \neq \beta$. Let φ the inner automorphism on $M_t(F)$ defined by $\varphi(x) = (1 + e_{21})x(1 - e_{21})$. Thus $f(\underline{s}) = f(\varphi(\underline{r})) = f(\underline{r}) + (\alpha - \beta)e_{21}$ is a valuation of f on R.

By calculation, it follows that

$$[q, f(\underline{s})]_2 = (\alpha - \beta)^2 \begin{bmatrix} -q_{12} & q_{12} \\ q_{22} - q_{11} - 2q_{12} & q_{12} \end{bmatrix}.$$

If $a \neq 0$, since $a[q, f(\underline{s})]_2 = 0$ and $([q, f(\underline{s})]_2)^2 \in F$ we have that $([q, f(\underline{s})]_2)^2 = 0$. This implies that

$$q_{12}^2 + q_{12}(q_{22} - q_{11} - 2q_{12}) = 0, \text{ that is } q_{12}(q_{22} - q_{11} - q_{12}) = 0$$

and, since $q_{12} \neq 0$,

$$q_{22} - q_{11} - q_{12} = 0, \quad \text{that is} \quad q_{22} - q_{11} - 2q_{12} = -q_{12}.$$

Therefore

$$[q, f(\underline{s})]_2 = q_{12}(\alpha - \beta)^2 \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$$

and

$$0 = a[q, f(\underline{s})]_2 = q_{12}(\alpha - \beta)^2 \begin{bmatrix} -a_{11} & a_{11} \\ -a_{21} & a_{21} \end{bmatrix}$$

that is $a_{11} = a_{21} = 0$ and we get the contradiction that $a = 0$.

Of course we get the same contradiction in the case $q_{21} \neq 0$. To do this we choose the inner automorphism $\psi(x) = (1 + e_{12})x(1 - e_{12})$, replace $f(\varphi(\underline{r}))$ by $f(\psi(\underline{r}))$ and proceed as before.

Thus we conclude that if $k = 2$ then q must be a diagonal matrix.

Case 2: $t \geq 3$. Also in this case we want to prove that q is a diagonal matrix. Suppose that there exists some non-zero entry q_{ji} of q, for $i \neq j$. As we said above the i-th column of a is zero. Let $m \neq i, j$ and $\varphi_{mi}(x) = (1 + e_{mi})x(1 - e_{mi})$. Consider the following valutations of $f(x_1, \ldots, x_n)$:

$$f(\underline{r}) = \gamma e_{ij}, \quad f(\underline{s}) = \varphi_{mi}(f(\underline{r})) = \gamma e_{ij} + \gamma e_{mj}, \quad \gamma \neq 0.$$

Since $f(\underline{s})^2 = 0$ we have

$$0 = a[q, f(\underline{s})]_2 = -2\gamma^2 a(e_{ij} + e_{mj})q(e_{ij} + e_{mj}).$$

Moreover, since the i-th column of a is zero, we obtain $-2\gamma^2 a(q_{ji} + + q_{jm})e_{mj} = 0$. Notice that if $q_{ji} + q_{jm} = 0$, then $q_{jm} = -q_{ji} \neq 0$, so the m-th column of a is zero. On the other hand, if $q_{ji} + q_{jm} \neq 0$, it follows again that the m-th column of a is zero. Hence we can say that a has at most one non-zero column, the j-th one.

Let now ψ any F-automorphism of $M_t(F)$, then

$$0 = \psi(a)[\psi(q), \psi(f(r_1, \dots, r_n))]_2 = \psi(a)[\psi(q), f(s_1, \dots, s_n)]_2$$

for all $s_1, \dots, s_n \in M_t(F)$. Therefore, as above, we can conclude that, if the (j,i)-entry of $\psi(q)$ is non-zero, for some $j \neq i$, then $\psi(a)$ has at most one non-zero column, the j-th one.

Let now $\psi(x) = (1 + e_{jm})x(1 - e_{jm}))$, with $m \neq j, i$. Hence $\psi(q) = q + e_{jm}q - qe_{jm} - e_{jm}qe_{jm}$ and so its (j,i)-entry is $q_{ji} + q_{mi}$.

If $q_{ji} + q_{mi} = 0$ then $q_{ji} = -q_{mi} \neq 0$, that is the (m,i)-entry of q is non-zero. In this case a has at most one non-zero column, the m-th one; but $m \neq j$ and so any column of a is zero.

If $q_{ji} + q_{mi} \neq 0$ then the (j,i)-entry of $\psi(q)$ is non-zero, hence $\psi(a)$ has at most one non-zero column, the j-th one.

Since $\psi(a) = (\sum_h a_{hj}e_{hj} + a_{mj}e_{jj}) - (\sum_h a_{hj}e_{hm} + a_{mj}e_{jm})$, then, for any $h \neq j$ must be $a_{hj} = 0$ and also $a_{jj} + a_{mj} = 0$. But in this situation we get $a = 0$. Therefore, if $a \neq 0$, then $q_{ji} = 0$, for all $j \neq i$.

The previous two cases show that q is a diagonal matrix, $q = = \sum q_{kk}e_{kk}$. Moreover if φ is an automorphism of $M_t(F)$, the same conclusion holds for $\varphi(q)$, since as above

$$0 = \varphi(a)[\varphi(q), \varphi(f(r_1, \dots, r_n))]_2 = \varphi(a)[\varphi(q), f(s_1, \dots, s_n)]_2.$$

Therefore, for any $i \neq j$, $\varphi(q) = (1 + e_{ij})q(1 - e_{ij})$ must be a diagonal matrix. Thus $(q_{jj} - q_{ii})e_{ij} = 0$, that is $q_{jj} = q_{ii}$ and q is a central element. This contradiction implies $a = 0$ and we are done. ◇

References

[1] BEIDAR K.I., MARTINDALE III, W.S. and MIKHALEV A.V.: Rings with generalized identities, Pure and Applied Math. Dekker, New York, 1996.

[2] CHUANG, C.L.: GPIs having coefficients in Utumi quotient rings, *Proc. Amer. Mat. Soc.* **103**/3 (1988), 723–728.

[3] De FILIPPIS, V. and Di VINCENZO, O.M.: Derivations on multilinear polynomials in semiprime rings *Comm. in Algebra* **27**/12 (1999), 5975–5983.

[4] HERSTEIN, I.N.: A theorem on invariant subrings, *J. Algebra* **83** (1983), 26–32.

[5] JACOBSON, N.: P.I. Algebras, an introduction, Lecture Notes in Mathematics, No. 441, Springer Verlag, New York 1975.

[6] Kharchenko, V.K.: Differential identities of prime rings, *Algebra and Logic* **17** (1978), 155–168.

[7] LANSKI, C.: An Engel condition with derivation, *Proc. Amer. Math. Soc.* **118**/3 (1993) 731–734.

[8] LEE, P.H. and LEE, T.K.: Derivations with Engel conditions on multilinear polynomials, *Proc. Amer. Math. Soc.* **124**/9 (1996) 2625–2629.

[9] LEE, T.K.: Derivations with Engel conditions on polynomials, *Algebra Colloq.* **5**/1 (1998), 13–24.

[10] LEE, T.K.: Semiprime ring with differential identities, *Bull. Inst. Math. Acad. Sinica* **20**/1 (1992), 27–38.

[11] LERON, U.: Nil and power central polynomials in rings, *Trans. Amer. Math. Soc.* **202** (1975), 97–103.

[12] MARTINDALE III, W.S.: Prime rings satisfying a generalized polynomial identity, *J. Algebra* **12** (1969), 576–584.

[13] POSNER, E.C.: Derivations in prime rings, *Proc. Amer. Math. Soc.* **8** (1957), 1093–1100.

[14] ROWEN, L.: Polynomial identities in Ring Theory, Pure and Applied Math. vol. 84, Academic Press, New York 1980.